Math 713 Summary

October 25, '10 (and Oct 27 combined)

<u>Definition</u> of Lebesgue outer measure

$$\lambda^*(A) = \left\{ \sum_n \ell(I_n) : I_n \text{ are open intervals and } A \subseteq \bigcup_n I_n \right\}$$

Recall the properties

(a) $\lambda^*(A) \geq 0$

(b) $\lambda^*(\emptyset) = 0$

(c) $A \subseteq B$ implies $\lambda^*(A) \subseteq \lambda^*(B)$

(d) $\lambda^*(x + A) = \lambda^*(A)$

(e) $\lambda^*(\bigcup A_n) \leq \sum \lambda^*(A_n)$

(f) $\lambda^*(I) = \ell(I)$

We also want additivity. Thus if $A \cap B = \emptyset$ we want

$$\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$$

and if $A_i \subseteq \mathbb{R}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$ then we want

$$\lambda^*(\bigcup A_n) = \sum \lambda^*(A_n).$$

However additivity does not hold for all collections of subsets of $\mathbb{R}$. That is why $\mathcal{B}$ and other σ-algebras were developed.

The construction on page 116 is of a strange set that shows we must restrict the domain of λ^* to obtain additivity.

The construction is complicated so we will procede slowly.

First define an equivalence relation on $\mathbb{R}$ by $x \sim y$ for $x, y \in \mathbb{R}$

means $x - y \in \mathbb{Q}$.

To show this is an equivalence relation we must have

(i) Reflexivity: $x \sim x$
(ii) Symmetry: $x \sim y$ implies $y \sim x$
(iii) Transitivity: $x \sim y$ and $y \sim z$ implies $x \sim z$.

Any relation that satisfies these three properties is called an equivalence relation.

<u>Claim</u> $x \sim y$ defined by $x-y \in \mathbb{Q}$ is an equivalence relation.

(i) Reflexivity. Since $x-x=0 \in \mathbb{Q}$ then $x \sim x$,

(ii) Symmetry. If $x-y \in \mathbb{Q}$ then since $\mathbb{Q}$ is closed under multiplication and $-1 \in \mathbb{Q}$ we obtain that

$$(-1)(x-y) = y-x \in \mathbb{Q}.$$

Therefore $y \sim x$.

(iii) Transitivity: Suppose $x \sim y$ and $y \sim z$. Then $r_1 = x-y \in \mathbb{Q}$ and $r_2 = y-z \in \mathbb{Q}$.

Since $\mathbb{Q}$ is closed under addition

$$r_1 + r_2 = x-y+y-z = x-z \in \mathbb{Q}$$

Therefore $x \sim z$.

This verifies the three properties of an equivalence relation.

Theorem Any equivalence relation on a set Ω partitions Ω into a collection of disjoint subsets.

This is an exercise in the book.

Exercise 1.34 Let Ω be a non-empty set and $\equiv$ an equivalence relation on Ω. For each $x \in \Omega$ define $E_x = \{y \in \Omega : y \equiv x\}$ and let $\mathcal{C} = \{E_x : x \in \Omega\}$. Each member of $\mathcal{C}$ is called an equivalence class of Ω under $\equiv$. Moreover

(a) Either $E_x \cap E_y = \emptyset$ or $E_x = E_y$

(b) $\Omega = \bigcup_{A \in \mathcal{C}} A$

Therefore $\equiv$ partitions Ω into disjoint equivalence classes; that is, Ω is a disjoint union of equivalence classes under $\equiv$.

This kind of partitioning occurs often enough to introduce the concept of an equivalence relation so we don't have to keep reproving the same theorem over and over. In particular, the definition of the equivalence relation is exactly the properties a relation needs to have to prove there is a disjoint partition of Ω into equivalence classes.

