Math 713 Summary

Oct 11, '10

We are now starting chapter 3. So far the course has discussed things like uniform limits of sequences of function which also appear in most undergraduate programs. In chapter 3 we start something that will be new for most people.

Definitions

Define: $C = \{ f : \mathbb{R} \to \mathbb{R} \text{ st. } f \text{ is continuous} \}$

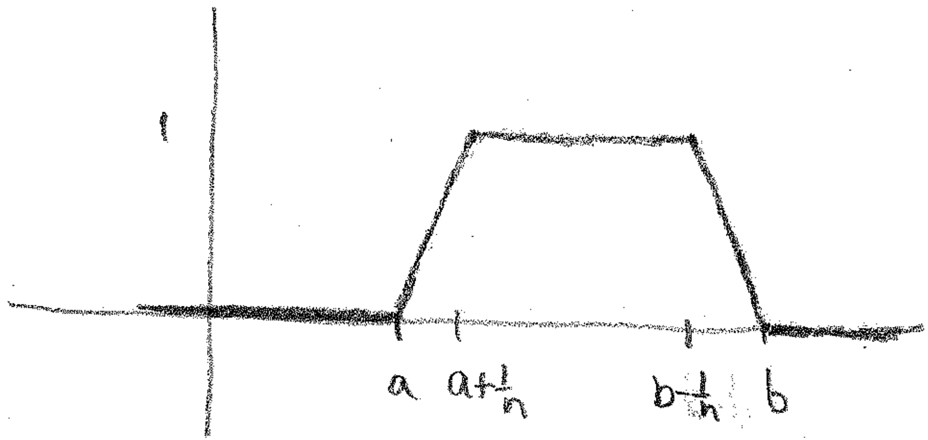
Define: $\hat{C}$ to be the smallest collection of functions that contains C and is closed under pointwise limits.

Example: For $a < b$ define

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [a+\frac{1}{n}, b-\frac{1}{n}] \\ 0 & \text{if } x \leq a \text{ or } x \geq b \\ n(x-a) & \text{if } x \in (a, a+\frac{1}{n}) \\ -n(x-b) & \text{if } x \in (b-\frac{1}{n}, b) \end{cases}$$

Then f_n is continuous and so $f_n \in \hat{C}$.

Graph of f_n



What is the limit of f_n as $n \to \infty$?

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x \in (a,b) \\ 0 & \text{if } x \le a \text{ or } x \ge b \end{cases}$$

$$= \chi_{(a,b)}(x).$$

Therefore, for any $a<b$ we have $\chi_{(a,b)} \in \hat{C}$.

Theorem $\hat{C}$ is an algebra of functions.
That is, if $f, g \in \hat{C}$ and $\alpha \in \mathbb{R}$ then

(a) $f+g \in \hat{C}$

(b) $\alpha f \in \hat{C}$

(c) $fg \in \hat{C}$.

Proof: Since (a) is proven in the book and (b) follows from (c) we will prove (c) here.

The proof is similar to the proof of the monotone class theorem. This is because the definition of $\hat{C}$ as the smallest of some type of set. If you recall the proof of the monotone class theorem you may remember that is started with a half step. We do the same here.

Let $g \in C$ and define

$$\mathcal{D} = \{ f \in \hat{C} : fg \in \hat{C} \}.$$

<u>Claim</u> $C \subseteq \mathcal{D}$. Let $f \in C$. Then $fg \in C$ since the product of continuous functions is continuous. Since $C \subseteq \hat{C}$ we have

$$f \in \hat{C} \text{ and } fg \in \hat{C}.$$

This implies, by definition, that $f \in \mathcal{D}$. Therefore $C \subseteq \mathcal{D}$.

Claim $\mathcal{D}$ is closed under pointwise limits.

Let $f_n \in \mathcal{D}$ be such that $f_n \to f$ pointwise.

Since $\mathcal{D} \subseteq \hat{C}$ then $f_n \in \hat{C}$.

Since $\hat{C}$ is closed under pointwise limits then $f \in \hat{C}$.

By definition $f_n \in \mathcal{D}$ means $f_n g \in \hat{C}$.

Since $\lim_{n\to\infty} a x_n = a \lim_{n\to\infty} x_n$ from Calculus, it follows that

$$\lim_{n\to\infty} f_n(x) g(x) = g(x) \lim_{n\to\infty} f_n(x) = g(x) f(x)$$

so that $f_n g \to fg$ pointwise.

Since $\hat{C}$ is closed under pointwise limits then $fg \in \hat{C}$.

It follows that $f \in \hat{C}$ and $fg \in \hat{C}$. This implies, by definition that $f \in \mathcal{D}$.

We now use the fact that $\hat{C}$ is the smallest set closed under pointwise limits that contains the continuous functions C to conclude $\mathcal{D} = \hat{C}$.

In otherwords, we have proven that

$$f \in \hat{C} \text{ and } g \in C \text{ implies } fg \in \hat{C}.$$

This is the half step toward what we are trying to prove. Now

Let $f \in \hat{C}$ and define

$$\mathcal{E} = \{ g \in \hat{C} : fg \in \hat{C} \}.$$

we prove the same claims:

<u>claim</u> $C \subseteq \mathcal{E}$ and $\mathcal{E}$ is closed under pointwise limits.

