Math 713 Summary Sept 20, 10

Finish the proof of the structure theorem of $\mathbb{R}$:

Each open set O is a countable union of disjoint open intervals. The representation is unique in the sense that if $\mathcal{C}$ and $\mathcal{D}$ are two pairwise disjoint collections of open intervals whose union is O, then $\mathcal{C} = \mathcal{D}$.

We have already showed the existance of a countable collection

$$\mathcal{I} = \{ I_x : x \in O \}$$

of intervals. The only thing left is to show uniqueness.

Before showing uniqueness I would like to revisit the proof of countability. The proof in the text and the one mentioned last lecture used the axiom of choice to select one rational number from each I_x.

It is possible to show countability using Proposition 1.7 whose proof is based on induction instead.

Let $V = O \cap \mathbb{Q}$. Then $V \subseteq \mathbb{Q}$ is countable, since $\mathbb{Q}$ is countable. Define $h : V \longrightarrow \mathcal{I}$ as $h(r) = I_r$.

Claim h is onto. Let $I_x \in \mathcal{I}$. Then since I_x is an interval it contains a rational number r.

Now $r \in I_r \cap I_x \neq \emptyset$ implies $I_r = I_x$ and so $h(r) = I_x$. Therefore h is onto. It follows that $\mathcal{I}$ is countable.

Math 7?B Summary Sept 20, '10

Proof of uniqueness

Let $\mathcal{I} = \{I_1, I_2, \dots\}$ and $\mathcal{J} = \{J_1, J_2, \dots\}$ be two countable collections of disjoint open intervals such that

$$\bigcup I_n = \bigcup J_m = O.$$

Claim the endpoints of the intervals are not in O.

In particular if $I = (a,b) \in \mathcal{I}$ and $J = (\alpha, \beta) \in \mathcal{J}$ then

$$a \notin O, \quad b \notin O, \quad \alpha \notin O \text{ and } \beta \notin O.$$

I will only show that $b \notin O$ since each of these are similar. For contradiction suppose $b \in O$. Then there $I_n \in \mathcal{I}$ such that $b \in I_n$. Since I_n is open there is $\varepsilon > 0$ such that $(b-\varepsilon, b+\varepsilon) \subseteq I_n$.

Take $r = \max(b-\varepsilon, a)$. Since $a < b$ then $r < b$. Therefore $r \in (a,b) \cap (b-\varepsilon, b+\varepsilon) = I \cap I_n \neq \emptyset$. This implies that $I = I_n$. But then $b \in I_n = I = (a,b)$ is a contradiction.

Claim $I \cap J \neq \emptyset$ implies $I = J$ for $I \in \mathcal{I}$ and $J \in \mathcal{J}$.

Let $I = (a,b)$, $J = (\alpha, \beta)$. Then

$I \cap J = \emptyset$ means either $b \leq \alpha$ or $\beta \leq a$.

Therefore

$I \cap J \neq \emptyset$ means both $b > \alpha$ and $\beta > a$.

Now

$b \notin O$ implies $b \notin (\alpha, \beta)$, since $b > \alpha$ then $b \geq \beta$.
$\beta \in O$ implies $\beta \notin (a, b)$. Since $\beta > a$ then $\beta \geq b$.

It follows that $b = \beta$. Similarly

$a \notin O$ implies $a \notin (\alpha, \beta)$. Since $a < \beta$ then $a \leq \alpha$.
$\alpha \notin O$ implies $\alpha \notin (a, b)$. Since $\alpha < b$ then $\alpha \leq a$.

It follows that $a = \alpha$. Therefore $I = J$.

<u>Claim</u> $\mathcal{I} = \mathcal{J}$

"$\subseteq$" If $I \in \mathcal{I}$ then since $I \subseteq O = \bigcup J_m$ there is some $J \in \mathcal{J}$ such that $I \cap J \neq \emptyset$. Then $I = J$ and so $I \in \mathcal{J}$.

"$\supseteq$" similar.

Therefore $\mathcal{I} = \mathcal{J}$ which finishes the proof of uniqueness.

<u>Definition of open sets and relatively open sets</u>

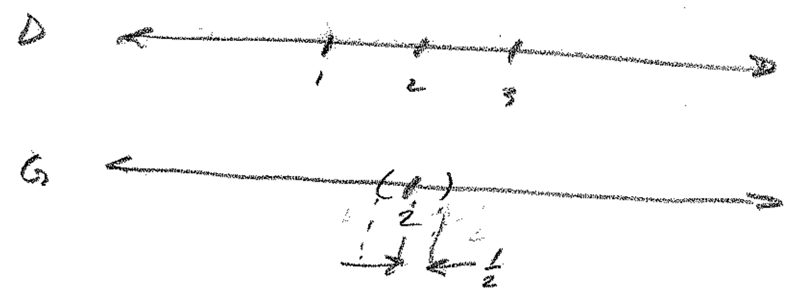
Let $G \subseteq \mathbb{R}$ then G is open if and only if for every $x \in G$ there is $r > 0$ such that $(x-r, x+r) \subseteq G$.

Let $G \subseteq D \subseteq R$. Then G is open relative D if and only if for every $x \in G$ there is $r > 0$ such that $(x-r, x+r) \cap D \subseteq G$.

<u>Read the proof of Theorem 2.5 for next time</u>

Example of an open set relative to D

Let $D = \{1, 2, 3\}$ and $G = \{2\}$



Choose $r = \frac{1}{2}$ then $(2-\frac{1}{2}, 2+\frac{1}{2}) \cap D = \{2\} \subseteq G$.
and so G is open relative to D.