Review Summary Sept 24-25, '10

Discussion of

$\forall$ for every or all

$\exists$ there exists.

Recall the definition of limit. Let $f: D \to \mathbb{R}$ then

$$\lim_{x \to a} f(x) = L$$

means

$\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $x \in D$ and $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$.

Breaking this down in pieces

$\forall \varepsilon > 0$ then $p(\varepsilon)$ is true

where $p(\varepsilon)$ is the proposition

$p(\varepsilon) =$ "$\exists \delta > 0$ s.t. $x \in D$ and $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$."

Let's make a simpler example.

Suppose

$p(n) =$ " n is even "

and $D = \{1, 2, 3, 4\}$. Thus

$q =$ " $\forall n \in D$ then $p(n)$ is true "

is a false proposition. That is

q is false

means

not q is true.

What is not q? To negate

$\forall$ changes to $\exists$

and $\exists$ changes to $\forall$. Thus

not $q =$ " $\exists n \in D$ s.t. $p(n)$ is false "

In particular

$3 \in D$ and $p(3)$ is false

So not q is true.

Let $E = \{2, 4, 16\}$. Then

r = "$\forall n \in E$ then $p(n)$ is true"

means

$p(2)$ is true
$p(4)$ is true
$p(16)$ is true.

Therefore r is a true proposition. More information about the universal and existential quantifiers $\forall$ and $\exists$ is taught in math 373.

Discussion about the Lindelöf's theorem

Let $\mathcal{O}$ be a collection of open sets. Then there is a countable subcollection $\{O_1, O_2, O_3, \dots\} \subseteq \mathcal{O}$ such that

$$\bigcup_{O \in \mathcal{O}} O = \bigcup_{n} O_n$$

How to find a countable subcollection.

Try something to do with $\mathbb{Q}$.

Here is an idea.

Let $P = \mathbb{Q} \cap \bigcup_{O \in \mathcal{O}} O$.

Then P is countable and $P \subseteq \bigcup_{O \in \mathcal{O}} O$. Then

$$\forall r \in P \text{ there is } O_r \in \mathcal{O} \text{ s.t } r \in O_r.$$

Is it true that

$$\bigcup_{r \in P} O_r = \bigcup_{O \in \mathcal{O}} O \ ?$$

Clearly $\bigcup_{r \in P} O_r \subseteq \bigcup_{O \in \mathcal{O}} O$.

What about the reverse inequality?

$$\mathbb{Q} \cap \bigcup_{O \in \mathcal{O}} O = P = \bigcup_{r \in P} \{r\} \subseteq \bigcup_{r \in P} O_r$$

Therefore

$$\mathbb{Q} \cap \bigcup_{O \in \mathcal{O}} O \subseteq \bigcup_{r \in P} O_r \subseteq \bigcup_{O \in \mathcal{O}} O$$

Thus $A = \bigcup_{r \in P} O_r$ and $B = \bigcup_{O \in \mathcal{O}} O$ are open sets such that $A \cap \mathbb{Q} = B \cap \mathbb{Q}$.

Does it follow that $A = B$?

Unfortunately, in general

$$\bigcup_{r \in P} O_r \neq \bigcup_{O \in \mathcal{O}} O .$$

Can you find a counter example?

Let $\mathcal{O} = \{(0,\sqrt{2}), (\sqrt{2},2), (0,2)\}$.

Then

$$\bigcup_{O \in \mathcal{O}} O = (0,2).$$

For each $r \in \mathbb{Q} \cap (0,2)$ define

$$O_r = \begin{cases} (0,\sqrt{2}) & \text{if } r < \sqrt{2} \\ (\sqrt{2},2) & \text{if } r > \sqrt{2} \end{cases}$$

then $r \in O_r$ but

$$\bigcup_{r \in \mathbb{Q} \cap (0,2)} O_r = (0,\sqrt{2}) \cup (\sqrt{2},2)$$

which is different that $(0,2)$.

Does anyone want to see a correct proof of the Lindelöf theorem?

Proof of Lindelöf theorem.

Need to do something for each $x \in \bigcup_{O \in \mathcal{O}} O$ since it didn't work to only work with rationals here.

