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Math 714: Assignment #1

2 February 2011

3.69. *Let f be a nonnegative Lebesgue measurable function. Show that*

$$\lim_{n\to\infty}\int_{[-n,n]} f\,d\lambda = \int_{\mathcal{R}} f\,d\lambda.$$

Consider the sequence $\{f_n\}_{n=1}^{\infty}$ of real-valued functions on $\mathcal{R}$, where $f_n = \chi_{[-n,n]}f$. Note that $[-n,n] \in \mathcal{M}$ since it is an interval; consequently $\chi_{[-n,n]}$ is a Lebesgue measurable function. Hence each f_n is Lebesgue measurable, being the product of two Lebesgue measurable functions. For each $x \in \mathcal{R}$ we have $f_n(x) = f(x)$ whenever $n \geq |x|$, so $\lim_{n\to\infty} f_n(x) = f(x)$. We also have

$$f_n(x) = \begin{cases} f(x) \geq 0 & \text{if } |x| \leq n, \\ 0 & \text{if } |x| > n \end{cases}$$

and so

$$0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq f_n(x) \leq \cdots$$

for each $x \in \mathcal{R}$. Therefore $\{f_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence of nonnegative functions that converges pointwise to f, and we can apply the Monotone Convergence Theorem. We have

$$\lim_{n\to\infty}\int_{[-n,n]} f\,d\lambda = \lim_{n\to\infty}\int_{\mathcal{R}} f_n\,d\lambda = \int_{\mathcal{R}} f\,d\lambda,$$

which is what we wanted to show. ■

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3.88. *Let $E \in \mathcal{M}$ with $\lambda(E) < \infty$ and $\{f_n\}_{n=1}^{\infty}$ a sequence of Lebesgue measurable functions that converges pointwise on E to a real-valued function. Further suppose that there is an $M \in \mathcal{R}$ such that $|f_n(x)| \leq M$ for $n \in \mathcal{N}$, $x \in E$. Show that*

$$\int_E \lim_{n\to\infty} f_n\,d\lambda = \lim_{n\to\infty}\int_E f_n\,d\lambda.$$

Let $g : E \to \mathcal{R}$ be given by $g(x) = M$ for all $x \in E$. Then g is Lebesgue measurable since it is a constant function. We have

$$\int_E |g|\,d\lambda = \int_E M\,d\lambda = M\int_E d\lambda = M \cdot \lambda(E) < \infty,$$

which means that g is Lebesgue integrable. We also have $|f_n| \leq g$ for all $n \in \mathcal{N}$. Thus the Dominated Convergence Theorem implies that

$$\int_E \lim_{n\to\infty} f_n \, d\lambda = \lim_{n\to\infty} \int_E f_n \, d\lambda,$$

as desired. ■

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3.101. *For Lebesgue measurable functions, f and g, define $f \sim g$ if and only if $f = g$ λ-ae. Prove that $\sim$ is an equivalence relation.*

We must show that $\sim$ is reflexive, symmetric, and transitive. Let f, g, h be Lebesgue measurable functions and note that $f \sim g$ if and only if $\lambda(\{x : f(x) \neq g(x)\}) = 0$. Now $\{x : f(x) \neq f(x)\} = \varnothing$ and $\lambda(\varnothing) = 0$, showing that $f \sim f$, i.e., $\sim$ is reflexive. Next, since

$$\{x : f(x) \neq g(x)\} = \{x : g(x) \neq f(x)\},$$

we have

$$\lambda(\{x : f(x) \neq g(x)\}) = \lambda(\{x : g(x) \neq f(x)\}).$$

This means that if $\lambda(\{x : f(x) \neq g(x)\}) = 0$ then $\lambda(\{x : g(x) \neq f(x)\}) = 0$, so if $f \sim g$ then $g \sim f$. Thus $\sim$ is symmetric. Finally, we must show that $\sim$ is transitive. Suppose that $f \sim g$ and $g \sim h$. Put

$$A = \{x : f(x) \neq g(x)\}, \qquad B = \{x : g(x) \neq h(x)\}, \qquad C = \{x : f(x) \neq h(x)\};$$

then by assumption $\lambda(A) = 0$ and $\lambda(B) = 0$. We claim that $C \subseteq A \cup B$. To see this, suppose $x \in C$, so that $f(x) \neq h(x)$. If $x \notin A$ then we have $f(x) = g(x)$; hence $g(x) \neq h(x)$, so $x \in B$. Thus $x \in A \cup B$, proving the claim. We now have

$$\lambda(C) \leq \lambda(A \cup B) \leq \lambda(A) + \lambda(B) = 0 + 0 = 0.$$

But $\lambda(C) = 0$ means that $f \sim h$, so $\sim$ is transitive. The proof is therefore complete. ■

