Math 715 Homework 1 Solutions

1. [Carrier, Krook and Pearson Section 2-1 Exercise 1] Show that no purely real function can be analytic, unless it is a constant.

Consider a function $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$ and where u and v are real functions. For f to be purely real means $v(x, y) = 0$. For f to be analytic means $u_x = v_y$ and $u_y = -v_x$ hold at every point in the complex plane. Therefore, if f is a purely-real analytic function, it follows that $u_x = 0$ and $u_y = 0$. Now $u_x = 0$ implies for y_0 fixed that the function $x \to u(x, y_0)$ is constant and similarly $u_y = 0$ implies for x_0 fixed that the function $y \to u(x_0, y)$ is constant.

Let $c = u(x_0, y_0)$. Consider any point x_1+iy_1 in the complex plane. Since $x \to u(x, y_0)$ is constant holding y_0 fixed, then $u(x_1, y_0) = u(x_0, y_0) = c$. Since $y \to u(x_1, y)$ is constant holding x_1 fixed, then $u(x_1, y_1) = u(x_1, y_0) = c$. It follows that u is identically equal to c throughout the entire complex plane. Therefore f is constant.

2. [Carrier, Krook and Pearson Section 2-2 Exercise 1] Evaluate

$$\int_{1+i}^{3-2i} \sin z\, dz$$

in two ways. First by choosing any path between the two end points and using real integrals as in

$$\int_C f(z)\, dz = \int_C (u\, dx - v\, dy) + i \int_C (u\, dy + v\, dx) \tag{2-5}$$

and second by use of an indefinite integral. Show that the inequality

$$\Big| \int_C f(z)\, dz \Big| \le \int_C \big| f(z) \big|\, |dz| \le ML \tag{2-6}$$

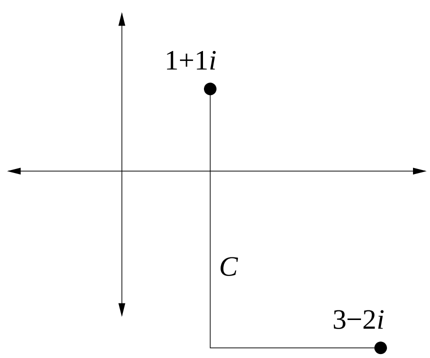
where $|f| \le M$ and L is the length of C is satisfied.

The trigonometric identities

$$\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$
$$\cos z = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$$

will be used in this exercise.

First, consider the path C given by



which may be written as the sum of the paths $[\gamma_1]$ and $[\gamma_2]$ where $\gamma_1(t) = 1 + i(1 - 3t)$ and $\gamma_2(t) = (1 + 2t) - 2i$. Since

$$\int_C \sin z\, dz = \int_C (\sin x \cosh y\, dx - \cos x \sinh y\, dy) + i \int_C (\sin x \cosh y\, dy + \cos x \sinh y\, dx),$$

compute the real path integrals

$$\int_{[\gamma_1]} (\sin x \cosh y\, dx - \cos x \sinh y\, dy) = -\int_1^{-2} \cos 1 \sinh y\, dy = -\cos 1(\cosh 2 - \cosh 1)$$

$$\int_{[\gamma_2]} (\sin x \cosh y\, dx - \cos x \sinh y\, dy) = \int_1^3 \sin x \cosh 2\, dx = (\cos 1 - \cos 3) \cosh 2$$

$$i\int_{[\gamma_1]}(\sin x\cosh y\,dy+\cos x\sinh y\,dx)=i\int_1^{-2}\sin 1\cosh y\,dy=-i\sin 1(\sinh 2+\sinh 1)$$

$$i\int_{[\gamma_2]}(\sin x\cosh y\,dy+\cos x\sinh y\,dx)=-i\int_1^{3}\cos x\sinh 2\,dx=i(\sin 1-\sin 3)\sinh 2$$

and add them to obtain

$$\int_C \sin z\,dz=\cos 1\cosh 1-\cos 3\cosh 2-i\big(\sin 1\sinh 1+\sin 3\sinh 2\big).$$

This finishes the computation using real path integrals.

Second, compute using an indefinite integral to obtain

$$\begin{aligned}\int_{1+i}^{3-2i}\sin z\,dz&=-\cos(3-2i)+\cos(1+i)\\&=-\cos 3\cosh 2-i\sin 3\sinh 2+\cos 1\cosh 1-i\sin 1\sinh 1.\end{aligned}$$

This answer is the same as the answer found using the path integrals.

