Beau Smith

82.2: $I = \int_0^\infty \frac{x \sin x}{a^2+x^2}\, dx$.

$\Gamma_1 = \{Re^{it} : t \in [0,\pi]\}$

$\Gamma_2 = \{t : t \in [-R,R],\ R >> |a|\}$

$$I = \int_0^\infty \frac{x \sin x}{(x+ai)(x-ai)}\, dx = \int_\Gamma \frac{z \sin z \, dz}{(z+ai)(z-ai)} = \int_\Gamma \frac{z\left(\frac{e^{iz}-e^{-iz}}{2i}\right) dz}{(z+ai)(z-ai)}.$$

$$= \frac{1}{2i}\Bigg[\underbrace{\int_\Gamma \frac{z}{a^2+z^2} \cdot e^{iz}\, dz}_{A_1} + \underbrace{\int_\Gamma \frac{z}{a^2+z^2} \cdot e^{-iz}\, dz}_{A_2}\Bigg].$$ $g(z) = \frac{z}{a^2+z^2}$; poles at $z = \pm ai$.

- $A_1 = \int_{\Gamma_1} \# \, dz + \int_{\Gamma_2} \# \, dz$. $\left|\int_{\Gamma_2} \# dz\right| \le \int_0^\pi \left| ie^{iR\cos\theta - R\sin\theta} \cdot \frac{R^2 e^{2i\theta}}{a^2+R^2e^{2i\theta}}\, d\theta \right| \to 0$ as $R \to \infty$.

  Thus, $A_1 = \int_{\Gamma_1} \frac{z}{a^2+z^2} e^{iz}\, dz$. Singularity in the contour is ai for $a>0$.

  So $A_1 = 2\pi i f(ai) = 2\pi i \left[\frac{ai e^{-a}}{2ai}\right] = \boxed{\pi i e^{-a}}$.

  If $a<0$, the singularity is at $-ai$. So $A_1 = 2\pi i f(-ai) = 2\pi i \frac{(-ai)e^{a}}{-2ai} = \boxed{\pi i e^{a}}$.

- Now $A_2 = \int_{\Gamma_1} \# \, dz + \int_{\Gamma_2} \# \, dz$. $\left|\int_{\Gamma_2} \# dz\right| \to 0$ as $R \to \infty$.

  So $A_2 = \int \frac{z}{a^2+z^2} e^{-iz}\, dz$. Singularity if $a>0$ is ai, so

  $A_2 = 2\pi i f(ai) = \boxed{\pi i e^{a}}$. If $a<0$, $A_2 = 2\pi i f(-ai) = \boxed{\pi i e^{-a}}$.

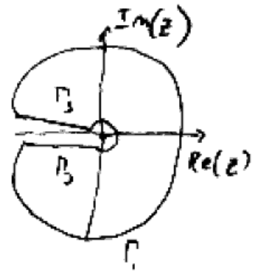
Jordan Blocher

Square of Logarithm

$$\int_0^\infty \frac{\log(x)}{(1+x^2)^2}\,dx$$

$$f(z) = \left(\frac{\log(z)}{1+z^2}\right)^2 \quad \text{w/ branch} \quad -\pi < \arg(z) \le \pi$$

w/ contour



$$\Rightarrow f(z) = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3}$$

By Jordan's Lemma, $\displaystyle\int_{\Gamma_1} \to 0$ as $R \to \infty$ since

$$\left|\int_R f(z)\,dz\right| \le 2\pi R \frac{(\log(R))^2 + \pi^2}{(R^2-1)^2} \to 0$$

Now, $z = -x + i\varepsilon$ on Γ_3, $z = -x - i\varepsilon$ on Γ_2, so

$$-i\pi^2 = \left(\int_{\Gamma_1} + \int_{\Gamma_2}\right) f(z)\,dz = -\int_\infty^0 \left(\frac{\log(-x+i\varepsilon)}{1+(-x+i\varepsilon)^2}\right)^2 dx$$

$$-\int_0^\infty \left(\frac{\log(-x-i\varepsilon)}{1+(-x-i\varepsilon)^2}\right)^2 dx = \int_0^\infty \left(\frac{\log(x)+i\pi}{1+x^2}\right)^2 dx - \int_0^\infty \left(\frac{\log(x)-i\pi}{1+x^2}\right)^2 dx$$

as $\varepsilon \to 0$

$$= \int_0^\infty \frac{(\log(x)+i\pi)^2 - (\log(x)-i\pi)^2}{(1+x^2)^2}\,dx = \int_0^\infty \frac{4\pi i \log(x)}{(1+x^2)^2}\,dx$$

$$= 4\pi i \int_0^\infty \frac{\log(x)}{(1+x^2)^2}\,dx = -\frac{\pi}{4}$$

for 715 Final.