Before doing this let's take a 3 minute break...

After the break. How many want to see the proof and how many want to just go on to another theorem?

results: 4 want to prove this theorem.

Claim: $C \subseteq \mathcal{E}$. Let $g \in C$. Then $fg \in \hat{C}$ by the result just proved. Since $C \subseteq \hat{C}$ we have that $g \in \hat{C}$ and $fg \in \hat{C}$. Therefore $g \in \mathcal{E}$.

Claim: $\mathcal{E}$ is closed under pointwise limits.

The proof is the same as the similar claim about $\mathcal{D}$ except with the roles of g and f reversed. I'll try to keep the formatting of the proof the same for comparison.

Let $g_n \in \mathcal{E}$ be such that $g_n \to g$ pointwise

Since $\mathcal{E} \subseteq \hat{C}$ then $g_n \in \hat{C}$.

Since $\hat{C}$ is closed under pointwise limits then $g \in \hat{C}$.

By definition $g_n \in \mathcal{E}$ means $fg_n \in \hat{C}$

Since $fg_n \to fg$ pointwise and since $\hat{C}$ is closed under pointwise limits then $fg \in \hat{C}$

It follows that $g \in \hat{C}$ and $fg \in \hat{C}$. This implies, by definition, that $g \in \mathcal{D}$.

We again use the fact that $\hat{C}$ is the smallest set closed under pointwise limits tha contains the continuous functions C to conclude $\mathcal{E} = \hat{C}$.

Thus $f \in \hat{C}$ and $g \in \hat{C}$ implies $fg \in \hat{C}$.

Example:

$$\chi_{(1,2)} + \chi_{(3,4)} = \chi_{(1,2)\cup(3,4)} \in \hat{C}.$$

Theorem: Let $O \subseteq \mathbb{R}$ be an open set, then $\chi_O \in \hat{C}$.

Since O is open, by the structure theorem for the open subsets of the real line we have

$$O = \bigcup_n I_n$$

Where I_n is a countable collection of disjoint open intervals.

Now, since I_n are disjoint, we have

$$\chi_O = \chi_{\bigcup_n I_n} = \sum_n \chi_{I_n}.$$

Define

$$f_n = \sum_{k=1}^{n} \chi_{I_k}$$

Then $f_n \in \hat{C}$ since $\chi_{I_k} \in \hat{C}$ and $\hat{C}$ is an algebra of functions.

$$\lim_{n\to\infty} f_n(x) = \sum_{k=1}^{\infty} \chi_{I_k}(x) = \chi_O(x)$$

Shows $f_n \to \chi_O$ pointwise. Therefore $\chi_O \in \hat{C}$.

We now turn to the handout on the Weierstrass approximation theorem. This theorem may have been proved in you undergraduate analysis class. The proof in the handout is nice because it includes an explicit construction of the approximating polynomials.

<u>Theorem</u>: There exists polynomials p_n such that $p_n(x) \to |x|$ pointwise for all $x \in \mathbb{R}$.

<u>Proof</u>: The Weierstrass approximation is used to give a uniform approximation to $|x|$ on a closed and bounded interval. To prove the theorem, take the interval larger while at the same time making the approximation better.

Thus, let $A_n = [-n, n]$. By the Weierstrass approximation theorem there is a polynomial p_n such that $x \in A_n$ implies $|p_n(x) - |x|| < \frac{1}{n}$.

<u>Claim</u> $p_n \to |x|$ pointwise on $\mathbb{R}$.

Let $x \in \mathbb{R}$ and let $\varepsilon > 0$ be arbitrary.

Choose $n_1 \in \mathbb{N}$ so that $x \in [-n_1, n_1]$ and $n_2 \in \mathbb{N}$ so that $\frac{1}{n_2} < \varepsilon$.

Let $N = \max\{n_1, n_2\}$. Then $n \geq N$ implies $x \in A_{n_1} \subseteq A_n$ and therefore

$$|P_n(x) - |x|| < \frac{1}{n} \leq \frac{1}{n_2} < \varepsilon.$$

It follows that $p_n \to |x|$ pointwise on $\mathbb{R}$.

Theorem: If $f \in \hat{C}$ then $|f| \in \hat{C}$.

Proof: Let p_n be a sequence of polynomials such that $p_n \to |x|$ pointwise.

Define

$$f_n(x) = p_n(f(x)).$$

Since $\hat{C}$ is an algebra and a polynomial only performs algebraic operations, then $f_n \in \hat{C}$ for each $n \in \mathbb{N}$.

Since

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} p_n(f(x)) = |x|$$

and $\hat{C}$ is closed under pointwise limits, then $|x| \in \hat{C}$.