We now show inequality (2-6) is satisfied. The term of the left hand side is

$$\begin{aligned}\Big|\int_C \sin z\,dz\Big|&=\big|\cos(1+i)-\cos(3-2i)\big|\\&\approx|4.558275530-1.500720276i|\approx 4.798962091.\end{aligned}$$

To approximate the integral

$$\int_C|\sin z||dz|$$

use numerical techniques. The Maple script

```
# Compute integral |f(z)||dz| along the path gamma1+gamma2
restart;
f:=z->sin(z);
gamma1:=t->1+I*(1-3*t);
gamma2:=t->(1+2*t)-2*I;
I1:=Integrate(abs(f(gamma1(t))*diff(gamma1(t),t)),t=0..1);
F1:=evalf(I1);
I2:=Integrate(abs(f(gamma2(t))*diff(gamma2(t),t)),t=0..1);
F2:=evalf(I2);
print("The integral |f(z)||dz| is approximately", F1+F2);
```

gives the output

```
f := z -> sin(z)

gamma1 := t -> 1 + (1 - 3 t) I

gamma2 := t -> 2 t + 1 - 2 I
```

```
                  1
                 /
                |
      I1 :=     |   3 | sin(1 + (1 - 3 t) I) | dt
                |
               /
                 0

                     F1 := 4.468016320

                   1
                  /
                 |
      I2 :=      |   2 | sin(2 t + 1 - 2 I) | dt
                 |
                /
                  0

                     F2 := 7.429875986

"The integral |f(z)||dz| is approximately", 11.89789231
```

Therefore

$$\int_C |\sin z||dz| \approx 11.89789231.$$

Finally we compute M using Maple

```
# Find the maximum value of |f(z)| on gamma1 and gamma2
restart;
_EnvAllSolutions:=true;

f:=z->sin(z);
g:=t->1+I*(1-3*t);
ff:=f(g(t))*conjugate(f(g(t))) assuming t::real;
dff:=diff(ff,t);
cp:=solve(dff=0,t);
k:=0;
for j from 1 to nops([cp])
do
    for i from -3 to 3
    do
        c:=evalf(subs(_Z1=i,cp[j]));
        if(abs(Im(c))<0.00001 and Re(c)<1 and Re(c)>0)
        then
            k:=k+1;
            cpn[k]:=abs(c);
        end
    end
```

```
end;
cps:=[0.0,seq(cpn[n],n=1..k),1.0];
v1:=seq(abs(f(g(cps[n]))),n=1..nops(cps));
print("The maximum of |f(z)| on gamma1 is",max(v1));

g:=t->(1+2*t)-2*I;
ff:=f(g(t))*conjugate(f(g(t))) assuming t::real;
dff:=diff(ff,t);
cp:=solve(dff=0,t);
k:=0;
for j from 1 to nops([cp])
do
    for i from -3 to 3
    do
        c:=evalf(subs(_Z2=i,cp[j]));
        if(abs(Im(c))<0.00001 and Re(c)<1 and Re(c)>0)
        then
            k:=k+1;
            cpn[k]:=abs(c);
        end
    end
end;
cps:=[0.0,seq(cpn[n],n=1..k),1.0];
v2:=seq(abs(f(g(cps[n]))),n=1..nops(cps));
print("The maximum of |f(z)| on gamma2 is",max(v2));

print("The maximum of |f(z)| on C is", max(v1,v2));
print("LM is",5*max(v1,v2));
```