Chen Chen

Type 3 Integrals

This class is similar to the previous one, but with a trigonometric function involved in the integrand:

$$I = \int_{-\infty}^{+\infty} \frac{\text{trig fn}}{\text{polynomial}} dx$$

In this case we have to take special care over the choice of the complex function, in other words the *continuation* of the trigonometric function away from the real axis. The three functions $\cos z$, (e^{iz}) and (e^{-iz}) all have the same real parts on the real axis, but are different elsewhere. In particular, when $z = iR$ and $R \to \infty$ the first and last become infinite, while the second tends to zero. Consequently the methods described for Type 2 integrals will work only if we adopt the second continuation.

Example:

$$I = \int_{-\infty}^{+\infty} \frac{\cos x dx}{x^2 + a^2}$$

For the reasons just described we find this by contour integration of

$$\frac{e^{iz}}{z^2 + a^2}$$

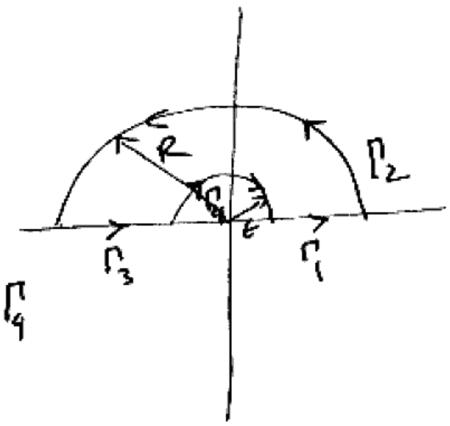
and since in polar coordinates $e^{iz} = e^{ir\cos\theta} e^{-r\sin\theta}$ the numerator tends to zero as r becomes large everywhere in the upper half-plane where $\sin\theta$ is positive. Using the same D-shaped contour as before, the semi-circular arc contributes

$$\int_{arc} \frac{e^{iz} dz}{z^2 + a^2} = \lim_{R \to \infty} \int_0^{+\pi} \frac{e^{iz} iRe^{i\theta} d\theta}{R^2 e^{2i\theta} + a^2} = 0$$

Syed

~~Residue th^n~~ $\boxed{\text{Contour Integral}}$

Show that, $$\int_0^\infty \frac{\sin x \, dx}{x} = \frac{\pi}{2}$$



Solution :

let us consider the integral

$$\oint_\Gamma \frac{e^{iz}}{z} dz \qquad ; \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$$

as the function is analytic in the contour.

$$\oint_\Gamma \frac{e^{iz}}{z} dz = 0$$

$$\Rightarrow \int_{\Gamma_3} \frac{e^{iz}}{z} dz + \int_{\Gamma_4} \frac{e^{iz}}{z} dz + \int_{\Gamma_1} \frac{e^{iz}}{z} dz + \int_{\Gamma_2} \frac{e^{iz}}{z} dz = 0$$

Hence, $$\int_{\Gamma_3} \frac{e^{iz}}{z} dz = \int_{-R}^{-\epsilon} \frac{e^{ix}}{x} dx = \int_\epsilon^R \frac{e^{-ix}}{x} dx$$

$$\int_{\Gamma_1} \frac{e^{iz}}{z} dz = \int_\epsilon^R \frac{e^{ix}}{x} dx$$

$$\therefore \int_\epsilon^R \frac{e^{ix} - e^{-ix}}{x} dx + \int_{\Gamma_4} \frac{e^{iz}}{z} dz + \int_{\Gamma_2} \frac{e^{iz}}{z} dz = 0$$

Syed

$$\Rightarrow 2i\int_{\epsilon}^{R}\frac{\sin x}{x}\,dx = -\int_{\Gamma_4}\frac{e^{iz}}{z}\,dz - \int_{\Gamma_2}\frac{e^{iz}}{z}\,dz$$

Now, $\displaystyle\int_{\Gamma_2}\frac{e^{iz}}{z}\,dz = \int_{\Gamma_2} e^{iz}\,g(z)\,dz \quad , \; g(z) = \frac{1}{z}$

$$\therefore \; G(R) = \frac{1}{R} \longrightarrow 0 \quad \text{as } R \longrightarrow \infty$$

