1. [Carrier, Krook and Pearson Section 2-1 Exercise 1] Show that no purely real function can be analytic, unless it is a constant.

Consider a function $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$ and where $u$ and $v$ are real functions. For $f$ to be purely real means $v(x, y) = 0$. For $f$ to be analytic means $u_x = v_y$ and $u_y = -v_x$ hold at every point in the complex plane. Therefore, if $f$ is a purely-real analytic function, it follows that $u_x = 0$ and $u_y = 0$. Now $u_x = 0$ implies for $y_0$ fixed that the function $x \rightarrow u(x, y_0)$ is constant and similarly $u_y = 0$ implies for $x_0$ fixed that the function $y \rightarrow u(x_0, y)$ is constant.

Let $c = u(x_0, y_0)$. Consider any point $x_1 + iy_1$ in the complex plane. Since $x \rightarrow u(x, y_0)$ is constant holding $y_0$ fixed, then $u(x_1, y_0) = u(x_0, y_0) = c$. Since $y \rightarrow u(x_1, y)$ is constant holding $x_1$ fixed, then $u(x_1, y_1) = u(x_1, y_0) = c$. It follows that $u$ is identically equal to $c$ throughout the entire complex plane. Therefore $f$ is constant.
2. [Carrier, Krook and Pearson Section 2-2 Exercise 1] Evaluate 
\[ \int_{1+i}^{3-2i} \sin z \, dz \]
in two ways. First by choosing any path between the two end points and using real integrals as in
\[ \int_C f(z) \, dz = \int_C (u \, dx - v \, dy) + i \int_C (u \, dy + v \, dx) \]  \hspace{1cm} (2-5)
and second by use of an indefinite integral. Show that the inequality
\[ \left| \int_C f(z) \, dz \right| \leq \int_C |f(z)| \, |dz| \leq ML \]  \hspace{1cm} (2-6)
where \( |f| \leq M \) and \( L \) is the length of \( C \) is satisfied.

The trigonometric identities
\[
\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y \\
\cos z = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y
\]
will be used in this exercise.

First, consider the path \( C \) given by

\[
\begin{align*}
&1+1i \\
&\gamma_1(t) = 1 + i(1-3t) \\
&\gamma_2(t) = (1 + 2t) - 2i
\end{align*}
\]
which may be written as the sum of the paths \([\gamma_1]\) and \([\gamma_2]\) where \(\gamma_1(t) = 1 + i(1-3t)\) and \(\gamma_2(t) = (1 + 2t) - 2i\). Since
\[
\int_C \sin z \, dz = \int_C (\sin x \cosh y \, dx - \cos x \sinh y \, dy) + i \int_C (\sin x \cosh y \, dy + \cos x \sinh y \, dx),
\]
compute the real path integrals
\[
\begin{align*}
\int_{[\gamma_1]} (\sin x \cosh y \, dx - \cos x \sinh y \, dy) &= -\int_1^{-2} \cos 1 \sinh y \, dy = -\cos 1(\cosh 2 - \cosh 1) \\
\int_{[\gamma_2]} (\sin x \cosh y \, dx - \cos x \sinh y \, dy) &= \int_1^3 \sin x \cosh 2 \, dx = (\cos 1 - \cos 3) \cosh 2
\end{align*}
\]
Math 715 Homework 1 Solutions

\[
i \int_{[\gamma_1]} (\sin x \cosh y dy + \cos x \sinh y dx) = i \int_1^{-2} \sin 1 \cosh y dy = -i \sin 1(\sinh 2 + \sinh 1)
\]

\[
i \int_{[\gamma_2]} (\sin x \cosh y dy + \cos x \sinh y dx) = -i \int_1^{3} \cos x \sinh 2 dx = i(\sin 1 - \sin 3) \sinh 2
\]

and add them to obtain

\[
\int_C \sin z \, dz = \cos 1 \cosh 1 - \cos 3 \cosh 2 - i(\sin 1 \sinh 1 + \sin 3 \sinh 2).
\]

This finishes the computation using real path integrals.

Second, compute using an indefinite integral to obtain

\[
\int_{1+i}^{3-2i} \sin z \, dz = -\cos(3 - 2i) + \cos(1 + i)
\]

\[
= -\cos 3 \cosh 2 - i \sin 3 \sinh 2 + \cos 1 \cosh 1 - i \sin 1 \sinh 1.
\]

This answer is the same as the answer found using the path integrals.