with the results

```
                              _EnvAllSolutions := true

                                  f := z -> sin(z)

                             g := t -> 1 + (1 - 3 t) I

                  ff := sin(1 + (1 - 3 t) I) sin(1 - (1 - 3 t) I)

dff := -3 I cos(1 + (1 - 3 t) I) sin(1 - (1 - 3 t) I)

     + 3 I sin(1 + (1 - 3 t) I) cos(1 - (1 - 3 t) I)

               /                              2 1/2           \
               |               -1 + (1 + tan(2) )             |
cp := -1/3 I |1 + I - arctan(---------------------) - Pi _Z1~|,
               \                      tan(2)                  /

             /                                2 1/2           \
```

```
        |               1 + (1 + tan(2) )              |
-1/3 I |1 + I + arctan(--------------------) - Pi _Z1~|
        \                      tan(2)                  /

                            k := 0

                cps := [0., 0.3333333333, 1.0]

         v1 := 1.445396576, 0.8414709848, 3.723196185

       "The maximum of |f(z)| on gamma1 is", 3.723196185

                  g := t -> 2 t + 1 - 2 I

        ff := sin(2 t + 1 - 2 I) sin(2 t + 1 + 2 I)

dff := 2 cos(2 t + 1 - 2 I) sin(2 t + 1 + 2 I)

     + 2 sin(2 t + 1 - 2 I) cos(2 t + 1 + 2 I)

                                                 2 1/2
                                1 + (1 - tanh(4) )       Pi _Z2~
cp := -1/2 + I - 1/2 I arctanh(---------------------) + -------,
                                      tanh(4)              2

                                                2 1/2
                              -1 + (1 - tanh(4) )       Pi _Z2~
    -1/2 + I + 1/2 I arctanh(----------------------) + -------
                                     tanh(4)              2

                            k := 0

                cps := [0., 0.2853981635, 1.0]

         v2 := 3.723196185, 3.762195691, 3.629604837

       "The maximum of |f(z)| on gamma2 is", 3.762195691

          "The maximum of |f(z)| on C is", 3.762195691

                     "LM is", 18.81097846
```

Therefore $M \approx 3.762195691$ and $L = 3 + 2 = 5$ so that $LM \approx 18.81097846$. Since

$$4.798962091 \leq 11.89789231 \leq 18.81097846$$

we have shown that inequality (2-6) is satisfied for the curve C.

3. [Carrier, Krook and Pearson Section 2-2 Exercise 8a] Show that formal, term-by-term differentiation, or integration, of a power series yields a new power series with the same radius of convergence.

Term-by-term Differentiation. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R and $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ have radius of convergence r. Claim $R = r$.

First show $r \geq R$. Let z be such that $0 < |z| < R$. Choose w such that $|z| < |w| < R$. Then by (1-5) on page 9 we have that the series for $f(w)$ converges absolutely. Thus,

$$\sum_{n=0}^{\infty} |a_n||w|^n < \infty.$$

Now, let $\alpha = |z|/|w|$. Then $0 < \alpha < 1$ and so $n\alpha^n \to 0$ as $n \to \infty$. It follows that $n\alpha^n$ is bounded by some constant A so that $n\alpha^n \leq A$ for all n. Now,

$$\sum_{n=1}^{\infty} n|a_n||z|^{n-1} = \frac{1}{|z|}\sum_{n=1}^{\infty} n|a_n||w|^n\alpha^n \leq \frac{A}{|z|}\sum_{n=1}^{\infty} |a_n||w|^n < \infty$$

shows that the series for $g(z)$ converges for any $|z| < R$. It follows that $r \geq R$.

Second show that $R \leq r$. Suppose $|z| < r$. Then by (1-5) on page 9 we have that the series for $g(z)$ converges absolutely. Thus,

$$\sum_{n=1}^{\infty} n|a_n||z|^{n-1} < \infty.$$

Now,

$$\sum_{n=0}^{\infty} |a_n||z|^n = |a_0| + |z|\sum_{n=1}^{\infty} |a_n||z|^{n-1} \leq |a_0| + |z|\sum_{n=1}^{\infty} n|a_n||z|^{n-1} < \infty.$$

shows that the series for $f(z)$ converges for any $|z| < r$. It follows that $R \leq r$.

Term-by-term Integration. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R and $h(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} a_n z^{n+1}$ have radius of convergence ρ. Claim $R = \rho$.

First show that $R \leq \rho$. Suppose $|z| < R$. Then by (1-5) on page 9 we have that the series for $f(z)$ converges absolutely. Thus,

$$\sum_{n=0}^{\infty} |a_n||z|^n < \infty.$$

Now,

$$\sum_{n=0}^{\infty} \frac{1}{n+1}|a_n||z|^{n+1} \leq |z|\sum_{n=0}^{\infty} |a_n||z|^n < \infty.$$

shows that the series for $h(z)$ converges for any $|z| < R$. It follows that $R \leq \rho$.

Second show $R \geq \rho$. Let z be such that $0 < |z| < \rho$. Choose w such that $|z| < |w| < \rho$. Then by (1-5) on page 9 we have that the series for $h(w)$ converges absolutely. Thus,

$$\sum_{n=0}^{\infty} \frac{1}{n+1} |a_n||w|^{n+1} < \infty.$$

Now, let $\alpha = |z|/|w|$. Let A be the bound so that $n\alpha^n \leq A$ for all n. Now,

$$\sum_{n=0}^{\infty} |a_n||z|^n = \frac{1}{|z|} \sum_{n=0}^{\infty} \frac{1}{n+1} |a_n||w|^{n+1}(n+1)\alpha^{n+1} \leq \frac{A}{|z|} \sum_{n=0}^{\infty} \frac{1}{n+1} |a_n||w|^{n+1} < \infty$$

shows that the series for $f(z)$ converges for any $|z| < \rho$. It follows that $R \geq \rho$.