$\therefore$ from Jordan's lemma,

$$\int_{\Gamma_2}\frac{e^{iz}}{z}\,dz \leq \left|\int_{\Gamma_2}\frac{e^{iz}}{z}\,dz\right| \to 0 \quad \text{as } R\to\infty$$

Now, let in Γ_4 ; $z = \epsilon e^{i\theta}$, in the limit,

$$-\lim_{\epsilon\to 0}\int_{\pi}^{0} \cancel{\epsilon t}\,\frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}}\, i\epsilon e^{i\theta}\,d\theta = -\lim_{\epsilon\to 0}\int_{\pi}^{0} i e^{i\epsilon e^{i\theta}}\,d\theta$$

$$= \int_{0}^{\pi} i\,d\theta = \pi i$$

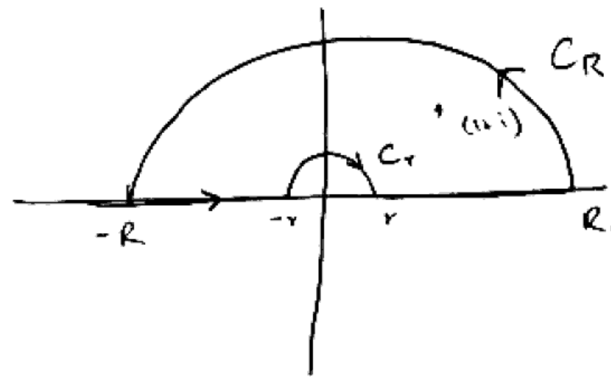
So, we have,

$$\lim_{\substack{R\to\infty\\ \epsilon\to 0}} 2i\int_{\epsilon}^{R}\frac{\sin x}{x}\,dx = \pi i$$

$$\therefore \boxed{\int_{0}^{\infty}\frac{\sin x}{x}\,dx = \frac{\pi}{2}}$$

Bishwash Shrestha

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)}\, dx$$



$$= \oint_C \frac{e^{iz}}{z(z^2-2z+2)}$$

It has simple pole @ $z=0$ & $z=1+i$ in the upper half plane.

From the fig above we can see that

$$\oint_C = \underbrace{\int_{C_R}}_{\substack{0 \\ \text{as} \\ R\to\infty}} + \int_{-R}^{-r} + \int_{C_r} + \int_r^R = 2\pi i \operatorname{Res}(f(z)e^{iz}, 1+i)$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2-2x+2)}\, dx - \pi i \operatorname{Res}(f(z)e^{iz}, 0) = 2\pi i \operatorname{Res}(f(z)e^{iz}, 1+i)$$

Here

$\operatorname{Res}(f(z)e^{iz}, 0) = \frac{1}{2}$ & $\operatorname{Res}(f(z)e^{iz}, 1+i) = -\frac{e^{-1+i}}{4}(1+i)$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2-2x+2)}\, dx = \pi i\left(\frac{1}{2}\right) + 2\pi i\left(-\frac{e^{-1+i}}{4}(1+i)\right)$$

(imaginary part) $\quad \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)}\, dx = \frac{\pi}{2} + e^{-1}(\sin 1 - \cos 1)$

Name: Suyesh Koyu

1) Contour Integral Problem

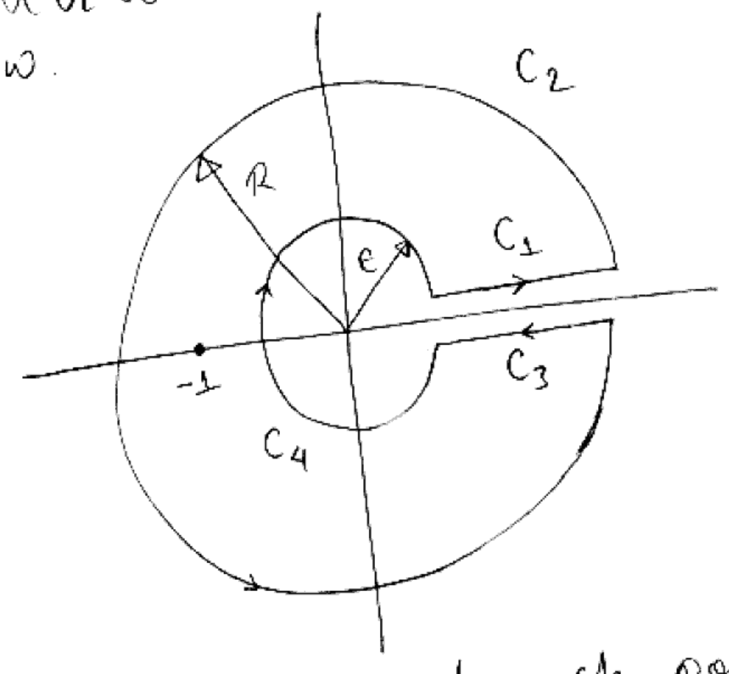
$$I = \int_0^\infty \frac{x^{P-1}}{1+x}\,dx \quad , \quad 0 < P < 1.$$