Proof of part (b) in Exercise 1.34

Reflexivity implies $x \equiv x$, so $x \in E_x$ for all $x \in \Omega$. Thus

$$\Omega = \bigcup_{x \in \Omega} \{x\} \subseteq \bigcup_{x \in \Omega} E_x = \bigcup_{A \in \mathcal{C}} A$$

Proof of part (a) in Exercise 1.34

Suppose $E_{x_1} \cap E_{x_2} \neq \emptyset$. Claim $E_{x_1} = E_{x_2}$.

Let $z \in E_{x_1} \cap E_{x_2}$.

Then $z \in E_{x_1}$ implies $z \equiv x_1$

and $z \in E_{x_2}$ implies $z \equiv x_2$

"$\subseteq$" Suppose $x \in E_{x_1}$. Then $x \equiv x_1$.

Since $z \equiv x_1$ then symmetry implies $x_1 \equiv z$.

Since $x \equiv x_1$ and $x_1 \equiv z$ transitivity

implies $x \equiv z$.

Since $x \equiv z$ and $z \equiv x_2$ transitivity again

implies $x \equiv x_2$

Therefore $x \in E_{x_2}$.

Therefore we have shown that $E_{x_1} \subseteq E_{x_2}$.

Similarly $E_{x_2} \subseteq E_{x_1}$. Therefore $E_{x_1} = E_{x_2}$.

This finishes the proof of part (a) and Exercise 1.34

For our equivalence relation $x \sim y$ when $x - y \in \mathbb{Q}$ we have $\Omega = \mathbb{R}$ and

$$E_x = \{ y \in \mathbb{R} : y \sim x \}$$

$$\mathcal{C} = \{ E_x : x \in \mathbb{R} \}.$$

By the axiom of choice there is

$$w : \mathcal{C} \to \mathbb{R} \quad \text{such that } w(E_x) \in E_x$$

for every $x \in \mathbb{R}$.

Define

$$T = w(\mathcal{C}) = \{ w(E_x) : x \in \mathbb{R} \}$$

and

$$S = \{ t - [\![t]\!] : t \in T \} = \{ w(E_x) - [\![w(E_x)]\!] : x \in \mathbb{R} \}$$

where $[\![t]\!]$ stands for the greatest integer less than or equal t.

By definition $t - [\![t]\!] \in [0,1)$ for every t, thus

$$S \subseteq [0,1)$$

Define

$$\mathcal{D} = \{ S + r : r \in \mathbb{Q} \}.$$

where $S + r = \{ s + r : s \in S \}$.

<u>Claim</u> that $\mathcal{D}$ is a collection of disjoint sets.

This property comes from the fact that $\mathcal{C}$ is a collection of disjoint sets.

Let $S+r \in \mathcal{D}$ and $S+q \in \mathcal{D}$ and $(S+r)\cap(S+q) \neq \emptyset$. Then there is $z \in (S+r)\cap(S+q)$ and $s_1, s_2 \in S$ such that $z = s_1 + r = s_2 + q$. Subtracting yields

$$s_1 - s_2 = q - r \in \mathbb{Q} \quad \text{so that } s_1 \sim s_2.$$

By definition

$$s_1 = t_1 - [\![t_1]\!] \quad \text{for some } t_1 \in T$$

$$s_2 = t_2 - [\![t_2]\!] \quad \text{for some } t_2 \in T.$$

By definition

$$t_1 = w(E_{x_1}) \quad \text{for some } x_1 \in \mathbb{R}$$

$$t_2 = w(E_{x_2}) \quad \text{for some } x_2 \in \mathbb{R}.$$

By the choice of w we have

$$w(E_{x_1}) \in E_{x_1} \quad \text{and} \quad w(E_{x_2}) \in E_{x_2}$$

Therefore $t_1 \in E_{x_1}$ and $t_2 \in E_{x_2}$ so that

$$t_1 \sim x_1 \quad \text{and} \quad t_2 \sim x_2.$$

Since
$$t_1 - t_2 = s_1 + [\![t_1]\!] - s_2 - [\![t_2]\!]$$
$$= q - r + [\![t_1]\!] - [\![t_2]\!] \in \mathbb{Q}$$

then $t_1 \sim t_2$.