4. [Carrier, Krook and Pearson Section 2-2 Exercise 8b] The uniform-convergence property of a power series implies that term-by-term integration yields the integral of the sum function. Show that the integrated sum function is single valued and analytic within the circle of convergence.

Let R be the radius of convergence of the infinite series defining f and h in part a. Let

$$f_n(z) = \sum_{k=0}^{n} a_k z^k \qquad \text{and} \qquad h_n(z) = \sum_{k=0}^{n} \frac{1}{k+1} a_k z^{k+1}.$$

Suppose ξ and z are such that $|\xi| < R$ and $|z| < R$. Let $\Gamma = \{\, \gamma(t) : t \in [0,t] \,\}$ be any path such that $\gamma(0) = z$, $\gamma(1) = \xi$ and $|\gamma(t)| < R$ for $t \in [0,1]$. Since f_n is a polynomial, it is analytic. Therefore the path integral along Γ is path independent and since $h_n' = f_n$ we obtain

$$\int_\Gamma f_n(\zeta)\, d\zeta = \int_z^\xi f_n(\zeta)\, d\zeta = h_n(\xi) - h_n(z).$$

Since $|\gamma(t)| < R$ for $t \in [0,1]$ then there is $\eta > 0$ such that $|\gamma(t)| \le R - \eta$ for $t \in [0,1]$. From the results on page 9 we obtain that

$$f_n\big(\gamma(t)\big) \to f\big(\gamma(t)\big) \qquad \text{uniformly in } t \text{ as } n \to \infty.$$

Therefore,

$$\lim_{n\to\infty} \int_\Gamma f_n(\zeta)\, d\zeta = \lim_{n\to\infty} \int_0^1 f_n(\gamma(t))\gamma'(t)\, dt$$
$$= \int_0^1 f(\gamma(t))\gamma'(t)\, dt = \int_\Gamma f(\zeta)\, d\zeta.$$

It follows that

$$\int_\Gamma f(\zeta)\, d\zeta = \lim_{n\to\infty} \big(h_n(\xi) - h_n(z)\big) = h(\xi) - h(z).$$

Since this equality holds for any Γ inside the radius of convergence, the integral is path independent. Therefore, we may write

$$\int_z^\xi f(\zeta)\, d\zeta = h(\xi) - h(z)$$

for any z and ξ such that $|\xi| < R$ and $|z| < R$ where the integral is to be interpreted as a path integral along along any path from z to ξ that lies strictly inside the radius of convergence. Since h is single valued then the integrated sum function is single valued.

We now claim h is analytic and $h'(z) = f(z)$. Let $\epsilon > 0$. Since f is continuous at z there is $\delta > 0$ so that $|\zeta - z| < \delta$ implies $|f(\zeta) - f(z)| < \epsilon$. Define $\gamma(t) = \xi(1-t) + zt$ so that $\gamma'(t) = z - \xi$. Now, $|\xi - z| < \delta$ implies $|\gamma(t) - z| < \delta$ for $t \in [0,1]$ and therefore

$$\Big| \frac{h(\xi) - h(z)}{\xi - z} - f(z) \Big| = \Big| \int_\Gamma \frac{f(\zeta) - f(z)}{z - \xi}\, d\zeta \Big| = \Big| \int_0^1 \frac{f(\gamma(t)) - f(z)}{z - \xi} \gamma'(t)\, dt \Big|$$
$$\le \int_0^1 \big| f(\gamma(t)) - f(z) \big|\, dt < \int_0^1 \epsilon\, dt = \epsilon.$$

Consequently $h'(z) = f(z)$ for every $|z| < R$.

5. [Carrier, Krook and Pearson Section 2-2 Exercise 8c] Show that a power series converges to an analytic function within its circle of convergence.

Consider the power series for $f(z)$ defined above with radius of convergence equal to R. By part a the function $g(z)$ also has radius of convergence equal to R. Since $f(z)$ may be obtained from $g(z)$ through term by term integration of $g(z)$ we have by part b that $f'(z) = g(z)$ for every z such that $|z| < R$. Thus, f is differentiable and therefore analytic in its circle of convergence.