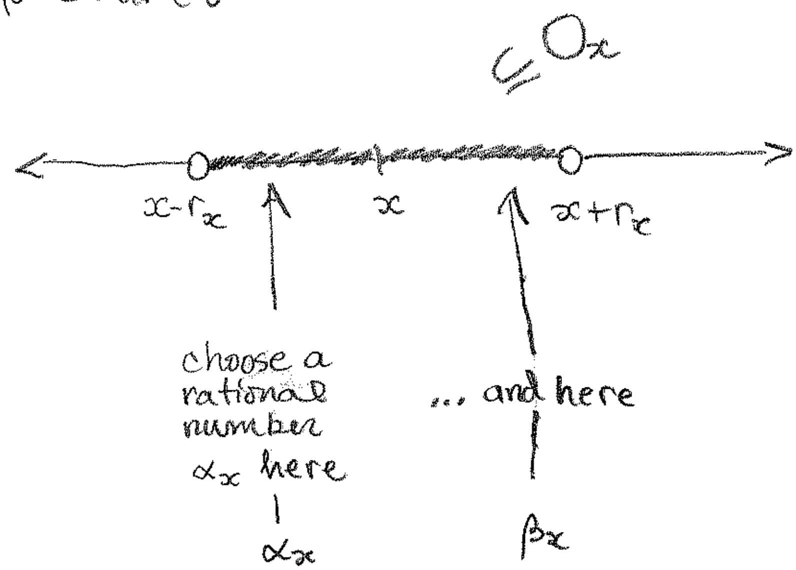
Thus,

$$\forall\, x \in \bigcup_{O \in \mathcal{O}} O \text{ there is } O_x \in \mathcal{O} \text{ s.t. } x \in O_x.$$

Since O_x is open there is $r_x > 0$ such that

$$(x - r_x, x + r_x) \subseteq O_x$$

In pictures



Since $\mathbb{Q}$ is dense there are $\alpha_x, \beta_x \in \mathbb{Q}$ such that

$$x \in (\alpha_x, \beta_x) \subseteq (x - r_x, x + r_x) \subseteq O_x.$$

Let $\mathcal{C} = \{ (\alpha_x, \beta_x) ; x \in \bigcup_{O \in \vartheta} O \}$. Since the endpoints are rational then $\mathcal{C}$ is countable. Thus

$$\mathcal{C} = \{ I_1, I_2, I_3, \dots \}$$

For each $n \in \mathbb{N}$ there is $x_n \in \bigcup_{O \in \vartheta} O$ such that

$$I_n = (\alpha_{x_n}, \beta_{x_n})$$

Setting $U = \bigcup_{O \in \vartheta} O$. Then

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} (\alpha_x, \beta_x) = \bigcup_{I \subseteq \mathcal{C}} I$$

$$= \bigcup_{n=1}^{\infty} I_n = \bigcup_{n=1}^{\infty} (\alpha_{x_n}, \beta_{x_n}) \subseteq \bigcup_{n=1}^{\infty} O_{x_n}$$

Therefore

$$\bigcup_{O \in \vartheta} O = \bigcup_{n=1}^{\infty} O_{x_n}$$

finishes the proof.

Discussion of Theorem 2.7

A bounded function on $[a,b]$ is Riemann integrable if and only if the set of points of discontinuity has measure zero.

Can you prove the simpler theorem?

A continuous function on $[a,b]$ is Riemann integrable.

What do we know about continuous functions
$$f : [a,b] \to \mathbb{R} \quad ?$$

1. bounded,
2. attains maximum and minimum,
3. uniformly continuous.

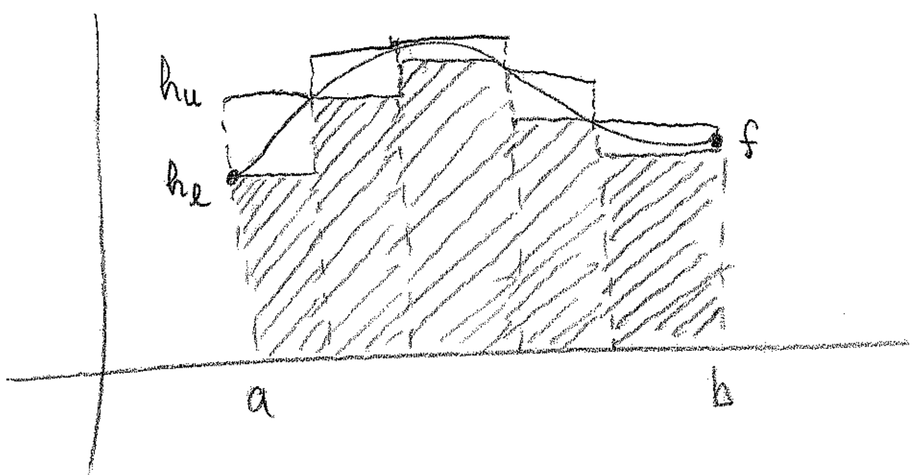
These are those hard theorems from the beginning of an advanced calculus class.

