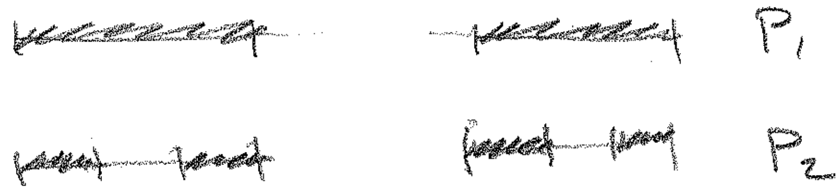
Math 713 Summary November 3, 2010

How did the reading go?

We are still needing to show $\mathcal{B} \subseteq \mathcal{M}$. I am not going to prove this today but instead finish the section on the Cantor set and singular function.

Please continue to read the section in the book from pages 113 to 116 that give the proof that $\mathcal{B} \subseteq \mathcal{M}$. We'll talk about this more next week. Hopefully there will not be much to explain and we can start working on the integral.

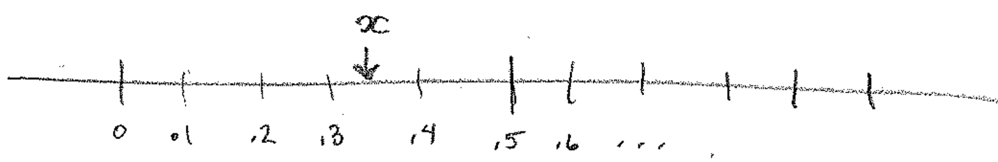
So far we have defined $P = \bigcap_{n=1}^{\infty} P_n$ where



etc ...

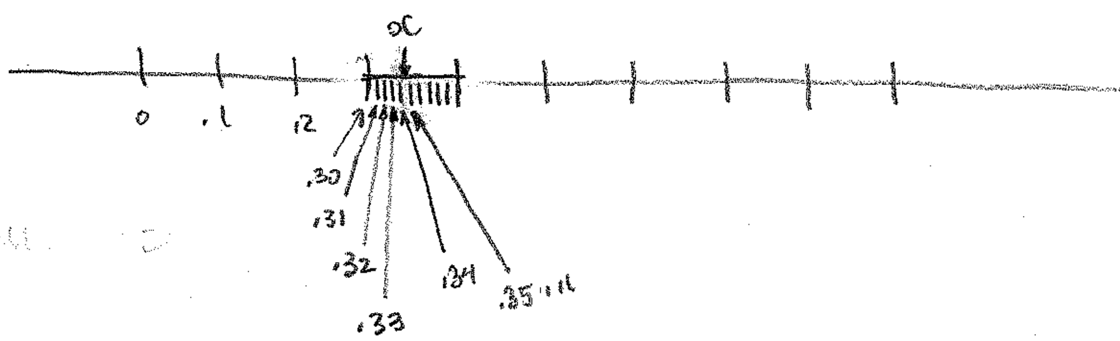
Before going further, lets characterize P in terms of what is called ternary expansions.

Recall that the decimal expansion of a point on the number line can be obtained by subdividing the units by 10 and then repeatedly subdividing by 10 as follows



Thus $x = .3\boxed{\ ?\ }$

Then further subdividing.



Thus $x = .34\boxed{\ ?\ }$

The only reason to subdivide by 10 is because people have 10 fingers.

Obvious other ways of subdividing each unit can be more convenient depending on the application.

<u>Example 1.</u>

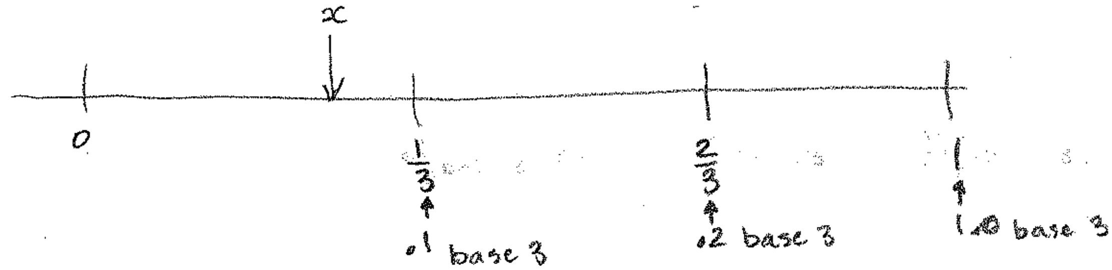
There are 60 seconds in a minute.
There are 60 minutes in an hour.

