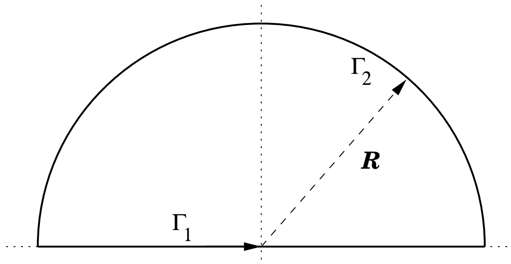
1. [Carrier, Krook and Pearson, Section 3-1 problem 1] Using the contour



show that if a, b and c are real with $b^2 < 4ac$, then

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2+bx+c} = \frac{2\pi}{\sqrt{4ac-b^2}}.$$

Let r_1 and r_2 be the roots of $ax^2+bx+c=0$. By hypothesis these roots are complex conjugates of each other with

$$r_1 = \frac{-b+i\sqrt{4ac-b^2}}{2a} \qquad \text{and} \qquad r_2 = \frac{-b-i\sqrt{4ac-b^2}}{2a}.$$

Setting $C = \Gamma_1 + \Gamma_2$ we obtain

$$\int_{-R}^{R} \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int_C \frac{dz}{(z-r_1)(z-r_2)} - \int_{\Gamma_2} \frac{dz}{az^2+bz+c}$$

First note that

$$\left|\int_{\Gamma_2} \frac{dz}{az^2+bz+c}\right| \le \int_0^\pi \left|\frac{iRe^{it}}{aR^2e^{2it}+bRe^{it}+c}\right| dt \le \int_0^\pi \frac{R}{|a|R^2-|b|R-|c|}dt$$
$$= \frac{\pi R}{|a|R^2-|b|R-|c|} \to 0 \qquad \text{as} \qquad R\to\infty.$$

Now, when $a>0$ and $R>|r_1|$ then r_1 is enclosed by the contour C and r_2 is outside it. Cauchy's formula yields

$$\frac{1}{a}\int_C \frac{dz}{(z-r_1)(z-r_2)} = \frac{2\pi i}{a}\frac{1}{r_1-r_2} = \frac{2\pi}{\sqrt{4ac-b^2}}.$$

Similarly, when $a<0$ and $R>|r_2|$ then r_2 is enclosed by the contour C and r_1 is outside it. Cauchy's formula yields

$$\frac{1}{a}\int_C \frac{dz}{(z-r_1)(z-r_2)} = \frac{2\pi i}{a}\frac{1}{r_2-r_1} = \frac{-2\pi}{\sqrt{4ac-b^2}}.$$

It follows that

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{2\pi}{\sqrt{4ac-b^2}} & \text{for } a>0 \\ \dfrac{-2\pi}{\sqrt{4ac-b^2}} & \text{for } a<0. \end{cases}$$

Note that the formula given in the text is only correct when $a>0$.

2. [Carrier, Krook and Pearson, Section 3-1 problem 2] Evaluate the integral

$$I = \int_0^\infty \frac{x \sin x}{a^2 + x^2} dx \qquad \text{where} \qquad a \in \mathbf{C}$$

using the contour in the previous problem and Jordan's lemma where needed.

We first consider the case $a > 0$. Since the integrand is even, then

$$\begin{aligned} I &= \frac{1}{2}\int_{-\infty}^\infty \frac{x \sin x}{a^2 + x^2} dx = \operatorname{Im}\left(\frac{1}{2}\int_{-\infty}^\infty \frac{x e^{ix}}{a^2 + x^2} dx\right) \\ &= \operatorname{Im}\left(\frac{1}{2}\int_C \frac{z e^{iz}}{a^2 + z^2} dz - \frac{1}{2}\int_{\Gamma_2} \frac{z e^{iz}}{a^2 + z^2} dz\right) \end{aligned}$$