Solution:

Let us consider the integral

$$J = \oint_C \frac{z^{P-1}}{1+z}\,dz$$

where $C = C_1 + C_2 + C_3 + C_4$ is the contour in counter clockwise direction as shown in the figure below.



Here $z = 0$ is a branch point. And, there exists a simple pole at $z = -1$ inside C.

Now,

$$J = \int_{C_1} \frac{x^{P-1}}{1+x}\,dx + \int_{C_2} \frac{z^{P-1}}{1+z}\,dz + \int_{C_3} \frac{x^{P-1}}{1+x}\,dx + \int_{C_4} \frac{z^{P-1}}{1+z}\,dz$$

$$= \int_\epsilon^R \frac{x^{P-1}}{1+x}\,dx + \int_0^{2\pi} \frac{(Re^{i\theta})^{P-1}}{1+Re^{i\theta}}\, iRe^{i\theta}\,d\theta + \int_R^\epsilon \frac{(xe^{2\pi i})^{P-1}}{1+xe^{2\pi i}}\,dx + \int_{2\pi}^0 \frac{(\epsilon e^{i\theta})^{P-1}}{1+\epsilon e^{i\theta}}\, i\epsilon e^{i\theta}\,d\theta$$

$$= J_1 + J_2 + J_3 + J_4$$

Along C_2, we have $z = Re^{i\theta}$. Along C_3, the argument of z increases by 2π in going around the circle C_2. i.e. $z = xe^{2\pi i}$ along C_3. Similarly along C_4, $z = \epsilon e^{i\theta}$.

The second integral is

$$J_2 = \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1}}{1+Re^{i\theta}} iRe^{i\theta}\, d\theta$$

$$= \int_0^{2\pi} \frac{iR^p e^{ip\theta}}{1+Re^{i\theta}}\, d\theta$$

$$= \int_0^{2\pi} \frac{i \cdot \frac{1}{R^{1-p}} e^{ip\theta}}{\frac{1}{R} + e^{i\theta}}\, d\theta$$

$$\to 0 \quad \text{as } R \to \infty$$

The fourth integral is

$$J_4 = \int_{2\pi}^{0} \frac{(\epsilon e^{i\theta})^{p-1}}{1+\epsilon e^{i\theta}} i\epsilon e^{i\theta}\, d\theta$$

$$= \int_{2\pi}^{0} \frac{i\epsilon^p e^{ip\theta}}{1+\epsilon e^{i\theta}}\, d\theta$$

$$\to 0 \quad \text{as } \epsilon \to 0.$$

for $\epsilon \to 0$, $R \to \infty$, we get

$$J = \int_0^\infty \frac{x^{p-1}}{1+x}\,dx + 0 + \int_\infty^0 \frac{e^{2\pi i(p-1)}}{1+x e^{2\pi i}}\,dx + 0$$

$$= \int_0^\infty \frac{x^{p-1}}{1+x}\,dx - \int_0^\infty \frac{e^{2\pi i(p-1)}}{1+x}\,dx$$

i.e. $J = \left(1 - e^{2\pi i(p-1)}\right) \int_0^\infty \frac{x^{p-1}}{1+x}\,dx$ ——— (1)

From cauchy Residue theorem, we have

$J = 2\pi i \sum \operatorname{Res}\left[f(z), -1\right]$ where $f(z) = \frac{z^{p-1}}{1+z}$

$$= 2\pi i \lim_{z \to -1} (z+1)\, f(z)$$

$$= 2\pi i \lim_{z \to -1} \cancel{(z+1)} \frac{z^{p-1}}{\cancel{(1+z)}}$$

$$= 2\pi i\, (-1)^{p-1}$$

$$= 2\pi i\, \left(e^{i\pi}\right)^{p-1}$$

i.e. $J = 2\pi i\, e^{i\pi(p-1)}$ ——— (2)