Now $t_1 \sim t_2$ and $t_2 \sim x_2$ implies $t_1 \sim x_2$

therefore $t_1 \in E_{x_2}$.

It follows that $t_1 \in E_{x_1} \cap E_{x_2}$ and so $E_{x_1} \cap E_{x_n} \neq \emptyset$.

Since $\mathcal{C}$ is a collection of disjoint sets we must have that $E_{x_1} = E_{x_2}$.

Thus $\quad w(E_{x_1}) = w(E_{x_2})$

Thus $\quad t_1 = t_2$

Thus $\quad \Delta_1 = \Delta_2$

Thus $\quad r = q$

Thus $\quad S + r = S + q$.

This proves $\mathcal{D}$ is a collection of disjoint sets.

Now define $W = (-1, 1) \cap \mathbb{Q} = \{q_1, q_2, \dots\}$.

and $A = \bigcup_{n=1}^{\infty} S + q_n = \bigcup_{n=1}^{\infty} F_n$

where

$$F_n = S + q_n$$

<u>Claim</u>: $(0,1) \subseteq A \subseteq (-1, 2)$.

First show $A \subseteq (-1,2)$.

Since $S \subseteq [0,1)$ and $q_n \in (-1,1)$ then $S+q_n \in (-1,2)$.

It follows that

$$A = \bigcup_{n=1}^{\infty} S + q_n \subseteq (-1,2).$$

Second show $(0,1) \subseteq A$.

Let $x \in (0,1)$.

By definition of E_x we have $x \in E_x$

By choice of w we have $w(E_x) \in E_x$.

Therefore $x \sim w(E_x)$ and

$$x \sim w(E_x) - [\![w(E_x)]\!]$$

Let $s = w(E_x) - [\![w(E_x)]\!]$ then

$$x \sim s \quad \text{where } s \in S.$$

Since $x \sim s$ we have $r = x - s \in \mathbb{Q}$

Since $x \in (0,1)$ and $s \in [0,1)$ then

$-s \in (-1,0]$ and $r = x - s \in (-1,1)$

Since $r \in (-1,1) \cap \mathbb{Q}$ then there is $n_0 \in \mathbb{N}$ such that $r = q_{n_0}$.

It follows that

$$x = s + q_{n_0} \subseteq S + q_{n_0} \subseteq \bigcup_{n=1}^{\infty} S + q_n = A.$$

Therefore $(0,1) \subseteq A$.

<u>Claim</u> is $F_n = S + q_n$ is a collection of strange disjoint sets for which λ^* is not even finitely additive.

<u>Proof</u>; For contradiction suppose that λ^* were finitely additive.

Then for any $N \in \mathbb{N}$ we would have

$$\lambda^*\left(\bigcup_{n=1}^{N} F_n\right) = \sum_{n=1}^{N} \lambda^*(F_n)$$

for every $N \in \mathbb{N}$ since the F_n are disjoint.

Since $(0,1) \subseteq A \subseteq (-1,2)$ then

$$\lambda^*((0,1)) \le \lambda^*(A) = \lambda^*\left(\bigcup_{n=1}^{\infty} F_n\right) \le \sum_{n=1}^{\infty} \lambda^*(F_n)$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \lambda^*(F_n)$$

$$= \lim_{N\to\infty} \lambda^*\left(\bigcup_{n=1}^{N} F_n\right) \le \lambda^*(A) \le \lambda^*((-1,2)).$$

Therefore

$$1 = \lambda^*((0,1)) \le \lim_{N\to\infty} \sum_{n=1}^{N} \lambda^*(F_n) \le \lambda^*((-1,2)) = 3$$

In particular

$$\sum_{n=1}^{N} \lambda^*(F_n) \le 3 \quad \text{for every } N \in \mathbb{N}.$$