We now show inequality (2-6) is satisfied. The term of the left hand side is

\[
\left| \int_C \sin z \, dz \right| = \left| \cos(1 + i) - \cos(3 - 2i) \right|
\]

\[
\approx |4.558275530 - 1.500720276i| \approx 4.798962091.
\]

To approximate the integral

\[
\int_C |\sin z| \, |dz|
\]

use numerical techniques. The Maple script

```maple
1 # Compute integral |f(z)||dz| along the path gamma1+gamma2
2 restart;
3 f:=z->sin(z);
4 gamma1:=t->1+I*(1-3*t);
5 gamma2:=t->(1+2*t)-2*I;
6 I1:=Integrate(abs(f(gamma1(t))*diff(gamma1(t),t)),t=0..1);
7 F1:=evalf(I1);
8 I2:=Integrate(abs(f(gamma2(t))*diff(gamma2(t),t)),t=0..1);
9 F2:=evalf(I2);
10 print("The integral |f(z)||dz| is approximately", F1+F2);
```

gives the output

\[
f := z \rightarrow \sin(z)
\]

\[
gamma1 := t \rightarrow 1 + (1 - 3 \, t) \, I
\]

\[
gamma2 := t \rightarrow 2 \, t + 1 - 2 \, I
\]
Math 715 Homework 1 Solutions

\[ \int_{0}^{\infty} \left| 3 \sin(1 + (1 - 3t) I) \right| dt \]

\[ I1 := 4.468016320 \]

\[ \int_{0}^{\infty} \left| 2 \sin(2t + 1 - 2I) \right| dt \]

\[ I2 := 7.429875986 \]

"The integral |f(z)||dz| is approximately", 11.89789231

Therefore

\[ \int_{C} |\sin z||dz| \approx 11.89789231. \]

Finally we compute \( M \) using Maple

\# Find the maximum value of |f(z)| on gamma1 and gamma2
restart;
_ENVAllSolutions:=true;

f:=z->sin(z);
g:=t->1+I*(1-3*t);
ff:=f(g(t))*conjugate(f(g(t))) assuming t::real;
dff:=diff(ff,t);
cp:=solve(dff=0,t);
k:=0;
for j from 1 to nops([cp])
do
  for i from -3 to 3
do
    c:=evalf(subs(_Z1=i,cp[j]));
    if(abs(Im(c))<0.00001 and Re(c)<1 and Re(c)>0)
      then
        k:=k+1;
        cpn[k]:=abs(c);
      end
  end
end
Math 715 Homework 1 Solutions

```
end;
cps:=\left[0.0,seq(cpn[n],n=1..k),1.0\right];
v1:=seq(abs(f(g(cps[n])))\text{,}\ n=1..\text{nops(cps)});
print("The maximum of } |f(z)| \text{ on gamma1 is}",\max(v1));
g:=t->(1+2*t)-2*I;
ff:=f(g(t))*conjugate(f(g(t)))\text{ assuming } t::\text{real};
dff:=diff(ff\text{,}\ t);
cp:=solve(dff=0\text{,}\ t);
k:=0;
for j from 1 to \text{nops([cp])}
do
  for i from -3 to 3
  do
    c:=evalf(subs(_Z2=i,\text{cp}[j]));
    if(abs(Im(c))<0.00001 and Re(c)<1 and Re(c)>0)
    then
      k:=k+1;
      cpn[k]:=abs(c);
    end
  end
end;
cps:=\left[0.0,seq(cpn[n],n=1..k),1.0\right];
v2:=seq(abs(f(g(cps[n])))\text{,}\ n=1..\text{nops(cps)});
print("The maximum of } |f(z)| \text{ on gamma2 is}",\max(v2));
print("The maximum of } |f(z)| \text{ on C is}",\max(v1,v2));
print("LM is\text{,}\ 5*max(v1,v2));
```