Since $\sin t \ge 2t/\pi$ for $t \in [0, \pi/2]$, by Jordan's lemma we obtain

$$\begin{aligned} \left| \int_{\Gamma_2} \frac{z e^{iz}}{a^2 + z^2} dz \right| &\le \int_0^\pi \left| \frac{Re^{it} e^{iR(\cos t + i \sin t)}}{a^2 + R^2 e^{i2t}} iRe^{it} \right| dt \le \int_0^\pi \frac{R^2 e^{-R \sin t}}{R^2 - a^2} dt \\ &= \frac{2R^2}{R^2 - a^2} \int_0^{\pi/2} e^{-R\sin t} dt \le \frac{2R^2}{R^2 - a^2} \int_0^{\pi/2} e^{-2Rt/\pi} dt \\ &= \frac{2R^2}{R^2 - a^2} \frac{\pi}{2R}(1 - e^{-2R}) \to 0 \qquad \text{as} \qquad R \to \infty. \end{aligned}$$

Now, provided $R > a$ we obtain

$$\int_C \frac{z e^{iz}}{a^2 + z^2} dz = \int_C \frac{z e^{iz}}{(z - ia)(z + ia)} dz = 2\pi i \frac{iae^{-a}}{2ia} = \pi i e^{-a}.$$

Consequently,

$$I = \frac{1}{2} \operatorname{Im}\left(\pi i e^{-a}\right) = \frac{\pi}{2} e^{-a}.$$

In the general case write $a = \alpha + i\beta$. Since $a^2 = (-a)^2$ we may assume without loss of generality that $\alpha \ge 0$. Thus, setting $\xi = -x$ we obtain

$$\begin{aligned} I &= \frac{1}{4i}\int_{-\infty}^\infty \frac{x(e^{ix} - e^{-ix})}{a^2 + x^2} dx = \frac{1}{4i}\int_{-\infty}^\infty \frac{x e^{ix}}{a^2 + x^2} dx - \frac{1}{4i}\int_{-\infty}^\infty \frac{x e^{-ix}}{a^2 + x^2} dx \\ &= \frac{1}{4i}\int_{-\infty}^\infty \frac{x e^{ix}}{a^2 + x^2} dx + \frac{1}{4i}\int_{\infty}^{-\infty} \frac{(-\xi) e^{i\xi}}{a^2 + (-\xi)^2} d\xi = \frac{1}{2i}\int_{-\infty}^\infty \frac{x e^{ix}}{a^2 + x^2} dx. \end{aligned}$$

When $\alpha > 0$ the analysis of this integral is exactly the same as above. Consequently

$$I = \frac{1}{2i}\left(\pi i e^{-\alpha - i\beta}\right) = \frac{1}{2i}\left(\pi i e^{-\alpha}(\cos\beta - i \sin\beta)\right) = \frac{\pi}{2} e^{-\alpha} \cos\beta.$$

When $\alpha = 0$ and $\beta \neq 0$ then $ia = -\beta$ and the singularities occur along the real axis and the integral must be interpreted as the Cauchy principle value integral. In particular, as the contour along the real axis is differentiable, equation (2-12) in the text yields that

$$\mathrm{PV}\int_C \frac{ze^{iz}}{(z+\beta)(z-\beta)}dz = \pi i\Big(\frac{\beta e^{i\beta}}{2\beta} + \frac{\beta e^{-i\beta}}{2\beta}\Big) = \pi i \cos\beta.$$

Consequently

$$I = \frac{1}{2i}\big(\pi i \cos\beta\big) = \frac{\pi}{2}\cos\beta.$$

We finish, by treating the case where $a = 0$. In this case the original integral reduces to

$$I = \int_0^\infty \frac{\sin x}{x}\,dx = \frac{1}{2i}\,\mathrm{PV}\int_{-\infty}^\infty \frac{e^{ix}}{x}dx.$$

As shown in the solution of item D of the contour integral problems for final exam, in this case $I = \pi/2$. In summary, for $a = \alpha + i\beta \in \mathbf{C}$ we obtain

$$\int_0^\infty \frac{x\sin x}{a^2+x^2}dx = \frac{\pi}{2}e^{-|\alpha|}\cos\beta$$

where the integral is interpreted in the sense of Cauchy's principle value when $\alpha = 0$.

