Math 713 Practice Quiz 1 Version A

13. Let $U \subseteq \mathbf{R}$ be a bounded open set and $x \in U$. Define

$$a = \inf\{\, y : y < x \text{ and } (y, x) \subseteq U \,\}.$$

Show that $a \notin U$.

Let $A = \{y : y < x \text{ and } (y,x) \subseteq U\}$. Since $a = \inf A$ there exists a sequence $y_n \in A$ such that $y_n \to a$. Thus $a \le y_n < x$ and $(y_n, x) \subseteq U$ for all $n \in \mathbb{N}$. Now

$$(a,x) = \bigcup_{n=1}^{\infty} (y_n, x) \subseteq U$$

implies $a \in A$. Suppose $a \in U$. Then for some $r > 0$ we would have $(a-r, a+r) \subseteq U$. Then

$$(a-r, a+r) \cup (a,x) \subseteq U$$

implies $(a-r, x) \subseteq U$ and so $a - r \in A$. This contradicts a being a lower bound of A. Therefore $a \notin U$.