From Eq^{ns} (1) & (2), we get

$$\left(1 - e^{2\pi i(p-1)}\right) \int_0^\infty \frac{x^{p-1}}{1+x}\,dx = 2\pi i\, e^{i\pi(p-1)}$$

or, $$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{2\pi i\, e^{i\pi(p-1)}}{\left(1 - e^{2\pi i(p-1)}\right)}$$

or, $$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{2\pi i\ \cancel{e^{i\pi(p-1)}}}{\cancel{e^{i\pi(p-1)}}\left(e^{-i\pi(p-1)} - e^{i\pi(p-1)}\right)}$$

$$= \frac{2\pi i}{\left(e^{-ip\pi}e^{i\pi} - e^{ip\pi}e^{-i\pi}\right)}$$

$$= \frac{2\pi i}{\left(-e^{-ip\pi} + e^{ip\pi}\right)}$$

$$= \frac{\cancel{2}\pi \cancel{i}}{\cancel{2i}\sin p\pi}$$

$$\therefore \int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}.$$

QED

Beau Smith

Find a conformal map of the disk $|z|<1$ onto the right half-plane $Re(\zeta)>0$.

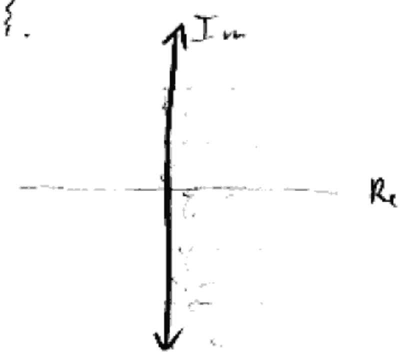
<u>Ans.:</u> Since $|z|<1$, we look for a map that puts $|z|=1$ onto the imaginary axis $=\{z\in\mathbb{C}: Re(z)=0\}$. So there must be a singularity on $|z|=1$, not at the origin.

- We can start with $\zeta=f(z)=\dfrac{z+1}{z-1}$. At $z=1$, ζ is undefined;

  if $z=0$, $\zeta=-1$, and if $z=-1$, $\zeta=0$.

- Now if $z=i$, $\zeta=\dfrac{i+1}{i-1}\cdot\dfrac{-i-1}{-i-1}=\dfrac{-i^2-2i-1}{-i^2+1}=\dfrac{-2i}{2}=-i$.

- We see from the points $z=-1$, $z=i$ that the straight line — the image of ζ — must be $\{z\in\mathbb{C}: Re(z)=0\}$.



- Note that if $z=0$, $\zeta=-1$, so right now this doesn't map to $Re(z)>0$, but to $Re(z)<0$.

- So if we set $\zeta=-\left(\dfrac{z+1}{z-1}\right)=\dfrac{1+z}{1-z}$,

  we now map to $Re(z)>0$.

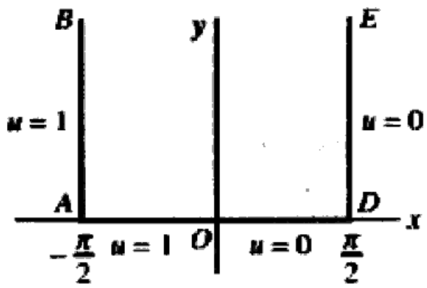
Our conformal map is $\boxed{\zeta=\dfrac{1+z}{1-z}}$.

EXAMPLE 6 Solving a Dirichlet Problem

The function $U(u, v) = (1/\pi)\ \text{Arg}\ w$ is harmonic in the upper half-plane $v > 0$ since it is the imaginary part of the analytic function $g(w) = (1/\pi)\ \text{Ln}\ w$. Use this function to solve the Dirichlet problem in **FIGURE 20.2.5(a)**.