<u>Example 2.</u>

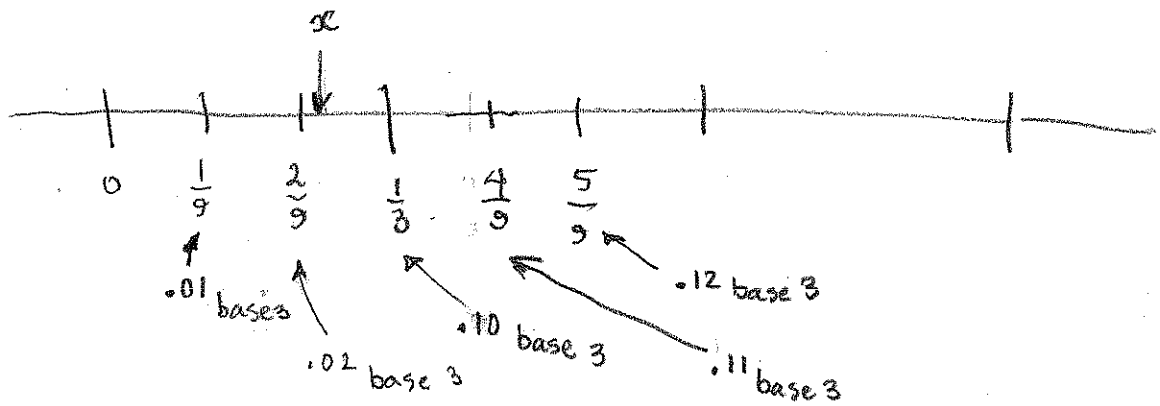
There are 2 pints in a quart
There are 2 cups in a pint
There are 16 tablespoons in a cup.

<u>Example 3.</u>

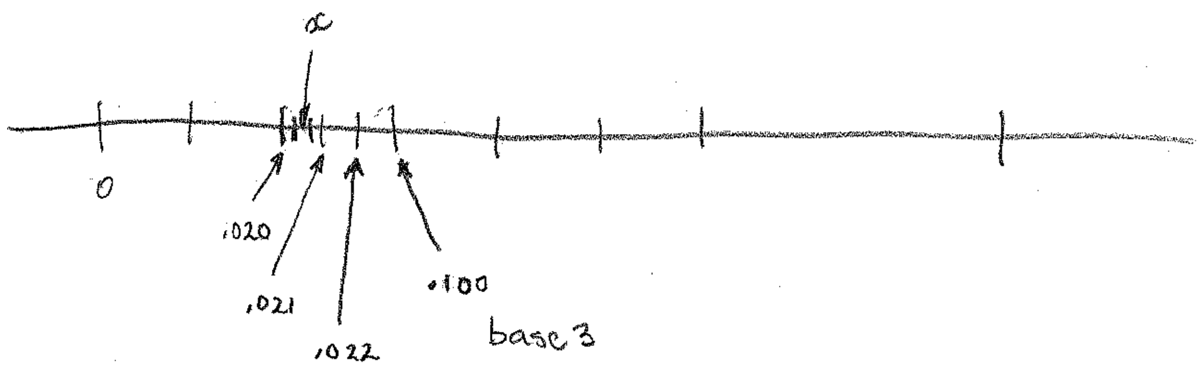
We will subdivide by 3's to obtain a nice representation of the cantor set.



Thus $x = .0$ $\boxed{\ ?\ }$ base 3.



Thus $x = .02$ $\boxed{\ ?\ }$ base 3



Thus $x = .0201$ $\boxed{\ ?\ }$ base 3

In general if $x \in [0,1]$ we may use this subdividing process to obtain an series $a_n \in \{0,1,2\}$ such that

$$x = \sum_{n=1}^{\infty} a_n \frac{1}{3^n}$$

An if $a_n \in \{0,1,2\}$ is any sequence then

$$x = \sum_{n=1}^{\infty} a_n \frac{1}{3^n}$$

Sums to some number $x \in [0,1]$.

We shall show that

$$P = \left\{ x \in [0,1] : \text{there exists } a_n \in \{0,2\} \text{ with } x = \sum_{n=1}^{\infty} a_n \frac{1}{3^n} \right\}.$$

and moreover if $x \in P$ then there is only one sequence $a_n \in \{0,2\}$ such that $x = \sum_{n=1}^{\infty} a_n \frac{1}{3^n}$.

Observe that some numbers may be written using 2 different base 3 expansions in an analogous way that some numbers have 2 different decimal representations.