3. [Carrier, Krook and Pearson, Section 3-1 problem 8] Evaluate

$$I = \int_0^{2\pi} \log(a + b\cos\theta)d\theta \qquad \text{for} \qquad a > b > 0.$$

First note that

$$I = \int_{-\pi}^{\pi} \log(a + b\cos\theta)d\theta = \int_{-\pi}^{\pi} \log\Big(a + \frac{b}{2}(e^{i\theta} + e^{-i\theta})\Big)d\theta$$
$$= 2\pi \log\Big(\frac{b}{2}\Big) + \int_{-\pi}^{\pi} \log\Big(\frac{2a}{b} + e^{i\theta} + e^{-i\theta}\Big)d\theta.$$

By definition

$$\log(re^{i\theta}) = \log r + i\theta \qquad \text{for} \qquad r > 0 \qquad \text{and} \qquad \theta \in (-\pi, \pi].$$

Since $a > b$ then

$$r = \frac{2a}{b} + e^{i\theta} + e^{-i\theta} = \frac{2a}{b} + 2\cos\theta > 0.$$

Consequently

$$\log\Big(\Big(\frac{2a}{b} + e^{i\theta} + e^{-i\theta}\Big)e^{i\theta}\Big) = \log\Big(\frac{2a}{b} + e^{i\theta} + e^{-i\theta}\Big) + i\theta.$$

It follows that

$$\int_{-\pi}^{\pi} \log\Big(\frac{2a}{b} + e^{i\theta} + e^{-i\theta}\Big)d\theta = \int_{-\pi}^{\pi} \log\Big(\frac{2a}{b}e^{i\theta} + e^{2i\theta} + 1\Big)d\theta - i\int_{-\pi}^{\pi} \theta d\theta$$
$$= \oint_{|z|=1} \log\Big(z^2 + \frac{2a}{b}z + 1\Big)\frac{dz}{iz}.$$

To determine where the logarithm is analytic solve the inequality $p(z) \le 0$ where

$$p(z) = z^2 + \frac{2a}{b}z + 1.$$

Setting $z = x + iy$ we obtain

$$p(x+iy) = (x+iy)^2 + \frac{2a}{b}(x+iy) + 1 = x^2 - y^2 + \frac{2a}{b}x + 1 + iy\Big(2x + \frac{2a}{b}\Big).$$

Therefore, the $p(z) \in \mathbf{R}$ if and only if

$$x = -\frac{a}{b} \qquad \text{or} \qquad y = 0.$$

In the case $x = -a/b$ we obtain that

$$p(-a/b + iy) = -\frac{a^2}{b^2} - y^2 + 1 \le 0 \qquad \text{for all} \qquad y \in \mathbf{R}.$$

On the other hand, if $y = 0$ then completing the square yields

$$p(x) = x^2 + \frac{2a}{b}x + 1 = \left(x + \frac{a}{b}\right)^2 + 1 - \frac{a^2}{b^2} = (x - r_1)(x + r_1)$$

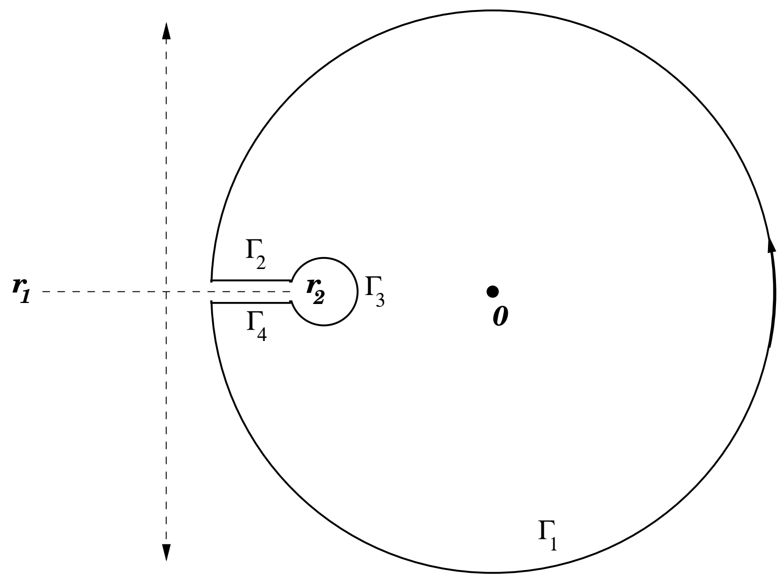
where

$$r_1 = -\frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1} \qquad \text{and} \qquad r_2 = -\frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1}.$$