with the results

\_EnvAllSolutions := true

\text{f := z -> sin(z)}

\text{g := t -> 1 + (1 - 3 t) I}

\text{ff := sin(1 + (1 - 3 t) I) sin(1 - (1 - 3 t) I)}

\text{dff := -3 I cos(1 + (1 - 3 t) I) sin(1 - (1 - 3 t) I)}

\text{+ 3 I sin(1 + (1 - 3 t) I) cos(1 - (1 - 3 t) I)}

\text{/ 2 1/2 \}
\text{| -1 + (1 + tan(2) ) |}
\text{cp := -1/3 I \{1 + I - arctan(------------------)} - Pi \_Z1\text{,}}
\text{tan(2) \}/}
\text{/ 2 1/2 \}
```
Math 715 Homework 1 Solutions

\[ | \frac{1}{1 + (1 + \tan(2))} - \frac{1}{3}i | 1 + i + \arctan\left(\frac{-\pi}{\tan(2)}\right) | \]

\[
k := 0
\]

cps := \[0., 0.3333333333, 1.0\]

\[v1 := 1.445396576, 0.8414709848, 3.723196185\]

"The maximum of \(|f(z)|\) on gamma1 is", 3.723196185

g := t -> 2t + 1 - 2i

\[ff := \sin(2t + 1 - 2i) \sin(2t + 1 + 2i)\]

\[dff := 2 \cos(2t + 1 - 2i) \sin(2t + 1 + 2i) + 2 \sin(2t + 1 - 2i) \cos(2t + 1 + 2i)\]

\[2^{1/2} \frac{1}{1 + (1 - \tanh(4))} \frac{-\pi}{\tanh(4)}\]

cp := \[-\frac{1}{2} + i - \frac{1}{2}i \arctanh\left(\frac{-\pi}{\tanh(4)}\right) + \frac{1}{2}\]

\[2^{1/2} \frac{-1 + (1 - \tanh(4))}{1 + 1/2 \arctanh\left(\frac{-\pi}{\tanh(4)}\right) + \frac{1}{2}}\]

\[k := 0\]

cps := \[0., 0.2853981635, 1.0\]

\[v2 := 3.723196185, 3.762195691, 3.629604837\]

"The maximum of \(|f(z)|\) on gamma2 is", 3.762195691

"The maximum of \(|f(z)|\) on C is", 3.762195691

"LM is", 18.81097846

Therefore \(M \approx 3.762195691\) and \(L = 3 + 2 = 5\) so that \(LM \approx 18.81097846\). Since \(4.798962091 \leq 11.89789231 \leq 18.81097846\)

we have shown that inequality (2-6) is satisfied for the curve \(C\).
3. [Carrier, Krook and Pearson Section 2-2 Exercise 8a] Show that formal, term-by-term differentiation, or integration, of a power series yields a new power series with the same radius of convergence.

**Term-by-term Differentiation.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) have radius of convergence \( R \) and \( g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \) have radius of convergence \( r \). Claim \( R = r \).

First show \( r \geq R \). Let \( z \) be such that \( 0 < |z| < R \). Choose \( w \) such that \( |z| < |w| < R \). Then by (1-5) on page 9 we have that the series for \( f(w) \) converges absolutely. Thus,

\[
\sum_{n=0}^{\infty} |a_n||w|^n < \infty.
\]

Now, let \( \alpha = |z|/|w| \). Then \( 0 < \alpha < 1 \) and so \( n\alpha^n \to 0 \) as \( n \to \infty \). It follows that \( n\alpha^n \) is bounded by some constant \( A \) so that \( n\alpha^n \leq A \) for all \( n \). Now,

\[
\sum_{n=1}^{\infty} n|a_n||z|^{n-1} = \frac{1}{|z|} \sum_{n=1}^{\infty} n|a_n||w|^n\alpha^n \leq \frac{A}{|z|} \sum_{n=1}^{\infty} |a_n||w|^n < \infty
\]

shows that the series for \( g(z) \) converges for any \( |z| < R \). It follows that \( r \geq R \).

Second show that \( R \leq r \). Suppose \( |z| < r \). Then by (1-5) on page 9 we have that the series for \( g(z) \) converges absolutely. Thus,

\[
\sum_{n=1}^{\infty} n|a_n||z|^{n-1} < \infty.
\]

Now,

\[
\sum_{n=0}^{\infty} |a_n||z|^n = |a_0| + |z| \sum_{n=1}^{\infty} |a_n||z|^{n-1} \leq |a_0| + |z| \sum_{n=1}^{\infty} n|a_n||z|^{n-1} < \infty.
\]

shows that the series for \( f(z) \) converges for any \( |z| < r \). It follows that \( R \leq r \).

**Term-by-term Integration.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) have radius of convergence \( R \) and \( h(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} a_n z^{n+1} \) have radius of convergence \( \rho \). Claim \( R = \rho \).

First show that \( R \leq \rho \). Suppose \( |z| < R \). Then by (1-5) on page 9 we have that the series for \( f(z) \) converges absolutely. Thus,