Recall

$$\underline{\int_a^b} f = \sup\left\{ \int_a^b h : h \text{ is a step function and } h \le f \right\}$$

$$\overline{\int_a^b} = \inf\left\{ \int_a^b h : h \text{ is a step function and } f \le h \right\}.$$

Draw a picture $\qquad h_l \le f \le h_u$



Then by definition

$$\int_a^b h_l \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le \int_a^b h_u$$

We are trying to show that

$$\underline{\int_a^b} f = \overline{\int_a^b} f .$$

Thus in the picture

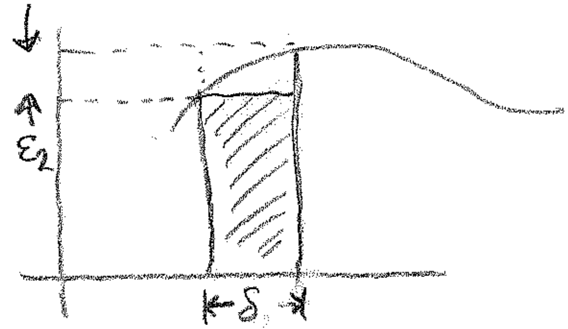
$$\overline{\int_a^b} f - \underline{\int_a^b} f \leq \int_a^b h_u - \int_a^b h_e = \sum_{i=1}^{n} \text{area of } \square_i$$

where $\square_i$ are the white rectangles which are made from the differences between the tall rectangles and the short ones.

Now use the continuity to make this sum less than ε.

Let $\varepsilon > 0$. Since $f : [a,b] \to \mathbb{R}$ is uniformly continuous, then for $\varepsilon_2 = \boxed{\ ?\ } > 0$ there is $\delta_2 > 0$ such that $|x-y| < \delta_2$ implies $|f(x) - f(y)| < \varepsilon_2$.

Choose n so large that $\delta = \frac{b-a}{n} < \delta_2$. Then



or in otherwords

area of $\square_i = \varepsilon_2 \delta$

Since there are $n = \frac{b-a}{\delta}$ rectangles, then

$$\overline{\int_a^b} f - \underline{\int_a^b} f \leq \sum_{i=1}^{n} \varepsilon_2 \delta = \frac{b-a}{\cancel{\delta}} \varepsilon_2 \cancel{\delta} = (b-a)\varepsilon_2 = \varepsilon$$

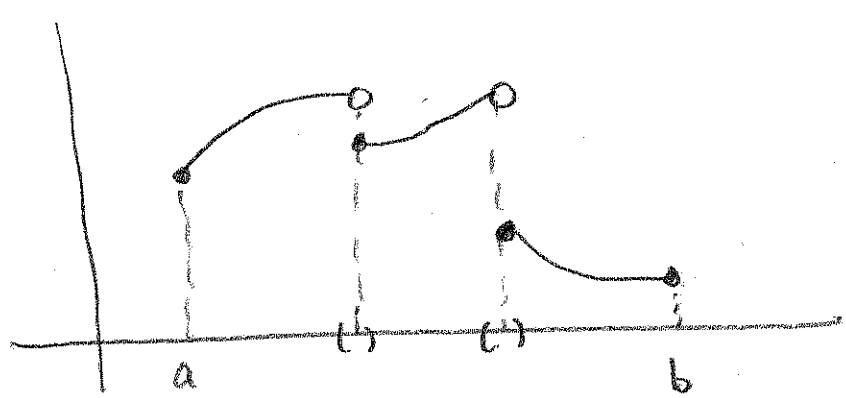
where we have chosen $\boxed{?}$ for $\varepsilon_2 = \frac{\varepsilon}{b-a}$.

Since $\varepsilon > 0$ is arbitrary, then

$$\overline{\int_a^b} f = \underline{\int_a^b} f$$

which shows that f is Riemann integrable,

What about discontinuities?



Let B be the bound for f. Then for an interval I about the discontinuity

$$\sup\{|f(x)-f(y)| : x,y \in I\} \leq 2B.$$

Let $A = \{x_0 \in [a,b] : \lim_{x\to x_0} f(x) \neq f(x_0)\}$ and

$$\lambda^*(A)=0.$$

Then for $\varepsilon>0$ there are open intervals I_n such that

$$A \subseteq \bigcup I_n \quad \text{and} \quad \sum \ell(I_n) < \varepsilon.$$

Leaving out a bunch of details we have an extra term in the previous estimate of the form

$$\sum_i \ell(I_n)\, 2B = 2B\varepsilon.$$

which can be made small because $\lambda^*(A)=0$.