Solution The analytic function $f(z) = \sin z$ maps the original region to the upper half-plane $v \geq 0$ and maps the boundary segments to the segments shown in Figure 20.2.5(b). The harmonic function $U(u, v) = (1/\pi)\ \text{Arg}\ w$ satisfies the transferred boundary conditions $U(u, 0) = 0$ for $u > 0$ and $U(u, 0) = 1$ for $u < 0$. Therefore, $u(x, y) = U(\sin z) = (1/\pi)\ \text{Arg}(\sin z)$ is the solution to the original problem. If $\tan^{-1}(v/u)$ is chosen to lie between 0 and π, the solution can also be written as

$$u(x, y) = \frac{1}{\pi} \tan^{-1}\left(\frac{\cos x \sinh y}{\sin x \cosh y}\right). \qquad \equiv$$



(a)

Conformal Mapping

Syed

Show thatb the function $x^2 - y^2 + 2y$ is harmonic in the w-plane under the transformation $z = w^3$

Solution:

If $z = w^3$ then,

$$x + iy = (u+iv)^3 = u^3 - 3uv^2 + i(3u^2v - v^3)$$

$\therefore$ equating,

$$x = u^3 - 3uv^2$$
$$y = 3u^2v - v^3$$

Now, $\Phi = x^2 - y^2 + 2y = (u^3 - 3uv^2)^2 - (3u^2v - v^3)^2 + 2(u^2v - v^3)$

$$= u^6 - 15u^4v^2 + 15u^2v^4 - v^6 + 6u^2v - 2v^3$$

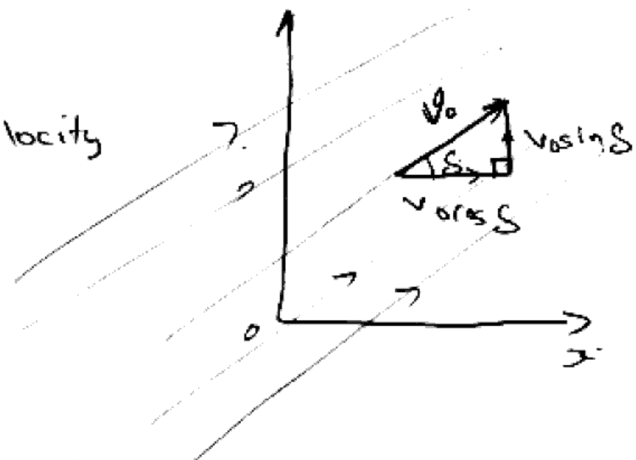
then, $\dfrac{\partial^2 \Phi}{\partial u^2} = 30u^4 - 180u^2v^2 + 30v^4 + 12v$

and, $\dfrac{\partial^2 \Phi}{\partial v^2} = -30u^4 + 180u^2v^2 - 30v^4 - 12v$

$\therefore \dfrac{\partial^2 \Phi}{\partial u^2} + \dfrac{\partial^2 \Phi}{\partial v^2} = 0$ — Showed —

Find the complex potential for a fluid moving with constant speed V_0 in a direction making an angle δ with the positive x-axis. Determine the velocity potential & stream function. Find the eq^n for the streamlines & equipotential lines.

Sol^n



The x & y component of velocity are

$v_x = V_0 \cos\delta$

$v_y = v_0 \sin\delta$

The complex velocity is

$$V = u_x + i v_y = V_0 \cos\delta + i V_0 \sin\delta = V_0 e^{i\delta}$$

The complex potential $\Omega(z)$ can be calculated as.

$$\frac{d\Omega(z)}{dz} = \bar{V} = V_0 e^{-i\delta}$$

Integrating

$$\Omega(z_0) = V_0 e^{-i\delta} z.$$

The complex potential can be expressed as

$$\Omega(z) = \phi + i\psi$$

where ϕ is velocity potential.

ψ is stream function.