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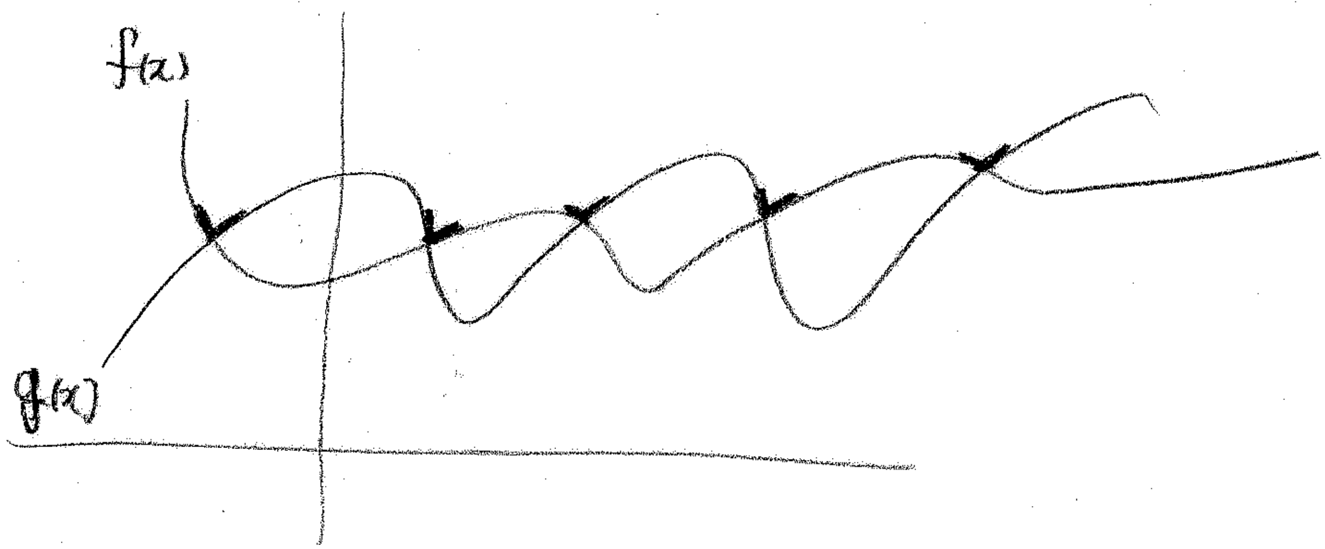
$$(f \vee g)(x) = \max\{f(x), g(x)\}$$

and

$$(f \wedge g)(x) = \min\{f(x), g(x)\}.$$

How to remember which is the maximum and which is the minimum?



For $f \vee g$ look for the shape "$\vee$" in every place the graphs cross. Now if you follow the "$\vee$"s at each crossing you obtain the maximum between the two functions. Thus

$$(f \vee g)(x) = \max\{f(x), g(x)\}.$$

Theorem: If $f, g \in \hat{C}$ then $f \vee g \in \hat{C}$ and $f \wedge g \in \hat{C}$.

This is a "rabbit out of a hat" proof. It means we write down some magic equation that makes the result obvious without explaining where that magic equation came from.

Proof: Since

$$(f \vee g)(x) = \frac{f(x) + g(x) + |f(x) - g(x)|}{2}$$

and

$$(f \wedge g)(x) = \frac{f(x) + g(x) - |f(x) - g(x)|}{2}$$

Then the fact that $\hat{C}$ is an algebra of functions along with the previous result shows that $f \vee g \in \hat{C}$ and $f \wedge g \in \hat{C}$.

Theorem: If $f_n \in \hat{C}$ then $\inf\{f_n(x) : n \in \mathbb{N}\}$ and $\sup\{f_n(x) : n \in \mathbb{N}\}$ are in $\hat{C}$.

This is obtained by taking limits in the previous theorem.

<u>Proof</u>: Let

$$g_n = f_1 \vee f_2 \vee \dots \vee f_n$$

Then $g_n \in \hat{C}$ and

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} (f_1 \vee f_2 \vee \dots \vee f_n)(x)$$

$$= \sup\{f_n(x) : n \in \mathbb{N}\}$$

implies $\sup\{f_n(x) : n \in \mathbb{N}\} \in \hat{C}$.

Similarly $\inf\{f_n(x) : n \in \mathbb{N}\} \in \hat{C}$.

We are not quite finished. Section 3.1 contains a final characterization of $\hat{C}$ in terms of σ-algebras.

This characterization provides an alternative definition of $\hat{C}$ that appears in many books.

Definition

$$\tau = \{ O \subseteq \mathbb{R} : O \text{ is an open set} \}$$

$\mathcal{B} = \mathcal{A}(\tau) =$ the smallest σ-algebra that contains τ.

Then

$$\mathcal{C} = \{ f : \mathbb{R} \to \mathbb{R} \text{ s.t. } f^{-1}(O) \in \tau \text{ for every } O \in \tau \}$$

$$\hat{\mathcal{C}} = \{ f : \mathbb{R} \to \mathbb{R} \text{ s.t. } f^{-1}(O) \in \mathcal{B} \text{ for every } O \in \tau \}.$$

We will prove the equivalence of these two definitions next time.