Since $\lambda^*(F_n) = \lambda^*(S+q_n) = \lambda^*(S)$ we have

$$N\lambda^*(S) \le 3 \text{ for every } N \in \mathbb{N}.$$

Therefore

$$\lambda^*(S) \le \frac{3}{N} \text{ for every } N \in \mathbb{N}.$$

Thus

$$0 \le \lambda^*(S) \le \frac{3}{N} \text{ for every } N \in \mathbb{N}$$

It follows that

$$\lambda^*(S) = 0.$$

But then

$$1 \le \lim_{N\to\infty} \sum_{n=1}^{N} \lambda^*(F_n) = \lim_{N\to\infty} \sum_{n=1}^{N} 0 = 0$$

is a contradiction. Therefore, there is some $N \in \mathbb{N}$ such that

$$\lambda^*\left(\bigcup_{n=1}^{N} F_n\right) \ne \sum_{n=1}^{N} \lambda^*(F_n)$$

Thus λ^* is not additive.

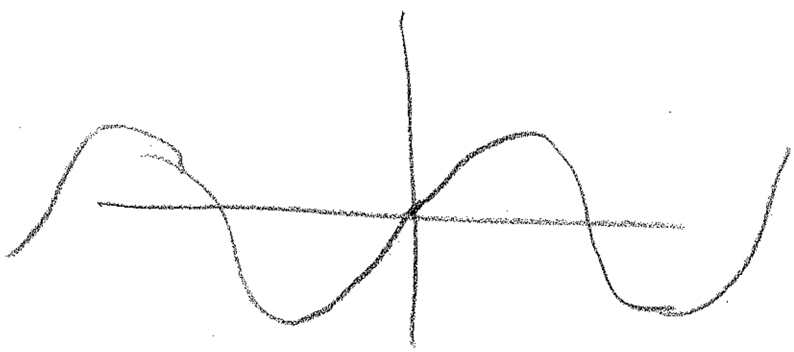
<u>Corollary</u>: There exists $A, B \subseteq \mathbb{R}$ such that $A \cap B = \emptyset$ and

$$\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B).$$

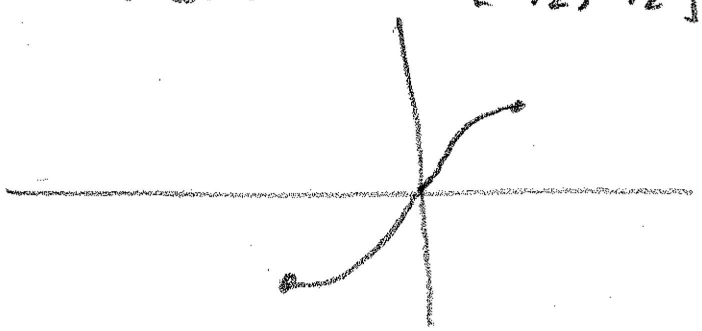
λ^* is defined for all subsets of $\mathbb{R}$.

In particular the domain of λ^* is $\mathcal{P}(\mathbb{R})$ but λ^* is not additive on this domain.

In calculus class we consider the function $\sin(x)$ for $x \in \mathbb{R}$



which has no inverse unless we restrict the domain to $[-\pi/2, \pi/2]$



so that the reflection of the graph about the 45° line passes the vertical line test.

Our function λ^* is similar. We want to restrict the domain of λ^* so that λ^* is additive on that domain.

All the work done with σ-algebras has been in preparation for finding a σ-algebra which is a good domain for λ^* such that λ^* is additive.

The Borel σ-algebra is the smallest σ-algebra that contains the open sets. As such, it is the smallest σ-algebra that would be useful for the analysis needed to develop a theory of integration. We shall show that $\lambda^*|_{\mathcal{B}}$ is additive by constructing a bigger σ-algebra $\mathcal{M}$ on which λ^* is additive and then showing $\mathcal{B} \subseteq \mathcal{M}$.

