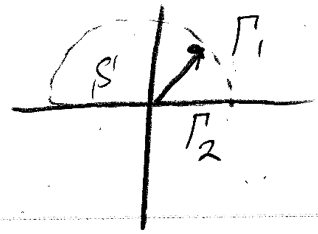

82.2 $I=\int_0^\infty \frac{x\sin x}{a^2+x^2}\,dx = \int_\Gamma \frac{z\sin z}{a^2+z^2}\,dz$

$$=\frac{1}{2i}\int_\Gamma \frac{z(e^{iz}-e^{-iz})}{(z+ai)(z-ai)}\,dz = \frac{1}{2i}\left[\int_{\Gamma'} \frac{ze^{iz}\,dz}{(z+ai)(z-ai)} + \int_{\Gamma'}\frac{ze^{-iz}\,dz}{(z+ai)(z-ai)}\right].$$

• Re$(a)>0$; the pole is at $z=ai$. Then

$I=\frac{1}{2i}\int_{\Gamma_1}\frac{ze^{iz}}{z^2+a^2}\,dz+\frac{1}{2i}\int_{\Gamma_2}\frac{ze^{iz}}{z^2+a^2}\,dz$. If $g(z)=\frac{z}{z^2+a^2}$, then $\int_{\Gamma_1} \# \,dz \to 0$ as $R\to\infty$ (by Jordan's Lemma).

So $I=\frac{1}{2i}\int_{\Gamma_2} \# \,dz = \frac{1}{2i}\cdot 2\pi i\cdot\frac{ze^{iz}}{z+ai} = \frac{2\pi i}{2i}\cdot\frac{aie^{-a}}{2ai} = \boxed{\frac{\pi e^{-a}}{2}}$

and $\int_{\Gamma}\frac{ze^{-iz}}{z^2+a^2}\,dz = 0$ since $-ai\notin S$.

• Re$(a)<0$; the pole is at $-ai$, and

$\frac{1}{2i}\int_{\Gamma_1}\frac{ze^{-iz}}{z^2+a^2}\,dz\to 0$ by Jordan's Lemma, and thus

$I=\frac{1}{2i}\int_{\Gamma_2}\frac{ze^{-iz}}{z^2+a^2}\,dz = \frac{2\pi i}{2i}f(-ai) = -\frac{\pi aie^{a}}{-2ai} = \boxed{\frac{-\pi e^{a}}{2}}.$

(a) $\int_0^\infty \frac{\log(x)}{(1+x^2)^2}\,dx$ using $f(z) = \left(\frac{\log z}{1+z^2}\right)^2$ w/ branch-cut

& contour 

We have, $\int_{C_R} f(z) \to 0$ as $R\to\infty$ & $\int_{C_\epsilon} f(z) \to 0$ as $\epsilon \to 0$

Using Cauchy formula, $\int_C f = 2\pi i\left(\text{Res}_{z=i} f(z) + \text{Res}_{z=-i} f(z)\right)$

since $\int_C \frac{(\ln z)^2}{(z-i)(z+i)}\,dz = \int_{B_\epsilon(i)} f + \int_{B_\epsilon(-i)} f$

$$= \frac{1}{2\pi i}\int \frac{(\ln z)^2/(z+i)^2}{(z-i)^2} + \frac{1}{2\pi i}\int \frac{(\ln z)^2/(z-i)^2}{(z+i)^2}$$

Now, with Residue theorem

$$\int_C f = 2\pi i\left[\frac{d}{dz}\frac{(\ln z)^2}{(z+i)^2}\Big|_{z=i} + \frac{d}{dz}\frac{(\ln z)^2}{(z-i)^2}\Big|_{z=-i}\right]$$ by Cauchy deriv. formula.

$$= 2\pi i\left[\frac{(z+i)^2\frac{2\ln z}{z} - (\ln z)^2 2(z+i)}{(z+i)^4}\Big|_{z=i} + \frac{(z-i)^2\frac{2\ln z}{z} - (\ln z)^2 2(z-i)}{(z-i)^4}\Big|_{z=-i}\right]$$

$$= 2\pi i\left(\frac{(2i)^2\frac{2\pi i}{2i} - \frac{\pi^2}{4}4i}{(2i)^4} + \frac{(-2i)^2\frac{2\pi i}{-2i} - \frac{\pi^2}{4}(-4i)}{(-2i)^4}\right)$$

$$= 2\pi i\left(\frac{(2i)^2\cdot\pi}{(2i)^4 2} + \frac{\pi^2 i}{(2i)^4} + \frac{(-2i)^2\pi}{(-2i)^4 2} + \frac{\pi^2 i}{(-2i)^4}\right) = 2\pi i\left(\frac{-\pi}{4} + \frac{\pi^2 i}{16} -\right.$$

$= -\pi i$

Now, $\quad -\pi i = -\int_{-\infty}^{0} \frac{\log(-x+\varepsilon i)^2}{1+(-x+\varepsilon i)^2}\,dx + \int_{0}^{\infty} \frac{\log(-x-\varepsilon i)^2}{1+(-x-\varepsilon i)^2}\,dx$

contour $\nearrow$
clockwise

Take $\varepsilon \to 0$

$$\Rightarrow \int_0^\infty \left(\frac{\log(x)+i\pi}{1+x^2}\right)^2 dx + \int_0^\infty \left(\frac{\log(x)-i\pi}{1+x^2}\right)^2 dx$$

$$= \int_0^\infty \left(\frac{(\log(x)+i\pi)^2-(\log(x)-i\pi)^2}{(1+x^2)^2}\right) dx = 4\pi i \int_0^\infty \frac{\log(x)}{(1+x^2)^2}\,dx$$

$$\therefore \int_0^\infty \frac{\log(x)}{(1+x^2)^2}\,dx = \frac{-i\pi^2}{4\pi i} = -\frac{\pi}{4}$$