In particular,

$$.0\overline{2}_{\text{base}3} = \sum_{n=2}^{\infty} \frac{2}{3^n} = 2\,\frac{\frac{1}{9}}{1-\frac{1}{3}} = 2\,\frac{\left(\frac{1}{9}\right)}{\left(\frac{2}{3}\right)} = \frac{1}{3}$$

and

$$.1_{\text{base}3} = \frac{1}{3}.$$

Also

$$.1\overline{2}_{\text{base}3} = \frac{1}{3} + \sum_{n=2}^{\infty} \frac{2}{3^n} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

and

$$.2_{\text{base }3} = \frac{2}{3}.$$

Note also that if a number has two different base 3 expansions then at least one of the representations has a 1 in it if not both.

This observation that if $a_n, b_n \in \{0,1,2\}$ are two different sequences and

$$\sum_{n=1}^{\infty} a_n \frac{1}{3^n} = \sum_{n=1}^{\infty} b_n \frac{1}{3^n}$$

then one of the sequences must contain the digit 1 in it implies the uniqueness of the representation of any number in the set

$$\left\{ x \in [0,1] : \text{there exists } a_n \in \{0,2\} \text{ with } x = \sum_{n=1}^{\infty} a_n \frac{1}{3^n} \right\}.$$

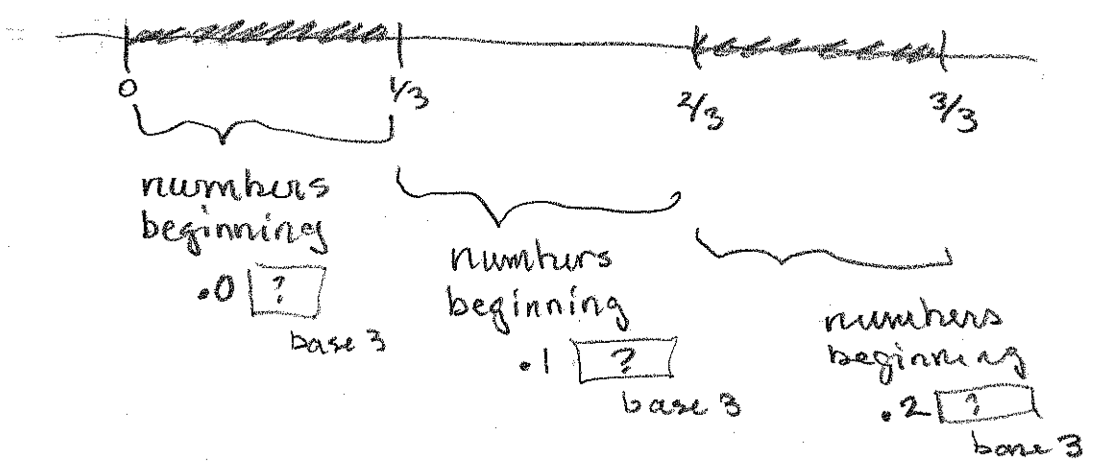
We now show that this set is, in fact P.

First note

$$P_0 = [0,1] = \left\{ \sum_{n=1}^{\infty} a_n \frac{1}{3^n} : a_n \in \{0,1,2\} \right\}.$$

Now

$$P_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1]$$



So

$$[0,\tfrac{1}{3}] = \{x : x = .0????\ldots \text{ base } 3\}$$
$$[\tfrac{1}{3},\tfrac{2}{3}] = \{x : x = .1????\ldots \text{ base } 3\}$$
$$[\tfrac{2}{3},1] = \{x : x = .2????\ldots \text{ base } 3\}.$$

Recall again that the above three sets overlap at their boundaries because the non uniqueness of base 3 representations.

Therefore

$$P_1 = \left\{\sum_{n=1}^{\infty} a_n \frac{1}{3^n} : a_1 \in \{0,2\} \text{ and } a_k \in \{0,1,2\} \text{ for } k>1\right\}$$

Similarly

$$P_2 = \left\{\sum_{n=1}^{\infty} a_n \frac{1}{3^n} : a_1, a_2 \in \{0,2\} \text{ and } a_k \in \{0,1,2\} \text{ for } k>2\right\}.$$

Clearly, then,

$$P = \bigcap_{n=1}^{\infty} P_n = \left\{\sum_{n=1}^{\infty} a_n \frac{1}{3^n} : a_k \in \{0,2\} \text{ for all } k\right\}.$$

$$= \left\{x \in [0,1] : \text{there exists } a_n \in \{0,2\} \text{ with } x = \sum_{n=1}^{\infty} a_n \frac{1}{3^n}\right\}$$

Note the word "clearly" on the previous page means as it does in most mathematical writing that some details have been left out of the argument which the author does not feel like writting down, but which are felt should be possible to figure out by whoever is reading this.