Example of a function $f: \mathbb{R} \to \mathbb{R}$ that is continuous at every irrational number and discontinuous at every rational.

$$f(x) = \begin{cases} \frac{1}{q} & : \text{ if } x = \frac{p}{q} \text{ where } p \text{ and } q \text{ have no common divisors} \\ 0 & : \text{ if } x \text{ is irrational.} \end{cases}$$

If $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ then

$$\lim_{x \to x_0} f(x) = 0 = f(x_0)$$

therefore f is continuous on $\mathbb{R} \setminus \mathbb{Q}$.

Claim if a is irrational and $\frac{p_n}{q_n} \to a$ as $n \to \infty$ then $q_n \to \infty$ as $n \to \infty$.

How to prove the claim?

No idea? Try proof by contradiction.

Suppose for some $N \geq 1$ that $q_n \in \{1, 2, 3, \dots, N\}$ for all $n \in \mathbb{N}$. Then define

$$H_1 = \{n : q_n = 1\}$$
$$H_2 = \{n : q_n = 2\}$$
$$\vdots$$
$$H_N = \{n : q_n = N\}.$$

Since

$$\mathbb{N} = \bigcup_{n=1}^{N} H_n$$

then the pigeonhole principle implies there is k_0 such that H_{k_0} is infinite.

<u>Pigeon hole principle</u>: If an infinite number of pigeons fly home into a finite number of pigeonholes, then at least one pigeonhole must contain an infinite number of pigeons.

Since H_{k_0} is countable then

$$H_{k_0} = \{n_1, n_2, n_3, \dots\}$$

where $n_1 < n_2 < n_3 < \cdots$.

It follows that the subsequence

$$\frac{P_{n_k}}{q_{n_k}} = \frac{P_{n_k}}{K_0} \to a \quad \text{as} \quad k \to \infty.$$

<u>Claim that P_{n_k} is eventually constant</u>

Since $\frac{P_{n_k}}{K_0}$ is convergent it is Cauchy.

Let $\varepsilon = \frac{1}{2K_0}$. Then there is $K \geq 1$ such

that $\left| \frac{P_{n_k}}{K_0} - \frac{P_{n_\ell}}{K_0} \right| < \frac{1}{2K_0}$ for $k, \ell \geq K.$

This means

$$|P_{n_k} - P_{n_\ell}| < \frac{1}{2} \quad \text{for } k, \ell \geq K.$$

Since P_{n_k} are integers, the only way this is possible is if P_{n_k} is constant for $k \geq K$.

But then

$$\frac{P_{n_K}}{K_0} = a$$

contradicts that a is irrational. Therefore $q_n \to \infty$ as $n \to \infty$.

Let $E \subseteq \mathbb{R}$. Show that the set of accumulation points

$$E' = \{x \in \mathbb{R} : \forall \varepsilon > 0 \; \exists y \in E \text{ s.t. } 0 < |y - x| < \varepsilon\}$$

is closed.

Thus we want to show $E' = \overline{E'}$.

Clearly $E' \subseteq \overline{E'}$.

Thus it is enough to show $\overline{E'} \subseteq E'$.

By definition

$$\overline{E'} = \{x \in \mathbb{R} : \forall \varepsilon > 0 \; \exists z \in E' \text{ s.t. } |z - x| < \varepsilon\}.$$

Let $x \in \overline{E'}$. Claim $x \in E'$.

Let $\varepsilon > 0$. Since $x \in \overline{E'}$ then for $\varepsilon_2 = \boxed{\varepsilon/2} > 0$ there is $z \in E'$ s.t. $|z - x| < \varepsilon_2$.

Since $z \in E'$ then for $\varepsilon_3 = \boxed{\varepsilon/2} > 0$ there is $y \in E$ s.t. $|y - z| < \varepsilon_3$.

Now $|x - y| \leq |x - z| + |z - y| < \varepsilon_2 + \varepsilon_3 = \varepsilon$

Implies that $x \in E'$. Thus E' is closed.

There is an error here because we need to also show that |x-y| > 0. A correction appears in the addendum to these notes.

That proof is like the proof that

If $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are continuous then $f\circ g:\mathbb{R}\to\mathbb{R}$ is continuous.

If we use the ε-δ definition of continuity there is ε_2 and ε_3 which come from the two hypothesis involving limits.