6. [Carrier, Krook and Pearson Section 2-3 Exercise 1] Use Cauchy's integral formula to evaluate the integral around the unit circle $|z| = 1$ of

$$\frac{\sin z}{2z+i}, \qquad \frac{\ln(z+2)}{z+2}, \qquad \frac{z^3 + \operatorname{asinh}(z/2)}{z^2+iz+i} \qquad \text{and} \qquad \cot z.$$

Let $D = \{\, z : |z| < 1 \,\}$ be the unit disk and $\Gamma = \partial D$ be its boundary oriented in the positive (counterclockwise) direction. Since $\sin z$ is analytic on D and $-i/z$ is contained within it, then Cauchy's formula implies

$$\int_\Gamma \frac{\sin z}{2z+i} = \frac{1}{2}\int_\Gamma \frac{\sin z}{z+i/2} = \pi i \sin(-i/2) = -\pi i \sin(i/2).$$

Since

$$\sin z = \frac{\exp(iz) - \exp(-iz)}{2i}$$

then

$$\sin(i/2) = \frac{\exp(-1/2) - \exp(1/2)}{2i} = i\frac{\exp(1/2) - \exp(-1/2)}{2} = i\sinh(1/2).$$

It follows that

$$\int_\Gamma \frac{\sin z}{2z+i} = \pi\sinh(1/2).$$

For the next integral note that, by definition, the principle branch $\ln z$ of the logarithm is the inverse of the function

$$z \to \exp(z) \colon \{\, x+iy : x \in \mathbf{R} \text{ and } y \in (-\pi,\pi] \,\} \to \mathbf{C}.$$

Therefore $\ln z$ is analytic on the domain $\mathbf{C} \setminus (-\infty, 0]$ and consequently $\ln z + 2$ is analytic on $\mathbf{C} \setminus (-\infty, -2]$. It follows that

$$\frac{\ln z + 2}{z+2}$$

is analytic in an open set containing $\bar{D}$. By Cauchy's theorem we have

$$\int_\Gamma \frac{\ln(z+2)}{z+2} = 0.$$

For the next integral note that

$$\sinh z = \frac{\exp(z) - \exp(-z)}{2}$$

Setting $\psi = \sinh z$ and $\omega = \exp z$ yields

$$2\omega\psi = \omega^2 - 1. \tag{$*$}$$

Completing the square and factoring yields

$$(\omega - \psi)^2 = \psi^2 + 1.$$

We would like to take square roots to solve for ω. Recall that the principle branch of the square root function is defined as

$$re^{i\theta} \to \sqrt{r}e^{i\theta/2} \qquad \text{for} \qquad \theta \in (-\pi, \pi].$$

This function is analytic on $\mathbf{C} \setminus (-\infty, 0]$. The only time $\psi^2 + 1 \in (-\infty, 0]$ is when

$$\psi \in i(-\infty, -1] \qquad \text{or} \qquad \psi \in i[1, \infty).$$

Therefore the map

$$\psi \to \psi + \sqrt{\psi^2 + 1}$$

is analytic on $\mathbf{C} \setminus \big(i(-\infty, -1] \cup i[1, \infty)\big)$.

From $(*)$ observe that if $\omega \le 0$ then $\psi \in \mathbf{R}$. However, $\psi \in \mathbf{R}$ implies $\omega = \psi + \sqrt{\psi^2 + 1} > 0$, which is a contradiction. Therefore, it can't happen that $\omega \le 0$. Consequently the function

$$\psi \to \log(\psi + \sqrt{\psi^2 + 1})$$

is analytic on $\mathbf{C} \setminus \big(i(-\infty, -1] \cup i[1, \infty)\big)$. As a result

$$\operatorname{asinh}(z) = \log(z + \sqrt{z^2 + 1})$$

is an inverse to $\sinh(z)$ which is analytic on $\mathbf{C} \setminus \big(i(-\infty, -1] \cup i[1, \infty)\big)$. In particular $\sinh(z/2)$ is analytic on a neighborhood of the unit disk D.