\[
\sum_{n=0}^{\infty} |a_n||z|^n < \infty.
\]

Now,

\[
\sum_{n=0}^{\infty} \frac{1}{n+1}|a_n||z|^{n+1} \leq |z| \sum_{n=0}^{\infty} |a_n||z|^n < \infty.
\]

shows that the series for \( h(z) \) converges for any \( |z| < R \). It follows that \( R \leq \rho \).
Math 715 Homework 1 Solutions

Second show $R \geq \rho$. Let $z$ be such that $0 < |z| < \rho$. Choose $w$ such that $|z| < |w| < \rho$. Then by (1-5) on page 9 we have that the series for $h(w)$ converges absolutely. Thus,

$$
\sum_{n=0}^{\infty} \frac{1}{n+1} |a_n||w|^{n+1} < \infty.
$$

Now, let $\alpha = |z|/|w|$. Let $A$ be the bound so that $n\alpha^n \leq A$ for all $n$. Now,

$$
\sum_{n=0}^{\infty} |a_n||z|^n = \frac{1}{|z|} \sum_{n=0}^{\infty} \frac{1}{n+1} |a_n||w|^{n+1} (n+1)\alpha^{n+1} \leq \frac{A}{|z|} \sum_{n=0}^{\infty} \frac{1}{n+1} |a_n||w|^{n+1} < \infty
$$

shows that the series for $f(z)$ converges for any $|z| < \rho$. It follows that $R \geq \rho$. 

Math 715 Homework 1 Solutions

4. [Carrier, Krook and Pearson Section 2-2 Exercise 8b] The uniform-convergence property of a power series implies that term-by-term integration yields the integral of the sum function. Show that the integrated sum function is single valued and analytic within the circle of convergence.

Let $R$ be the radius of convergence of the infinite series defining $f$ and $h$ in part a. Let

$$f_n(z) = \sum_{k=0}^{n} a_k z^k \quad \text{and} \quad h_n(z) = \sum_{k=0}^{n} \frac{1}{k+1} a_k z^{k+1}.$$

Suppose $\xi$ and $z$ are such that $|\xi| < R$ and $|z| < R$. Let $\Gamma = \{ \gamma(t) : t \in [0, t] \}$ be any path such that $\gamma(0) = z$, $\gamma(1) = \xi$ and $|\gamma(t)| < R$ for $t \in [0, 1]$. Since $f_n$ is a polynomial, it is analytic. Therefore the path integral along $\Gamma$ is path independent and since $h'_n = f_n$ we obtain

$$\int_{\Gamma} f_n(\zeta) \, d\zeta = \int_{z}^{\xi} f_n(\zeta) \, d\zeta = h_n(\xi) - h_n(z).$$

Since $|\gamma(t)| < R$ for $t \in [0, 1]$ then there is $\eta > 0$ such that $|\gamma(t)| \leq R - \eta$ for $t \in [0, 1]$. From the results on page 9 we obtain that

$$f_n(\gamma(t)) \to f(\gamma(t)) \quad \text{uniformly in } t \text{ as } n \to \infty.$$

Therefore,

$$\lim_{n \to \infty} \int_{\Gamma} f_n(\zeta) \, d\zeta = \lim_{n \to \infty} \int_{0}^{1} f_n(\gamma(t)) \gamma'(t) \, dt = \int_{0}^{1} f(\gamma(t)) \gamma'(t) \, dt = \int_{\Gamma} f(\zeta) \, d\zeta.$$

It follows that

$$\int_{\Gamma} f(\zeta) \, d\zeta = \lim_{n \to \infty} (h_n(\xi) - h_n(z)) = h(\xi) - h(z).$$

Since this equality holds for any $\Gamma$ inside the radius of convergence, the integral is path independent. Therefore, we may write

$$\int_{z}^{\xi} f(\zeta) \, d\zeta = h(\xi) - h(z)$$

for any $z$ and $\xi$ such that $|\xi| < R$ and $|z| < R$ where the integral is to be interpreted as a path integral along any path from $z$ to $\xi$ that lies strictly inside the radius of convergence. Since $h$ is single valued then the integrated sum function is single valued.

We now claim $h$ is analytic and $h'(z) = f(z)$. Let $\epsilon > 0$. Since $f$ is continuous at $z$ there is $\delta > 0$ so that $|\xi - z| < \delta$ implies $|f(\xi) - f(z)| < \epsilon$. Define $\gamma(t) = \xi (1 - t) + z t$ so that $\gamma'(t) = z - \xi$. Now, $|\xi - z| < \delta$ implies $|\gamma(t) - z| < \delta$ for $t \in [0, 1]$ and therefore