Note that $r_1, r_2 \in \mathbf{R}$ and $r_1 < -1 < r_2 < 0$. Therefore, it is also the case that $p(x+iy) \leq 0$ when

$$x \in [r_1, r_2] \qquad \text{and} \qquad y = 0.$$

We therefore consider the following contour



Since $\log p(z)$ is analytic inside the contour, we have that

$$\int_{\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4} \frac{\log p(z)}{iz} dz = 2\pi \log p(0) = 2\pi \log 1 = 0.$$

Choose $\eta > 0$ so small that $|z| < 1 + \eta$ implies $x + a/b > 0$. Therefore

$$\operatorname{Im} p(z) = 2y\left(x + \frac{a}{b}\right) > 0 \qquad \text{when} \qquad |x| < 1 + \eta \quad \text{and} \quad y > 0.$$

It follows for $|x| \leq 1 + \eta$ that

$$\lim_{y \to 0^+} \log p(x + iy) = \log p(x) + i\pi$$

and similarly that

$$\lim_{y \to 0^-} \log p(x + iy) = \log p(x) - i\pi.$$

Consequently, upon setting $\xi = -x$ we obtain

$$\begin{aligned}\int_{\Gamma_2+\Gamma_4} \frac{\log p(z)}{iz} dz &= \int_{-1}^{r_2-\varepsilon} \frac{\log p(x) + i\pi}{ix} dx + \int_{r_2-\varepsilon}^{-1} \frac{\log p(x) - i\pi}{ix} dx \\ &= 2\pi \int_{-1}^{r_2-\varepsilon} \frac{dx}{x} = -2\pi \int_{-r_2+\varepsilon}^{1} \frac{d\xi}{\xi} \\ &= 2\pi \log(-r2 + \varepsilon) \to 2\pi \log\left(\frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1}\right) \quad \text{as} \quad \varepsilon \to 0.\end{aligned}$$

Finally,

$$\int_{\Gamma_3} \frac{\log p(z)}{iz} dz = \int_{\pi}^{-\pi} \log p(\varepsilon e^{it}) dt \to \int_{\pi}^{-\pi} \log p(0) dt = 0 \qquad \text{as} \qquad \varepsilon \to 0.$$

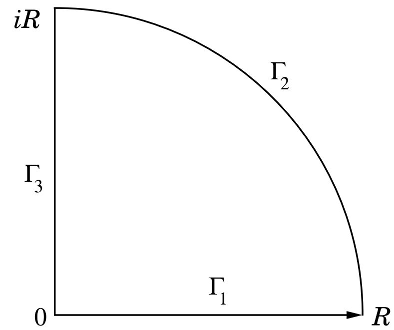
It follows that

$$\begin{aligned}I &= 2\pi \log\left(\frac{b}{2}\right) - 2\pi \log\left(\frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1}\right) \\ &= 2\pi \log\left(\frac{a + \sqrt{a^2 - b^2}}{2}\right).\end{aligned}$$

4. [Carrier, Krook and Pearson, Section 3-1 problem 10] If α and β are real and positive, show that
$$\int_0^\infty \frac{\cos\alpha x}{x+\beta}dx = \int_0^\infty \frac{xe^{-\alpha\beta x}}{1+x^2}dx.$$
Which of these two would be easier to compute numerically?