Define

$$\mathcal{M} = \left\{ E \subseteq \mathbb{R} : \lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W \cap E^c) \text{ for all } W \subseteq \mathbb{R} \right\}$$

<u>Claim</u> $\mathcal{M}$ is a σ-algebra

$\mathcal{B} \subseteq \mathcal{M}$

and $\lambda^*\big|_{\mathcal{M}}$ is countably additive on $\mathcal{M}$.

<u>Start</u> by showing $\mathcal{M}$ is an algebra

Need to show

(i) If $E \in \mathcal{M}$ then $E^c \in \mathcal{M}$

(ii) if $A, B \in \mathcal{M}$ then $A \cup B \in \mathcal{M}$.

<u>Proof</u> of (i) Suppose $E \in \mathcal{M}$. Define $F = E^c$.
then

$$\lambda(W \cap F) + \lambda^*(W \cap F^c) = \lambda^*(W \cap E^c) + \lambda^*(W \cap (E^c)^c)$$
$$= \lambda^*(W \cap E) + \lambda^*(W \cap E^c) = \lambda^*(W)$$

for all $W \subseteq \mathbb{R}$. Therefore $F = E^c \in \mathcal{M}$.

Proof of (iii) Suppose $A, B \in \mathcal{M}$. Define $F = A \cup B$.
Then

$$\lambda^*(W \cap F) + \lambda^*(W \cap F^c)$$
$$= \lambda^*(W \cap (A \cup B)) + \lambda^*(W \cap (A \cup B)^c)$$
$$= \lambda^*(W \cap (A \cup B)) + \lambda^*(W \cap A^c \cap B^c)$$
$$= \lambda^*(W \cap (A \cup B)) + \underbrace{\lambda^*((W \cap A^c) \cap B^c) + \lambda^*((W \cap A^c) \cap B)}_{\text{since } B \in \mathcal{M}} - \lambda^*((W \cap A^c) \cap B)$$
$$= \lambda^*(W \cap (A \cup B)) + \lambda^*(W \cap A^c) - \lambda^*(W \cap A^c \cap B)$$
$$= \lambda^*(W \cap (A \cup B)) + \underbrace{\lambda^*(W \cap A^c) + \lambda^*(W \cap A)}_{\text{since } A \in \mathcal{M}} - \lambda^*(W \cap A) - \lambda^*(W \cap A^c \cap B)$$
$$= \lambda^*(W \cap (A \cup B)) + \lambda^*(W) - \lambda^*(W \cap A) - \lambda^*(W \cap A^c \cap B)$$

want these terms to go away.

$$\lambda^*(W \cap (A \cup B)) = \lambda^*((W \cap A) \cup (W \cap B))$$
$$= \lambda^*((W \cap A) \cup (W \cap A^c \cap B)) \le \lambda^*(W \cap A) + \lambda^*(W \cap A^c \cap B)$$

Therefore $\lambda^*(W \cap (A \cup B)) - \lambda^*(W \cap A) - \lambda^*(W \cap A^c \cap B) \le 0$.

It follows that

$$\lambda^*(W \cap F) + \lambda^*(W \cap F^c) \le \lambda^*(W)$$

To show the reverse inequality is easy since by subadditivity

$$\lambda^*(W) = \lambda^*((W \cap F) \cup (W \cap F^c))$$
$$\leq \lambda^*(W \cap F) + \lambda^*(W \cap F^c)$$

Therefore

$$\lambda^*(W) = \lambda^*(W \cap F) + \lambda^*(W \cap F^c).$$

It follows that $F = A \cup B \in \mathcal{M}$.

Therefore $\mathcal{M}$ is an algebra.

Please read the proof that $\mathcal{M}$ is a σ-algebra in the book on page 120 and that it is additive on page 122.

Recitation will be cancelled Friday because of the holiday. We will have recitation on Saturday.