$$\left| \frac{h(\xi) - h(z)}{\xi - z} - f(z) \right| = \left| \int_{\Gamma} \frac{f(\zeta) - f(z)}{z - \xi} \, d\zeta \right| = \left| \int_{0}^{1} \frac{f(\gamma(t)) - f(z)}{z - \xi} \gamma'(t) \, dt \right|$$

$$\leq \int_{0}^{1} \left| f(\gamma(t)) - f(z) \right| \, dt < \int_{0}^{1} \epsilon \, dt = \epsilon.$$

Consequently $h'(z) = f(z)$ for every $|z| < R$. 

9
5. [Carrier, Krook and Pearson Section 2-2 Exercise 8c] Show that a power series converges to an analytic function within its circle of convergence.

Consider the power series for $f(z)$ defined above with radius of convergence equal to $R$. By part a the function $g(z)$ also has radius of convergence equal to $R$. Since $f(z)$ may be obtained from $g(z)$ through term by term integration of $g(z)$ we have by part b that $f'(z) = g(z)$ for every $z$ such that $|z| < R$. Thus, $f$ is differentiable and therefore analytic in its circle of convergence.
Math 715 Homework 1 Solutions

6. [Carrier, Krook and Pearson Section 2-3 Exercise 1] Use Cauchy’s integral formula to evaluate the integral around the unit circle $|z| = 1$ of

$$\frac{\sin z}{2z + i}, \frac{\ln(z + 2)}{z + 2}, \frac{z^3 + \text{asinh}(z/2)}{z^2 + iz + i} \text{ and } \cot z.$$ 

Let $D = \{ z : |z| < 1 \}$ be the unit disk and $\Gamma = \partial D$ be its boundary oriented in the positive (counterclockwise) direction. Since $\sin z$ is analytic on $D$ and $-i/z$ is contained within it, then Cauchy’s formula implies

$$\int_{\Gamma} \frac{\sin z}{2z + i} = \frac{1}{2} \int_{\Gamma} \frac{\sin z}{z + i/2} = \pi i \sin(-i/2) = -\pi i \sin(i/2).$$

Since

$$\sin z = \frac{\exp(iz) - \exp(-iz)}{2i}$$

then

$$\sin(i/2) = \frac{\exp(-1/2) - \exp(1/2)}{2i} = \frac{i}{2} \left( \exp(1/2) - \exp(-1/2) \right) = i \sinh(1/2).$$

It follows that

$$\int_{\Gamma} \frac{\sin z}{2z + i} = \pi \sinh(1/2).$$

For the next integral note that, by definition, the principle branch $\ln z$ of the logarithm is the inverse of the function

$$z \to \exp(z): \{ x + iy : x \in \mathbb{R} \text{ and } y \in (-\pi, \pi) \} \to \mathbb{C}.$$ 

Therefore $\ln z$ is analytic on the domain $\mathbb{C} \setminus (-\infty, 0]$ and consequently $\ln z + 2$ is analytic on $\mathbb{C} \setminus (-\infty, -2]$. It follows that

$$\frac{\ln z + 2}{z + 2}$$

is analytic in an open set containing $D$. By Cauchy’s theorem we have

$$\int_{\Gamma} \frac{\ln(z + 2)}{z + 2} = 0.$$ 

For the next integral note that

$$\sinh z = \frac{\exp(z) - \exp(-z)}{2}$$

Setting $\psi = \sinh z$ and $\omega = \exp z$ yields

$$2\omega \psi = \omega^2 - 1. \quad (*)$$
Math 715 Homework 1 Solutions

Completing the square and factoring yields

\[(\omega - \psi)^2 = \psi^2 + 1.\]

We would like to take square roots to solve for \(\omega\). Recall that the principle branch of the square root function is defined as

\[re^{i\theta} \mapsto \sqrt{re^{i\theta}/2} \quad \text{for} \quad \theta \in (-\pi, \pi].\]

This function is analytic on \(\mathbb{C} \setminus (-\infty, 0]\). The only time \(\psi^2 + 1 \in (-\infty, 0]\) is when \(\psi \in i(-\infty, -1]\) or \(\psi \in i[1, \infty)\).

Therefore the map

\[\psi \mapsto \psi + \sqrt{\psi^2 + 1}\]

is analytic on \(\mathbb{C} \setminus (i(-\infty, -1] \cup i[1, \infty))\).

From (*) observe that if \(\omega \leq 0\) then \(\psi \in \mathbb{R}\). However, \(\psi \in \mathbb{R}\) implies \(\omega = \psi + \sqrt{\psi^2 + 1} > 0\), which is a contradiction. Therefore, it can’t happen that \(\omega \leq 0\).