Consider the following contour

$$\Gamma_1 = \{\, t : t \in [0,R]\,\}$$
$$\Gamma_2 = \{\, Re^{it} : t \in [0,\pi]\,\}$$
$$\Gamma_3 = \{\, i(R-t) : t \in [0,R]\,\}$$



Since the mapping $z \to e^{\alpha z}/(z+\beta)$ is analytic on a neighborhood of the area enclosed by this contour, it follows that
$$\int_{\Gamma_1} \frac{e^{i\alpha z}}{z+\beta}dz = -\int_{\Gamma_2} \frac{e^{i\alpha z}}{z+\beta}dz - \int_{\Gamma_3} \frac{e^{i\alpha z}}{z+\beta}dz.$$
By Jordan's lemma
$$\begin{aligned}\left|\int_{\Gamma_2} \frac{e^{i\alpha z}}{z+\beta}dz\right| &\le \int_0^{\pi/2} \left|\frac{e^{i\alpha R(\cos t + i\sin t)}}{Re^{it}+\beta} iRe^{it}\right| dt \\ &= \int_0^{\pi/2} \frac{e^{-\alpha R\sin t}}{|Re^{it}+\beta|} R\,dt \le \frac{R}{R-\beta}\int_0^{\pi/2} e^{-2\alpha Rt/\pi}dt \\ &= \frac{R}{R-\beta}\frac{\pi}{2\alpha R}\big(1-e^{-\alpha R}\big) \to 0 \qquad \text{as} \qquad R\to\infty.\end{aligned}$$
Setting $R-t=\beta s$ so that $-dt=\beta ds$ yields
$$\begin{aligned}\int_{\Gamma_3} \frac{e^{i\alpha z}}{z+\beta}dz &= -i\int_0^R \frac{e^{-\alpha(R-t)}}{i(R-t)+\beta}dt = -i\int_0^{R/\beta} \frac{e^{-\alpha\beta s}}{is+1}ds \\ &= -i\int_0^{R/\beta} \frac{(1-is)e^{-\alpha\beta s}}{1+s^2}ds = -\int_0^{R/\beta} \frac{(s+i)e^{-\alpha\beta s}}{1+s^2}ds.\end{aligned}$$
It follows that
$$\begin{aligned}\int_0^\infty \frac{\cos\alpha x}{x+\beta}dx &= \lim_{R\to\infty} \operatorname{Re}\int_{\Gamma_1} \frac{e^{i\alpha z}}{z+\beta}dz \\ &= \lim_{R\to\infty} \operatorname{Re}\int_0^{R/\beta} \frac{(s+i)e^{-\alpha\beta s}}{1+s^2}ds = \int_0^\infty \frac{xe^{-\alpha\beta x}}{1+x^2}dx.\end{aligned}$$
As the integrand in the last integral decays exponentially to zero as $x\to\infty$, it would be easier to evaluate numerically.

5. [Carrier, Krook and Pearson, Section 3-1 problem 14] Evaluate

$$\int_0^{\pi} \frac{x \sin x}{1 - 2\alpha \cos x + \alpha^2} dx$$

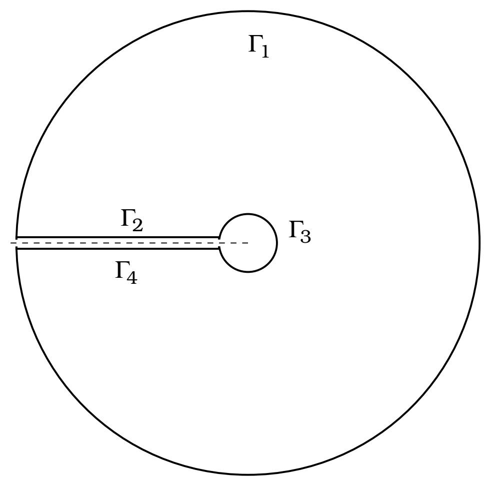
for α real and for each of the two cases $|\alpha| < 1$ and $|\alpha| > 1$.