or $\phi + i\psi = v_0 e^{-i\delta} z$

$= V_0 e^{-i\delta}(x + iy)$

$= v_0(\cos\delta - i\sin\delta)(x + iy.$

$= v_0 (x\cos\delta - ix\sin\delta + iy\cos\delta + y\sin\delta$

$= V_0\cos\delta x + V_0\sin\delta y + i\{ V_0\cos\delta\, y - V_0\sin\delta x$

comparing real & imaginary part we get

$\phi = V_0 (x\cos\delta + y\sin\delta)$

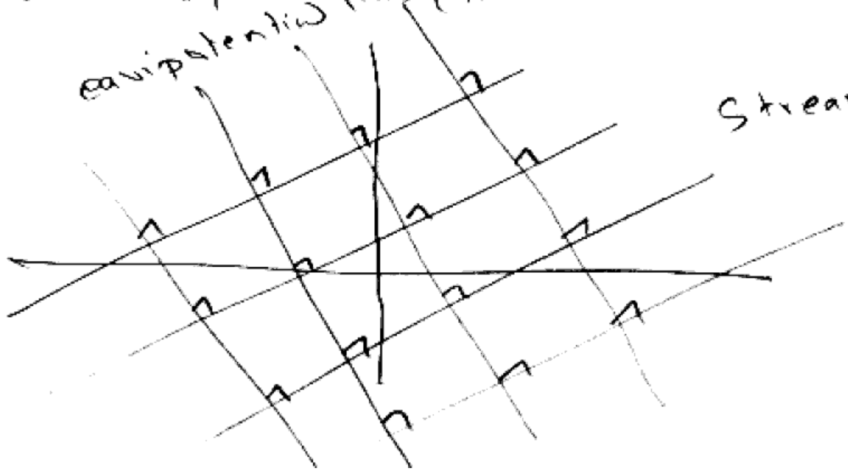
$\psi = V_0 (y\cos\delta - x\sin\delta)$

The stream ~~function~~ lines are given by stream function.

$\psi = V_0 (y\cos\delta - x\sin\delta) = B$ for diff. values of B.

Physically, a streamline represent the path followed by fluid particle at steady state condition. In our problem it is a straight line.

The equipotential lines are given by the velocity potential $\phi = V_0 (x\cos\delta + y\sin\delta) = \alpha$ for diff value of α. The equipotential lines are perpendicular to the stream lines geometrically. However all points on an equipotential lines are at equal potential (ϕ).



Name: Snyesh Koyu

2) Laplace Equation from conformal mapping.

Find a function harmonic in the upper half of the z-plane, $Im\{z\} > 0$, which takes the values on the x-axis given by $G(x) = \begin{cases} \phi_1 & x > 0 \\ \phi_0 & x < 0 \end{cases}$

Solution:

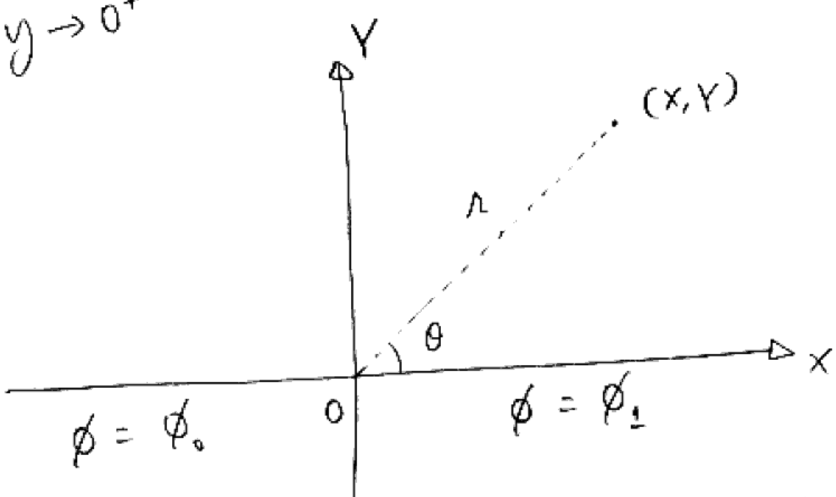
Let $\phi(x, y)$ be the required function. For $\phi(x, y)$ to be harmonic, it must satisfy Laplace's equation.

Mathematically, we have

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \ , \ y > 0$$

And,

$$\lim_{y \to 0^+} \phi(x, y) = G(x) = \begin{cases} \phi_1 & x > 0 \\ \phi_0 & x < 0 \end{cases}$$



The function $\phi = A\theta + B$ where A and B are constants is harmonic. This is because it is the imaginary part of $(A \ln z + B)$.

The boundary conditions are

$$\phi = \phi_1 \text{ for } X > 0$$

i.e. $\theta = 0$

So, $\phi_1 = A \cdot 0 + B$

$\therefore B = \phi_1$

And, $\phi = 0$ for $X < 0$

i.e. $\theta = \pi$

So, $\phi_0 = A \cdot \pi + B$

or, $\phi_0 = A \cdot \pi + \phi_1$

i.e. $A = \frac{1}{\pi}(\phi_0 - \phi_1)$

The required solution is

$$\phi(X,Y) = A\theta + B$$

$$= \frac{1}{\pi}(\phi_0 - \phi_1)\,\theta + \phi_1$$

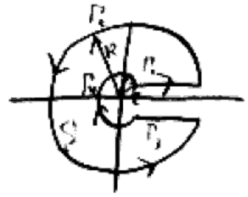
$$\therefore \phi(X,Y) = \phi_1 - \frac{1}{\pi}(\phi_1 - \phi_0) \cdot \tan^{-1}\left(\frac{Y}{X}\right).$$

Q E D

Benn Smith

Extra credit:

Calculate $\int_0^\infty \frac{1}{x^2+3x+2}\,dx = I$.