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3.108. *Let f and g be $\mathcal{M}$-measurable functions with $\int |f - g| \, d\lambda = 0$. Prove that $f = g$ λ-ae.*

Suppose to the contrary that we do not have equality almost everywhere. Let

$$E = \{x : f(x) \neq g(x)\} = \{x : |f(x) \neq g(x)| > 0\};$$

$$E_1 = \{x : |f(x) - g(x)| \geq 1\};$$
$$E_n = \{x : \tfrac{1}{n} \leq |f(x) - g(x)| < \tfrac{1}{n-1}\},\ n > 0.$$

Then the sets $\{E_n\}_{n=1}^{\infty}$ are disjoint and $E = \bigcup_{n=1}^{\infty} E_n$. Our assumption implies that $\lambda(E) > 0$. By the additivity of the measure, we have

$$\sum_{n=1}^{\infty} \lambda(E_n) = \lambda(E) > 0.$$

It follows that there is some $N \in \mathcal{N}$ such that $\lambda(E_N) > 0$. For $x \in E_N$, $\frac{1}{N} \leq |f(x) - g(x)|$. We have

$$\begin{aligned} 0 < \tfrac{1}{N}\lambda(E_N) &= \tfrac{1}{N} \int_{E_N} d\lambda \\ &= \int_{E_N} \tfrac{1}{N}\, d\lambda \\ &\leq \int_{E_N} |f(x) - g(x)|\, d\lambda \\ &\leq \int_{\mathcal{R}} |f - g|\, d\lambda. \end{aligned}$$

But this is a contradiction, since $\int |f - g|\, d\lambda = 0$. We conclude that $f = g$ λ-ae. ■

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3.109. *Show that, if f is Lebesgue integrable and $\int_E f\, d\lambda = 0$ for each $E \in \mathcal{M}$, then $f = 0$ λ-ae.*

Let $A = f^{-1}\big((0, \infty)\big) = \{x : f(x) > 0\}$ and $B = f^{-1}\big((-\infty, 0)\big) = \{x : f(x) < 0\}$. Since f is Lebesgue integrable it is Lebesgue measurable, and so A and B, being inverse images of open sets, are members of $\mathcal{M}$. By assumption we have $\int_A f\, d\lambda = 0$ and $\int_B f\, d\lambda = 0$. Note that

$$f^+(x) = \begin{cases} f(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin A \end{cases} = \chi_A f(x)$$

and

$$f^-(x) = \begin{cases} -f(x) & \text{if } x \in B, \\ 0 & \text{if } x \notin B \end{cases} = -\chi_B f(x).$$

Therefore,

$$\int_{\mathcal{R}} f^+\, d\lambda = \int_{\mathcal{R}} \chi_A f\, d\lambda = \int_A f\, d\lambda = 0$$

and

$$\int_{\mathcal{R}} f^-\, d\lambda = \int_{\mathcal{R}} \chi_B f\, d\lambda = -\int_B f\, d\lambda = 0.$$

Observe that

$$\int_{\mathcal{R}} |f| \, d\lambda = \int_{\mathcal{R}} f^+ \, d\lambda + \int_{\mathcal{R}} f^- \, d\lambda = 0 + 0 = 0.$$

Let $g : \mathcal{R} \to \mathcal{R}$ be given by $g(x) = 0$ for all $x \in \mathcal{R}$. Then g is $\mathcal{M}$-measurable (since it is a constant function), and $\int_{\mathcal{R}} |f - g| \, d\lambda = \int_{\mathcal{R}} |f| \, d\lambda = 0$. The result of the previous exercise implies that $f = g$ λ-ae. That is, $f = 0$ λ-ae, completing the proof. ■