Since $x \sin x$ and $\cos x$ are even, then

$$I = \int_0^{\pi} \frac{x \sin x}{1 - 2\alpha \cos x + \alpha^2} dx = \frac{1}{2} \int_{\pi}^{\pi} \frac{x \sin x}{1 - 2\alpha \cos x + \alpha^2} dx.$$

Consider the contour $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ where

$$\Gamma_1 = \left\{ e^{it} : t \in [-\pi, \pi] \right\}$$
$$\Gamma_2 = \left\{ t + i0^+ : t \in [-1, -\varepsilon] \right\}$$
$$\Gamma_3 = \left\{ \varepsilon e^{-it} : t \in [-\pi, \pi] \right\}$$
$$\Gamma_4 = \left\{ -t + i0^- : t \in [\varepsilon, 1] \right\}$$



Writing $z = e^{ix}$ yields that

$$\cos x = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{and} \qquad \sin x = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Therefore

$$I = \frac{1}{2} \int_{\Gamma_1} \frac{-i(\log z)\frac{1}{2i}\left(z - z^{-1}\right)}{1 - 2\alpha\frac{1}{2}\left(z + z^{-1}\right) + \alpha^2} \frac{dz}{iz} = \frac{1}{4i} \int_{\Gamma_1} \frac{-(\log z)\left(z - z^{-1}\right)}{1 - \alpha\left(z + z^{-1}\right) + \alpha^2} \frac{dz}{z}$$
$$= \frac{1}{4i} \int_{\Gamma_1} \frac{(\log z)\left(z - z^{-1}\right)}{\alpha z^2 - (1 + \alpha^2) z + \alpha} dz = \frac{1}{4i} \int_{\Gamma_1} \frac{(\log z)\left(z - z^{-1}\right)}{(\alpha z - 1)(z - \alpha)} dz$$

Provided $\alpha \notin (-\infty, 0]$ and $|\alpha| \neq 1$ we may choose ε so small that either α or $1/\alpha$ is inside the region bounded by Γ. In this case

$$I = \frac{1}{4i} \int_{\Gamma - \Gamma_2 - \Gamma_3 - \Gamma_4} \frac{(\log z)\left(z - z^{-1}\right)}{(\alpha z - 1)(z - \alpha)} dz$$

Now

$$\frac{1}{4i}\int_{\Gamma}\frac{(\log z)\big(z-z^{-1}\big)}{(\alpha z-1)(z-\alpha)}dz=\begin{cases}\dfrac{\pi}{2}\dfrac{\log(\alpha)(\alpha-\alpha^{-1})}{\alpha^2-1} & \text{for } |\alpha|<1\\[2mm] \dfrac{\pi}{2}\dfrac{\log(\alpha^{-1})(\alpha^{-1}-\alpha)}{\alpha(\alpha^{-1}-\alpha)} & \text{for } |\alpha|>1\end{cases}$$

$$=\begin{cases}-\dfrac{\pi}{2}\dfrac{\log(\alpha^{-1})}{\alpha} & \text{for } |\alpha|<1\\[2mm] -\dfrac{\pi}{2}\dfrac{\log(\alpha)}{\alpha} & \text{for } |\alpha|>1.\end{cases}$$

Moreover, since

$$\frac{1}{4i}\int_{\Gamma_2}\frac{(\log z)\big(z-z^{-1}\big)}{(\alpha z-1)(z-\alpha)}dz=\frac{1}{4i}\int_{-1}^{-\varepsilon}\frac{(\log|t|+i\pi)\big(t-t^{-1}\big)}{(\alpha t-1)(t-\alpha)}dt$$

$$=\frac{1}{4i}\int_{\varepsilon}^{1}\frac{(\log|t|+i\pi)\big(t^{-1}-t\big)}{(\alpha t+1)(t+\alpha)}dt$$

and

$$\frac{1}{4i}\int_{\Gamma_4}\frac{(\log z)\big(z-z^{-1}\big)}{(\alpha z-1)(z-\alpha)}dz=-\frac{1}{4i}\int_{\varepsilon}^{1}\frac{(\log|t|-i\pi)\big(-t+t^{-1}\big)}{(-\alpha t-1)(-t-\alpha)}dt$$

$$=\frac{1}{4i}\int_{\varepsilon}^{1}\frac{(-\log|t|+i\pi)\big(t^{-1}-t\big)}{(\alpha t+1)(t+\alpha)}dt,$$