Note that $z^2 + iz + i = (z - z_1)(z - z_2)$ where

$$|z_1| = \left| \frac{-i + \sqrt{-1 - 4i}}{2} \right| > 1 \qquad \text{and} \qquad |z_2| = \left| \frac{-i - \sqrt{-1 - 4i}}{2} \right| < 1.$$

Therefore

$$\frac{z^3 + \operatorname{asinh}(z/2)}{z - z_1}$$

is analytic on D. It follows from Cauchy's formula that

$$\int_\Gamma \frac{z^3 + \operatorname{asinh}(z/2)}{z^2 + iz + i} dz = 2\pi i \frac{z_2^3 + \operatorname{asinh}(z_2/2)}{z_2 - z_1} \approx -0.04880 + 1.8762i.$$

For the final integral note that

$$\cot z = \frac{\cos z}{\sin z}$$

and that

$$\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} z^{2k+1} = z\Big(\sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} z^{2k} \Big) = z \operatorname{sinc} z,$$

where $\operatorname{sinc} z$ is defined by the absolutely convergent series in parenthesis shown above. Since the only zeros of $\sin z$ are $z = k\pi$ where $k \in \mathbf{Z}$ then the only zeros of $\operatorname{sinc} z$ are for $z = k\pi$ where $k \in \mathbf{Z} \setminus \{0\}$. It follows that $\frac{\cos z}{\operatorname{sinc} z}$ is analytic on D. Consequently

$$\int_\Gamma \cot z \, dz = 2\pi i \Big(\frac{\cos 0}{\operatorname{sinc} 0} \Big) = 2\pi i.$$

7. [Carrier, Krook and Pearson Section 2-3 Exercise 2] If $\Phi(z)$ is analytic in a simply connected region in which a closed contour C is drawn, obtain all possible values of

$$\int_C \frac{\Phi(\zeta)}{\zeta^2 - z^2}$$

where z is not a point on C itself.

Let $C = \partial\Omega$ be the boundary of an open set and $S = \bar{\Omega}$ be a set on which the Green's theorem holds. Consider separately two cases. The first where $z = 0$ and the second where $z \neq 0$. If $z = 0$, the possible values of the integral in question are given by

$$\int_C \frac{\Phi(\zeta)}{\zeta^2} = \begin{cases} 2\pi i \Phi'(0) & \text{for } 0 \in \Omega \\ 0 & \text{for } 0 \notin \Omega. \end{cases}$$

If $z \neq 0$, we note that $\zeta^2 - z^2 = (\zeta - z)(\zeta + z)$. Therefore, the possible values are

$$\int_C \frac{\Phi(\zeta)}{\zeta^2 - z^2} = \begin{cases} \pi i \Phi(z) z^{-1} - \pi i \Phi(-z) z^{-1} & \text{for } z \in \Omega \text{ and } -z \in \Omega \\ \pi i \Phi(z) z^{-1} & \text{for } z \in \Omega \text{ and } -z \notin \Omega \\ -\pi i \Phi(-z) z^{-1} & \text{for } z \notin \Omega \text{ and } -z \in \Omega \\ 0 & \text{for } z \notin \Omega \text{ and } -z \notin \Omega. \end{cases}$$

If one also considers closed contours C that do not necessarily bound a domain, then the contours may wind around the singularities multiple times. In this case we obtain the following possibilities

$$2\pi i k \Phi'(0) \qquad \text{and} \qquad \pi i k_1 \Phi(z) z^{-1} - \pi i k_2 \Phi(-z) z^{-1}$$

where $k, k_1, k_2 \in \mathbf{Z}$.

8. [Carrier, Krook and Pearson Section 2-3 Exercise 3] If n is an integer, positive or negative, and if C is a closed contour around the origin, use

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \qquad (2-14)$$

to show that

$$\int_C \frac{dz}{z^n} = 0 \qquad \text{unless} \qquad n = 1.$$

If $n \leq 0$ then $1/z^n$ is analytic. Therefore, Cauchy's theorem implies

$$\int_C \frac{dz}{z^n} = 0.$$

If $n > 1$, we write $f(z) = 1$ so that $f^{(n-1)}(z) = 0$. Then (2-14) implies

$$\int_C \frac{1}{z^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(0) = 0.$$

Finally, if $n = 1$ then

$$\int_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{it}} i e^{it} dt = \int_0^{2\pi} i dt = 2\pi i$$

which is non-zero.

9. [Carrier, Krook and Pearson Section 2-3 Exercise 6] Show, that if $\Phi(z)$ is any function continuous on C and if a function $f(z)$ is defined for z include C by

$$f(z) = \frac{1}{2\pi i} \int_C \frac{\Phi(\zeta)}{\zeta - z} d\zeta$$

then $f(z)$ is analytic inside C. Invent an example to illustrate the fact that, as an interior point z approaches a boundary point z_0, then $f(z)$ need not approach $\Phi(z_0)$.