Consequently the function

\[\psi \mapsto \log(\psi + \sqrt{\psi^2 + 1})\]

is analytic on \(\mathbb{C} \setminus (i(-\infty, -1] \cup i[1, \infty))\). As a result

\[\text{asinh}(z) = \log(z + \sqrt{z^2 + 1})\]

is an inverse to \(\sinh(z)\) which is analytic on \(\mathbb{C} \setminus (i(-\infty, -1] \cup i[1, \infty))\). In particular \(\sinh(z/2)\) is analytic on a neighborhood of the unit disk \(D\).

Note that \(z^2 + iz + i = (z - z_1)(z - z_2)\) where

\[|z_1| = \left|\frac{-i + \sqrt{-1 - 4i}}{2}\right| > 1 \quad \text{and} \quad |z_2| = \left|\frac{-i - \sqrt{-1 - 4i}}{2}\right| < 1.\]

Therefore

\[\frac{z^3 + \text{asinh}(z/2)}{z - z_1}\]

is analytic on \(D\). It follows from Cauchy’s formula that

\[
\int_\Gamma \frac{z^3 + \text{asinh}(z/2)}{z^2 + iz + i} \, dz = 2\pi i \frac{z_2^3 + \text{asinh}(z_2/2)}{z_2 - z_1} \approx -0.04880 + 1.8762i.
\]

For the final integral note that

\[\cot z = \frac{\cos z}{\sin z}\]

and that

\[\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} z^{2k+1} = z \left( \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} z^{2k} \right) = z \text{sinc } z,\]

where \(\text{sinc } z\) is defined by the absolutely convergent series in parenthesis shown above.

Since the only zeros of \(\sin z\) are \(z = k\pi\) where \(k \in \mathbb{Z}\) then the only zeros of \(\text{sinc } z\) are for \(z = k\pi\) where \(k \in \mathbb{Z} \setminus \{0\}\). It follows that \(\frac{\cos z}{\text{sinc } z}\) is analytic on \(D\). Consequently

\[
\int_\Gamma \cot z \, dz = 2\pi i \left( \frac{\cos 0}{\text{sinc } 0} \right) = 2\pi i.
\]
7. [Carrier, Krook and Pearson Section 2-3 Exercise 2] If \( \Phi(z) \) is analytic in a simply connected region in which a closed contour \( C \) is drawn, obtain all possible values of

\[
\int_C \frac{\Phi(\zeta)}{\zeta^2 - z^2}
\]

where \( z \) is not a point on \( C \) itself.

Let \( C = \partial\Omega \) be the boundary of an open set and \( S = \Omega \) be a set on which the Green’s theorem holds. Consider separately two cases. The first where \( z = 0 \) and the second where \( z \neq 0 \). If \( z = 0 \), the possible values of the integral in question are given by

\[
\int_C \frac{\Phi(\zeta)}{\zeta^2} = \begin{cases} 
2\pi i\Phi'(0) & \text{for } 0 \in \Omega \\
0 & \text{for } 0 \notin \Omega.
\end{cases}
\]

If \( z \neq 0 \), we note that \( \zeta^2 - z^2 = (\zeta - z)(\zeta + z) \). Therefore, the possible values are

\[
\int_C \frac{\Phi(\zeta)}{\zeta^2 - z^2} = \begin{cases} 
\pi i\Phi(z)z^{-1} - \pi i\Phi(-z)z^{-1} & \text{for } z \in \Omega \text{ and } -z \in \Omega \\
\pi i\Phi(z)z^{-1} & \text{for } z \in \Omega \text{ and } -z \notin \Omega \\
-\pi i\Phi(-z)z^{-1} & \text{for } z \notin \Omega \text{ and } -z \in \Omega \\
0 & \text{for } z \notin \Omega \text{ and } -z \notin \Omega.
\end{cases}
\]

If one also considers closed contours \( C \) that do not necessarily bound a domain, then the contours may wind around the singularities multiple times. In this case we obtain the following possibilities

\[
2\pi ik\Phi'(0) \quad \text{and} \quad \pi i k_1\Phi(z)z^{-1} - \pi i k_2\Phi(-z)z^{-1}
\]

where \( k, k_1, k_2 \in \mathbb{Z} \).
8. [Carrier, Krook and Pearson Section 2-3 Exercise 3] If \( n \) is an integer, positive or negative, and if \( C \) is a closed contour around the origin, use

\[