we obtain

$$\frac{1}{4i}\int_{\Gamma_2+\Gamma_4}\frac{(\log z)\big(z-z^{-1}\big)}{(\alpha z-1)(z-\alpha)}dz=\frac{\pi}{2}\int_{\varepsilon}^{1}\frac{\big(t^{-1}-t\big)}{(\alpha t+1)(t+\alpha)}dt$$

$$=\frac{\pi}{2\alpha}\int_{\varepsilon}^{1}\Big(\frac{1}{t}-\frac{1}{t+\alpha^{-1}}+\frac{1}{t+\alpha}\Big)dt$$

$$=\frac{\pi}{2\alpha}\Big(\log(t)-\log(t+\alpha^{-1})-\log(t+\alpha)\Big)\Big|_{t=\varepsilon}^{1}$$

$$=\frac{\pi}{2\alpha}\Big(\log(\varepsilon^{-1})+\log(\varepsilon+\alpha^{-1})-\log(1+\alpha^{-1})+\log(\varepsilon+\alpha)-\log(1+\alpha)\Big)$$

Also

$$\frac{1}{4i}\int_{\Gamma_3}\frac{(\log z)\big(z-z^{-1}\big)}{(\alpha z-1)(z-\alpha)}dz=\frac{1}{4i}\int_{-\pi}^{\pi}\frac{(\log\varepsilon-it)\big(\varepsilon e^{-it}-\varepsilon^{-1}e^{it}\big)}{(\alpha\varepsilon e^{-it}-1)(\varepsilon e^{-it}-\alpha)}(-i\varepsilon e^{-it})dt$$

$$=J_\varepsilon+\frac{1}{4i}\int_{-\pi}^{\pi}\frac{-it\big(\varepsilon e^{-it}-\varepsilon^{-1}e^{it}\big)}{(\alpha\varepsilon e^{-it}-1)(\varepsilon e^{-it}-\alpha)}(-i\varepsilon e^{-it})dt$$

where

$$J_\varepsilon=\frac{\log\varepsilon}{4i}\int_{\Gamma_3}\frac{\big(z-z^{-1}\big)}{(\alpha z-1)(z-\alpha)}dz=-\frac{\log\varepsilon}{4i}\int_{-\Gamma_3}\frac{\big(z^2-1\big)}{(\alpha z-1)(z-\alpha)z}dz=-\frac{\pi\log(\varepsilon^{-1})}{2\alpha}.$$

We obtain that

$$\lim_{\varepsilon\to 0}\Big(\frac{1}{4i}\int_{\Gamma_2+\Gamma_4}\frac{(\log z)\big(z-z^{-1}\big)}{(\alpha z-1)(z-\alpha)}dz+J_\varepsilon\Big)$$
$$=\frac{\pi}{2\alpha}\Big(\log(\alpha^{-1})-\log(1+\alpha^{-1})+\log(\alpha)-\log(1+\alpha)\Big)$$
$$=-\frac{\pi}{2\alpha}\log\big(2+\alpha+\alpha^{-1}\big).$$

Finally, since

$$\lim_{\varepsilon\to 0}\frac{1}{4i}\int_{-\pi}^{\pi}\frac{-it\big(\varepsilon e^{-it}-\varepsilon^{-1}e^{it}\big)}{(\alpha\varepsilon e^{-it}-1)(\varepsilon e^{-it}-\alpha)}(-i\varepsilon e^{-it})dt=\frac{1}{4i}\int_{-\pi}^{\pi}\frac{-it\big(-e^{it}\big)}{(-1)(-\alpha)}(-ie^{-it})dt$$
$$=\frac{1}{4\alpha i}\int_{-\pi}^{\pi}tdt=0$$

it follows for

$$\alpha\in\mathbf{C}\setminus\big(\{x:x\le 0\}\cup\{z:|z|=1\}\big)$$

that

$$I(\alpha)=\frac{\pi}{2\alpha}\log\big(2+\alpha+\alpha^{-1}\big)-\begin{cases}\dfrac{\pi}{2}\dfrac{\log(\alpha^{-1})}{\alpha} & \text{for } |\alpha|<1\\[2mm] \dfrac{\pi}{2}\dfrac{\log(\alpha)}{\alpha} & \text{for } |\alpha|>1.\end{cases}$$

It remains to consider the cases when $\alpha\le 0$ and $\alpha=1$ or in general when α is real.