Can you think of any easier way to prove this result using what we have learned in 413?

Proof: Let $O\subseteq\mathbb{R}$ be open. Then $f^{-1}(O)\subseteq\mathbb{R}$ is open because $f:\mathbb{R}\to\mathbb{R}$ is continuous. Let $U=f^{-1}(O)$. Then $g^{-1}(U)\subseteq\mathbb{R}$ is open because $g:\mathbb{R}\to\mathbb{R}$ is continuous. Thus

$$(f\circ g)^{-1}(O)=g^{-1}(f^{-1}(O))=g^{-1}(U)$$

is open implies $f\circ g$ is continuous.

Examples of sets of measure zero

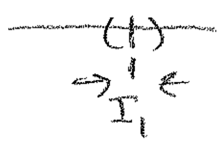
$$\lambda^*(A) = \inf\left\{ \sum_n \ell(I_n) : I_n \text{ are open intervals and } A \subseteq \bigcup_n I_n \right\}$$

Example 1

$A = \{1\}$

Let $\varepsilon > 0$ and

$I_1 = (1 - \frac{\varepsilon}{2}, 1 + \frac{\varepsilon}{2})$, then.



$\lambda^*(\{1\}) \leq \ell(I_1) = \varepsilon$.

Since ε is arbitrary then $\lambda^*(\{1\}) = 0$.

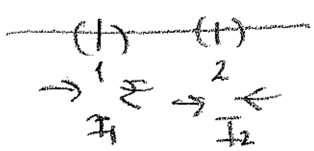
Example 2

$A = \{1, 2\}$

Let $\varepsilon > 0$ and

$I_1 = (1 - \frac{\varepsilon}{4}, 1 + \frac{\varepsilon}{4})$

$I_2 = (1 - \frac{\varepsilon}{4}, 1 + \frac{\varepsilon}{4})$ then



$\lambda^*(\{1,2\}) \leq \ell(I_1) + \ell(I_2) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Since ε is arbitrary then

$\lambda^*(\{1,2\}) = 0$

Example 3

$A = \mathbb{N}$

Let $\varepsilon > 0$ and

$I_n = (n - \frac{\varepsilon_n}{2}, n + \frac{\varepsilon_n}{2})$, then

$$\lambda^*(\mathbb{N}) \le \sum l(I_n) = \sum_{n=1}^{\infty} \varepsilon_n$$

Can you find a sequence ε_n such that

$$\sum_{n=1}^{\infty} \varepsilon_n \le \varepsilon \, ?$$

How about $\varepsilon_n = \frac{1}{n^2}$?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

which might be bigger than ε.

The sequence ε_n needs to depend on ε.

How about $\varepsilon_n = \frac{\varepsilon}{2^n}$?

$$\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Another sequence $\varepsilon_n = \frac{6\varepsilon}{\pi^2 n^2}$

$$\sum \frac{6\varepsilon}{\pi^2 n^2} = \varepsilon$$

In general if $a_n > 0$ and $\sum a_n = L < \infty$ then

$$\varepsilon_n = \frac{\varepsilon}{L} a_n$$

given

$$\sum \varepsilon_n = \frac{\varepsilon}{L} \sum a_n = \frac{\varepsilon}{L} L = \varepsilon$$

Hints for the extra credit problem

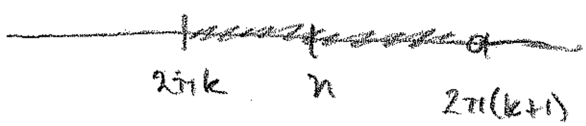
Show that the set of cluster points of $x_n = \sin(n)$ is $[-1,1]$.

Hint 1

Since x_n is bounded it has a convergent subsequence $x_{n_k} \to x \in [-1,1]$.

Hint 2

for each n define w_n to be the number such that $w_n \in [0, 2\pi]$ and $\sin(n) = \sin(w_n)$.
Thus $w_n = n - 2\pi k$ where n lies in the interval.



Claim w_n is a sequence of distinct points.

For contradiction suppose $w_n = w_m$ for some $n \neq m$. By definition there is K_n and K_m integers such that

$$w_n = n - 2\pi K_n \text{ and } w_m = m - 2\pi K_m.$$

But then

$$n - m = 2\pi(K_n - K_m)$$

Since $n \neq m$ then $K_n \neq K_m$. Dividing gives that

$$\pi = \frac{n-m}{2(K_n - K_m)}$$

Contradicting that π is irrational. Thus w_n is a sequence of distinct points