f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta
\]  

(2 - 14)

to show that

\[
\oint_C \frac{dz}{z^n} = 0 \quad \text{unless} \quad n = 1.
\]

If \( n \leq 0 \) then \( 1/z^n \) is analytic. Therefore, Cauchy’s theorem implies

\[
\oint_C \frac{dz}{z^n} = 0.
\]

If \( n > 1 \), we write \( f(z) = 1 \) so that \( f^{(n-1)}(z) = 0 \). Then (2-14) implies

\[
\oint_C \frac{1}{z^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(0) = 0.
\]

Finally, if \( n = 1 \) then

\[
\oint_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it} dt = \int_0^{2\pi} idt = 2\pi i
\]

which is non-zero.
9. [Carrier, Krook and Pearson Section 2-3 Exercise 6] Show, that if $\Phi(z)$ is any function continuous on $C$ and if a function $f(z)$ is defined for $z$ include $C$ by

$$f(z) = \frac{1}{2\pi i} \int_C \frac{\Phi(\zeta)}{\zeta - z} d\zeta$$

then $f(z)$ is analytic inside $C$. Invent an example to illustrate the fact that, as an interior point $z$ approaches a boundary point $z_0$, then $f(z)$ need not approach $\Phi(z_0)$.

To show that $f(z)$ is analytic, it is enough to show it has a derivative. By definition,

$$f'(z) = \lim_{\xi \to z} \frac{f(\xi) - f(z)}{\xi - z}.$$ 

Therefore, it remains to show that this limit exists. Now,

$$\frac{f(\xi) - f(z)}{\xi - z} = \frac{1}{\xi - z} \cdot \frac{1}{2\pi i} \int_C \Phi(\zeta) \left( \frac{1}{\zeta - \xi} - \frac{1}{\zeta - z} \right) d\zeta$$

$$= \frac{1}{2\pi i} \int_C \Phi(\zeta) \left( \frac{1}{(\zeta - \xi)(\zeta - z)} \right) d\zeta$$

$$= \frac{1}{2\pi i} \int_C \Phi(\zeta) \left( \frac{1}{(\zeta - z)^2} + \frac{1}{(\zeta - \xi)(\zeta - z)} - \frac{1}{(\zeta - z)^2} \right) d\zeta$$

$$= \frac{1}{2\pi i} \int_C \Phi(\zeta) \left( \frac{1}{(\zeta - z)^2} + \frac{\xi - z}{(\zeta - \xi)(\zeta - z)^2} \right) d\zeta$$

$$= \frac{1}{2\pi i} \int_C \frac{\Phi(\zeta)}{(\zeta - z)^2} \left( 1 + \frac{\xi - z}{\zeta - \xi} \right) d\zeta.$$

It follows that

$$\left| \frac{f(\xi) - f(z)}{\xi - z} \right| \leq \frac{1}{2\pi} \left| \int_C \frac{\Phi(\zeta)}{(\zeta - z)^2} \left( \frac{\xi - z}{\zeta - \xi} \right) d\zeta \right|$$

Now since $\Phi(\zeta)$ is continuous on $C$ it’s maximum exists and we may define

$$M = \max \left\{ |\Phi(\zeta)| : \zeta \in C \right\}.$$

Since $z \notin C$ then

$$\mu = \text{dist}(z, C) = \min \left\{ |z - \zeta| : \zeta \in C \right\} > 0.$$

Finally, if $\xi$ is close enough to $z$ then also $\xi \notin C$. Consequently,

$$\nu = \text{dist}(\xi, C) > 0.$$

By assumption $C = \{ \gamma(t) : t \in [0, 1] \}$ where $\gamma(t)$ is a differentiable function such that

$$L = \int_0^1 |\gamma'(t)| dt < 0.$$
Math 715 Homework 1 Solutions

Therefore, follows that

\[
\frac{1}{2\pi} \left| \int_C \frac{\Phi(\xi)}{(\xi - z)^2} \left( \frac{\xi - z}{\xi - \xi} \right) d\xi \right| \leq \frac{1}{2\pi} \int_0^1 \left| \frac{\Phi(\gamma(t))}{(\gamma(t) - z)^2} \right| \left| \frac{\xi - z}{\gamma(t) - \xi} \right| |\gamma'(t)| dt
\]

\[
\leq \frac{1}{2\pi} \frac{ML}{\mu^2 \nu} |\xi - z| \to 0
\]

as \( \xi \to z \). Therefore \( f(z) \) is analytic at each point inside \( C \).

For an example where \( f(z) \) does not approach \( \Phi(z_0) \) as the interior point \( z \) approaches a boundary point \( z_0 \), consider

\[
\Phi(z) = \text{Re}(z), \quad z_0 = 1 \quad \text{and} \quad C = \{ \exp(it) : t \in [-\pi, \pi] \}.
\]

Now

\[
\Phi(z) = \text{Re}(z) \to 1 \quad \text{as} \quad z \to 1
\]

However,

\[