$\Gamma_2 = \{Re^{it} : t \in [0, 2\pi]\}$;
$\gamma_2'(t) = iRe^{it}$
$\Gamma_4 = \{\varepsilon e^{-it} : t \in [0, 2\pi]\}$;
$\gamma_4'(t) = -i\varepsilon e^{-it}$,

Ans.: Use the contour as follows

$\varepsilon \ll 1$, $R \gg 1$.

Note that Γ_1, Γ_3 differ by $2\pi i$, so

$$\int_{\Gamma_1} \frac{\ln z}{z^2+3z+2}\,dz = \int_\varepsilon^R \frac{\ln_1(x)\,dx}{x^2+3x+2}$$

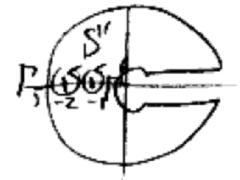
$$+ \int_{\Gamma_3} \# \,dz = -\int_\varepsilon^R \frac{\ln(x)+2\pi i}{x^2+3x+2}\,dx$$

$$\int_{\Gamma_1 \cup \Gamma_3} \# \,dz = -2\pi i \int_\varepsilon^R \frac{dx}{x^2+3x+2}.$$

Send $\varepsilon \to 0$ and $R \to \infty$. Then

- $\int_{\Gamma_2} \frac{\ln_1(z)}{z^2+3z+2}\,dz = \int_0^{2\pi} \frac{\ln_1(Re^{it})\, iRe^{it}}{R^2e^{2it}+3Re^{it}+2}\,dt \to 0$ as $R \to \infty$;
- $\int_{\Gamma_4} \# \,dz = \int_0^{2\pi} \frac{\ln_1(\varepsilon e^{-it})(-i\varepsilon e^{-it})}{\varepsilon^2 e^{-2it}+3\varepsilon e^{-it}+2}\,dt \to 0$ as $\varepsilon \to 0$.

Thus, $I = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{\bigcup_{i=1}^4 \Gamma_i} \frac{\ln_1 z}{z^2+3z+2}\,dz$.



Note that $z^2+3z+2 = 0$ when $z = -1, -2$. Since these lie in S'',

$$\int_{\Gamma_1} \# \,dz = -\int_{\Gamma_5} \# \,dz - \int_{\Gamma_6} \# \,dz = 2\pi i \left[\frac{1}{2\pi i}\int_{\partial\Gamma_5} \frac{\ln_1(z)}{(z-(-2))(z+2)}\,dz + \frac{1}{2\pi i}\int_{\partial\Gamma_6} \frac{\ln_1(z)}{(z+2)(z-(-1))}\,dz\right]$$

$= -2\pi i \left[\frac{\ln 2 + i\pi}{-1} + \frac{i\pi}{1}\right] = -2\pi i \ln 2$. Since $-2\pi i$ cancels with the $2\pi i$ in $\int_{\Gamma_3} \# \,dz$,

$$\int_0^\infty \frac{dx}{x^2+3x+2} = \boxed{\ln 2}.$$

Evaluate $\oint_c \tan z \, dz$ whr $|z| = 2$

The integrand $\tan z = \sin z / \cos z$ has simple poles at points whr $\cos z = 0$ we ~~[illegible]~~ know that only zeros for $\cos z$ are the real number $z = (2n+1)\pi/2$ $n = 0, \pm 1, \pm 2$ since only $-\pi/2$ & $\pi/2$ are within circle $|z| = 2$ we have

$$\oint_c \tan z = 2\pi i \left(\operatorname{Res}(f(z), -\pi/2) + \operatorname{Res}(f(z), \pi/2) \right]$$

~~Now from~~ we know.

$$\operatorname{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}$$

so

$$\operatorname{Res}(f(z), -\tfrac{\pi}{2}) = \frac{\sin(-\pi/2)}{-\sin(\pi/2)} = -1, \quad \operatorname{Res}(f(z), \pi/2) = \frac{\sin \pi/2}{-\sin \pi/2}$$

$$\therefore \oint_c \tan z \, dz = 2\pi i(-1-1) = -4\pi i \text{ A.}$$