Sometimes it takes a page or an entire book of theory to supply the missing details. When reading a passage that contains the words

"clearly", "obviously", "(why?)"
"it may easily be shown", etc...

these word should be taken as a warning indicating to the reader that details have been left out which need to be supplied in order to complete the argument.

In particular, this is a request to the reader to take up a pencil and write the missing details down on their own paper.

Anyway...

Except for the word "clearly" we now have a representation of the Cantor set as.

$$P = \left\{ x \in [0,1] : \text{there exists } a_n \in \{0,2\} \text{ with } x = \sum_{n=1}^{\infty} a_n \frac{1}{3^n} \right\}.$$

Using this representation we define

$$f : P \to [0,1]$$

by

$$\sum_{n=1}^{\infty} a_n \frac{1}{3^n} \longrightarrow \sum_{n=1}^{\infty} \frac{a_n/2}{2^n}.$$

This function is well defined, since the representation $\sum_{n=1}^{\infty} a_n \frac{1}{3^n}$ is unique

and also because $a_n \in \{0,2\}$ then $a_n/2 \in \{0,1\}$ so as a result $\sum_{n=1}^{\infty} \frac{a_n/2}{2^n}$ is a base-2 expansion of some number in $[0,1]$. So $f(P) \subseteq [0,1]$.

<u>Claim</u> $f : P \to [0,1]$ is onto.

Since any $y \in [0,1)$ has a base 2 representation of the form

$$y = \sum_{n=1}^{\infty} \frac{b_n}{2^n} \quad \text{where } b_n \in \{0,1\},$$

then setting $a_n = 2b_n$ we obtain

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \in P$$

such that $f(x) = y$. Note that this claim and the fact that $P \subseteq [0,1]$ shows that $P \sim [0,1]$. To see this use the Schroeder-Bernstein theorem or the varient of the Schroeder-Bernstein theorem from the homework.

It's time for a break...
While on break please think about the question:

<u>Question</u>: Is $f : P \to [0,1]$ one-to-one?

<u>Vote</u>:

One-to-one	not one-to-one
10	2

I do hope the results of yesterdays election were better that this vote, because f is not 1-to-1.

For example

$$f(0.2_{\text{base } 3}) = 0.1_{\text{base } 2} = \frac{1}{2}$$

$$f(0.0\overline{2}\overline{2}_{\text{base } 3}) = 0.0\overline{1}\overline{1}_{\text{base } 2}$$

$$= \sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{4}}{1-\frac{1}{2}} = \frac{1}{2}$$

We are now ready to define the Cantor singular function:

Since $[0,1]\setminus P$ is open then $[0,1]\setminus P = \bigcup_{n=1}^{\infty}(a_n, b_n)$ by the structure theorem of the open sets on $\mathbb{R}$.

Define

$$\psi(x) = \begin{cases} f(x) & \text{if } x \in P \\ f(a_n) & \text{if } x \in (a_n, b_n) \text{ for some } n \in \mathbb{N}. \end{cases}$$

There is a graph of $\Psi(x)$ in the book. It is missing some details because of all the ... which appear. Let's try to draw a more accurate graph based on the definition and base-3 expansions.

The idea. Given x convert it to base 3. If the expansion has only the digits 0 or 2 great, otherwise we round it down the greatest element $a_n \in P$ such that $a_n < x$. Since P is closed such an element exists. The rounding process looks like this

$$\begin{array}{ccc} x & \longrightarrow & a_n \\ .0202212102 & \longrightarrow & .020220\overline{2} \\ \text{base } 3 & & \text{base } 3 \end{array}$$

first 1 in the base 3 expansion

Therefore $\psi(x) = f(a_n)$ implies

$\psi(.0202212102_{\text{base } 3})$

$= f(.0202202\bar{2}_{\text{base } 3})$

$= .010110\bar{1}_{\text{base } 2}$

$= .010111_{\text{base } 2}$

So to simplify we map

$$\underset{\substack{\uparrow \\ \text{first } 1}}{.0202212102}_{\text{base } 3} \to \underbrace{.01011}_{\substack{\uparrow \\ \text{the preceding digits where the 2's have been changed to 1's.}}}\underset{\substack{\uparrow \\ \text{first } 1}}{1}_{\text{base } 2}$$

The Maple script developed in class to graph this function is on the website.