To show that $f(z)$ is analytic, it is enough to show it has a derivative. By definition,

$$f'(z) = \lim_{\xi \to z} \frac{f(\xi) - f(z)}{\xi - z}.$$

Therefore, it remains to show that this limit exists. Now,

$$\begin{aligned}
\frac{f(\xi) - f(z)}{\xi - z} &= \frac{1}{\xi - z} \cdot \frac{1}{2\pi i} \int_C \Phi(\zeta)\Big(\frac{1}{\zeta - \xi} - \frac{1}{\zeta - z}\Big) d\zeta \\
&= \frac{1}{2\pi i} \int_C \Phi(\zeta)\Big(\frac{1}{(\zeta - \xi)(\zeta - z)}\Big) d\zeta \\
&= \frac{1}{2\pi i} \int_C \Phi(\zeta)\Big(\frac{1}{(\zeta - z)^2} + \frac{1}{(\zeta - \xi)(\zeta - z)} - \frac{1}{(\zeta - z)^2}\Big) d\zeta \\
&= \frac{1}{2\pi i} \int_C \Phi(\zeta)\Big(\frac{1}{(\zeta - z)^2} + \frac{\xi - z}{(\zeta - \xi)(\zeta - z)^2}\Big) d\zeta \\
&= \frac{1}{2\pi i} \int_C \frac{\Phi(\zeta)}{(\zeta - z)^2}\Big(1 + \frac{\xi - z}{\zeta - \xi}\Big) d\zeta.
\end{aligned}$$

It follows that

$$\Big|\frac{f(\xi) - f(z)}{\xi - z} - \frac{1}{2\pi i} \int_C \frac{\Phi(\zeta)}{(\zeta - z)^2} d\zeta\Big| \le \frac{1}{2\pi} \Big| \int_C \frac{\Phi(\zeta)}{(\zeta - z)^2} \Big(\frac{\xi - z}{\zeta - \xi}\Big) d\zeta \Big|$$

Now since $\Phi(\zeta)$ is continuous on C it's maximum exists and we may define

$$M = \max\big\{ |\Phi(\zeta)| : \zeta \in C \big\}.$$

Since $z \notin C$ then

$$\mu = \operatorname{dist}(z, C) = \min\big\{ |z - \zeta| : \zeta \in C \big\} > 0.$$

Finally, if ξ is close enough to z then also $\xi \notin C$. Consequently,

$$\nu = \operatorname{dist}(\xi, C) > 0.$$

By assumption $C = \big\{ \gamma(t) : t \in [0,1] \big\}$ where $\gamma(t)$ is a differentiable function such that

$$L = \int_0^1 \big|\gamma'(t)\big| dt < 0.$$

Therefore, follows that

$$\begin{aligned}\frac{1}{2\pi}\left|\int_C \frac{\Phi(\zeta)}{(\zeta-z)^2}\Big(\frac{\xi-z}{\zeta-\xi}\Big)d\zeta\right| &\le \frac{1}{2\pi}\int_0^1 \left|\frac{\Phi(\gamma(t))}{(\gamma(t)-z)^2}\right|\left|\frac{\xi-z}{\gamma(t)-\xi}\right|\big|\gamma'(t)\big|dt\\ &\le \frac{1}{2\pi}\frac{ML}{\mu^2\nu}|\xi-z|\to 0\end{aligned}$$

as $\xi\to z$. Therefore $f(z)$ is analytic at each point inside C.

For an example where $f(z)$ does not approach $\Phi(z_0)$ as the interior point z approaches a boundary point z_0, consider

$$\Phi(z)=\mathrm{Re}(z),\qquad z_0=1\qquad\text{and}\qquad C=\{\,\exp(it):t\in[-\pi,\pi]\,\}.$$

Now

$$\Phi(z)=\mathrm{Re}(z)\to 1\qquad\text{as}\qquad z\to 1$$

However,

$$\begin{aligned}\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-z}d\zeta &= \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{\cos t}{e^{it}-z}ie^{it}dt = \frac{1}{4\pi i}\int_{-\pi}^{\pi}\frac{e^{it}+e^{-it}}{e^{it}-z}ie^{it}dt\\ &= \frac{1}{4\pi i}\int_C \frac{\zeta+\zeta^{-1}}{\zeta-z}d\zeta = \frac{1}{4\pi i}\int_C \frac{\zeta^2+1}{(\zeta-z)\zeta}d\zeta\\ &= \frac{1}{2}\Big(\frac{z^2+1}{z}-\frac{1}{z}\Big)=\frac{z}{2}\to\frac{1}{2}\qquad\text{as}\qquad z\to 1.\end{aligned}$$

10. [Carrier, Krook and Pearson Section 2-3 Exercise 7] A function $\Phi(z)$ is known to be continuous throughout a simply connected region and to have the property that

$$\int_C \Phi(z)dz = 0$$

for any closed contour in that region. Show that $\Phi(z)$ must be analytic in that region.