\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{\cos t}{e^{it} - z} ie^{it} dt = \frac{1}{4\pi i} \int_{-\pi}^{\pi} \frac{e^{it} + e^{-it}}{e^{it} - z} ie^{it} dt
\]

\[
= \frac{1}{4\pi i} \int_C \frac{\zeta + \zeta^{-1}}{\zeta - z} d\zeta = \frac{1}{4\pi i} \int_C \frac{\zeta^2 + 1}{(\zeta - z)\zeta} d\zeta
\]

\[
= \frac{1}{2} \left( \frac{z^2 + 1}{z} - \frac{1}{z} \right) = \frac{z}{2} \to \frac{1}{2} \quad \text{as} \quad z \to 1.
\]
Math 715 Homework 1 Solutions

10. [Carrier, Krook and Pearson Section 2-3 Exercise 7] A function \( \Phi(z) \) is known to be continuous throughout a simply connected region and to have the property that

\[
\int_C \Phi(z) dz = 0
\]

for any closed contour in that region. Show that \( \Phi(z) \) must be analytic in that region.

Let \( \Omega \) be the simple region mentioned above. For \( z_0 \in \Omega \) fixed, define

\[
F(\xi) = \int_{\Gamma} \Phi(\zeta) d\zeta \quad \text{for} \quad \xi \in \Omega
\]

where \( \Gamma = \{ \gamma(t) : t \in [0, 1] \} \) is any path that remains inside \( \Omega \) and such that \( \gamma(0) = z_0 \) and \( \gamma(1) = \xi \). Claim that \( F(\xi) \) is independent of the path \( \Gamma \). Consider two different paths \( \Gamma_i = \{ \gamma_i(t) : t \in [0, 1] \} \) such that \( \gamma_i(0) = z_0 \) and \( \gamma_i(1) = \xi \) for \( i = 1, 2 \). Let

\[
C = \{ c(t) : t \in [0, 2] \} \quad \text{where} \quad c(t) = \begin{cases} \gamma_1(t) & \text{for } t \in [0, 1] \\ \gamma_2(2 - t) & \text{for } t \in (1, 2] \end{cases}
\]

Then \( C \) is a closed contour and \( c(t) \) is pathwise differentiable. It follows that

\[
0 = \int_C \Phi(\zeta) d\zeta = \int_0^2 \Phi(c(t)) c'(t) dt = \int_0^1 \Phi(\gamma_1(t)) \gamma_1'(t) dt + \int_1^2 \Phi(\gamma_2(2 - t)) (-\gamma_2'(2 - t)) dt.
\]

Setting \( s = 2 - t \) so that \( ds = -dt \) yields

\[
\int_1^2 \Phi(\gamma_2(2 - t)) (-\gamma_2'(2 - t)) dt = -\int_0^1 \Phi(\gamma_2(s)) \gamma_2'(s) ds.
\]

Therefore

\[
\int_{\Gamma_1} \Phi(\zeta) d\zeta = \int_{\Gamma_2} \Phi(\zeta) d\zeta,
\]

which shows the integral defining \( F \) does not depend on the choice of path.

Claim \( F \) is differentiable on \( \Omega \). Let \( z \in \Omega \) be fixed. Since \( \Omega \) is open, there is \( \alpha > 0 \) such that \( B_\alpha(z) \subseteq \Omega \). Therefore, for \( \xi \) such that \( |\xi - z| < \alpha \), the straight line path

\[
\Gamma_3 = \{ \gamma_3(t) : t \in [0, 1] \} \quad \text{where} \quad \gamma_3(t) = z + t(\xi - z)
\]

lies entirely within \( \Omega \). It follows that

\[
\frac{F(\xi) - F(z)}{\xi - z} = \frac{1}{\xi - z} \left( \int_{z_0}^{\xi} \Phi(\zeta) d\zeta - \int_{z_0}^{z} \Phi(\zeta) d\zeta \right) = \frac{1}{\xi - z} \int_{z}^{\xi} \Phi(\zeta) d\zeta.
\]
Math 715 Homework 1 Solutions

Therefore

\[
\left| \frac{F(\xi) - F(z)}{\xi - z} - \Phi(z) \right| = \left| \frac{1}{\xi - z} \int_0^1 (\Phi(\gamma_3(t)) - \Phi(z)) \gamma_3'(t) dt \right| \\
\leq \int_0^1 |\Phi(\gamma_3(t)) - \Phi(z)| dt
\]

Given \( \varepsilon > 0 \), since \( \Phi \) is continuous, there exists \( \delta > 0 \) with \( \delta < \alpha \) such that \( |\xi - z| < \delta \) implies \( |\Phi(\xi) - \Phi(z)| < \varepsilon \). It follows that

\[
\left| \frac{F(\xi) - F(z)}{\xi - z} - \Phi(z) \right| \leq \varepsilon \quad \text{whenever} \quad |\xi - z| < \delta.
\]

Therefore the limit exists and \( F'(z) = \Phi(z) \) for every \( z \in \Omega \). It follows that \( F \) is analytic and consequently that \( \Phi \) is also analytic.