Let Ω be the simple region mentioned above. For $z_0 \in \Omega$ fixed, define

$$F(\xi) = \int_\Gamma \Phi(\zeta)d\zeta \qquad \text{for} \qquad \xi \in \Omega$$

where $\Gamma = \{\, \gamma(t) : t \in [0,1] \,\}$ is any path that remains inside Ω and such that $\gamma(0) = z_0$ and $\gamma(1) = \xi$. Claim that $F(\xi)$ is independent of the path Γ. Consider two different paths $\Gamma_i = \{\, \gamma_i(t) : t \in [0,1] \,\}$ such that $\gamma_i(0) = z_0$ and $\gamma_i(1) = \xi$ for $i = 1, 2$. Let

$$C = \{\, c(t) : t \in [0,2] \,\} \qquad \text{where} \qquad c(t) = \begin{cases} \gamma_1(t) & \text{for } t \in [0,1] \\ \gamma_2(2-t) & \text{for } t \in (1,2]. \end{cases}$$

Then C is a closed contour and $c(t)$ is pathwise differentiable. It follows that

$$\begin{aligned} 0 = \int_C \Phi(\zeta)d\zeta &= \int_0^2 \Phi\big(c(t)\big)c'(t)dt \\ &= \int_0^1 \Phi\big(\gamma_1(t)\big)\gamma_1'(t)dt + \int_1^2 \Phi\big(\gamma_2(2-t)\big)\big(-\gamma_2'(2-t)\big)dt. \end{aligned}$$

Setting $s = 2 - t$ so that $ds = -dt$ yields

$$\int_1^2 \Phi\big(\gamma_2(2-t)\big)\big(-\gamma_2'(2-t)\big)dt = -\int_0^1 \Phi\big(\gamma_2(s)\big)\gamma_2'(s)ds.$$

Therefore

$$\int_{\Gamma_1} \Phi(\zeta)d\zeta = \int_{\Gamma_2} \Phi(\zeta)d\zeta,$$

which shows the integral defining F does not depend on the choice of path.

Claim F is differentiable on Ω. Let $z \in \Omega$ be fixed. Since Ω is open, there is $\alpha > 0$ such that $B_\alpha(z) \subseteq \Omega$. Therefore, for ξ such that $|\xi - z| < \alpha$, the straight line path

$$\Gamma_3 = \{\, \gamma_3(t) : t \in [0,1] \,\} \qquad \text{where} \qquad \gamma_3(t) = z + t(\xi - z)$$

lies entirely within Ω. It follows that

$$\frac{F(\xi) - F(z)}{\xi - z} = \frac{1}{\xi - z}\Big(\int_{z_0}^{\xi} \Phi(\zeta)d\zeta - \int_{z_0}^{z} \Phi(\zeta)d\zeta\Big) = \frac{1}{\xi - z}\int_z^\xi \Phi(\zeta)d\zeta.$$

Therefore

$$\Big|\frac{F(\xi)-F(z)}{\xi - z} - \Phi(z)\Big| = \Big|\frac{1}{\xi - z}\int_0^1 \big(\Phi(\gamma_3(t)) - \Phi(z)\big)\gamma_3'(t)dt\Big|$$
$$\leq \int_0^1 \big|\Phi(\gamma_3(t)) - \Phi(z))\big|dt$$

Given $\varepsilon > 0$, since Φ is continuous, there exists $\delta > 0$ with $\delta < \alpha$ such that $|\xi - z| < \delta$ implies $|\Phi(\xi) - \Phi(z)| < \varepsilon$. It follows that

$$\Big|\frac{F(\xi)-F(z)}{\xi - z} - \Phi(z)\Big| \leq \varepsilon \qquad \text{whenever} \qquad |\xi - z| < \delta.$$

Therefore the limit exists and $F'(z) = \Phi(z)$ for every $z \in \Omega$. It follows that F is analytic and consequently that Φ is also analytic.