If $\alpha=-1$ then

$$I(-1)=\frac{1}{2}\int_0^{\pi}\frac{x\sin x}{1+\cos x}dx=\infty.$$

If $\alpha=1$ then

$$I(1)=\frac{1}{2}\int_0^{\pi}\frac{x\sin x}{1-\cos x}dx=\frac{1}{4}\int_{-\pi}^{\pi}\frac{x\sin x}{1-\cos x}dx=\pi\log 2.$$

If $\alpha=0$ then the integral can be done by parts as

$$I(0)=\int_0^{\pi}x\sin xdx=-x\cos x\Big|_0^{\pi}+\int_0^{\pi}\cos xdx=\pi.$$

if $\alpha<0$ and $\alpha\ne -1$ then define $\beta=-\alpha$ to obtain

$$I(\alpha)=\int_0^{\pi}\frac{x\sin x}{1+2\beta\cos x+\beta^2}dx=\int_{-\pi}^{0}\frac{x\sin x}{1+2\beta\cos x+\beta^2}dx$$
$$=\int_{-\pi}^{0}\frac{-x\sin(x+\pi)}{1-2\beta\cos(x+\pi)+\beta^2}dx=-\int_0^{\pi}\frac{(x-\pi)\sin x}{1-2\beta\cos x+\beta^2}dx$$
$$=-I(\beta)+\pi\int_0^{\pi}\frac{\sin x}{1-2\beta\cos x+\beta^2}dx$$

Substituting $u = 1 - 2\beta\cos x + \beta^2$ so that $du = 2\beta \sin x\, dx$ we obtain

$$\int_0^{\pi} \frac{\sin x}{1 - 2\beta\cos x + \beta^2} = \frac{1}{2\beta}\int_{(1-\beta)^2}^{(1+\beta)^2} \frac{du}{u} = \frac{\ln|1+\beta| - \ln|1-\beta|}{\beta}.$$

Consequently, when $\alpha < 0$ and $\alpha \neq -1$ then

$$I(\alpha) = \frac{\pi}{2\alpha}\left\{ \log\left(2 - \alpha - \alpha^{-1}\right) - 2\log\left|\frac{1-\alpha}{1+\alpha}\right| \right\} - \begin{cases} \dfrac{\pi}{2}\dfrac{\log(-\alpha^{-1})}{\alpha} & \text{for } \alpha \in (-1, 0) \\ \dfrac{\pi}{2}\dfrac{\log(-\alpha)}{\alpha} & \text{for } \alpha < -1. \end{cases}$$

Finally, it is worth noting that, given the definition of principle value for the logarithm, the original formula for $I(\alpha)$ is consistent with the above special cases when $\alpha < 1$. Therefore, in general

$$I(\alpha) = \frac{\pi}{2\alpha}\log\left(2 + \alpha + \alpha^{-1}\right) - \begin{cases} \dfrac{\pi}{2}\dfrac{\log(\alpha^{-1})}{\alpha} & \text{for } |\alpha| < 1 \text{ or } \alpha = 1 \\ \dfrac{\pi}{2}\dfrac{\log(\alpha)}{\alpha} & \text{for } |\alpha| > 1. \end{cases}$$

6. [Carrier, Krook and Pearson, Section 3-2 problem 1.] Replace the denominator of the integrand in

$$I_n = \frac{1}{2\pi i}\int_{C_n} \frac{d\xi}{\xi(\xi - z)\sin\xi} \tag{3-55}$$

by $(\xi - z)^2 \sin\xi$ so as to obtain an expansion for $(\cot z)(\csc z)$.

Define

$$J_n = \frac{1}{2\pi i}\int_{C_n} \frac{d\xi}{(\xi - z)^2 \sin\xi}$$

taking the contour C_n to be the square with corners at

$$(n + \tfrac{1}{2})(\pm 1 \pm i)\pi \qquad \text{where} \qquad n \in \mathrm{N}.$$

For the same reasons that $I_n \to 0$ as $n \to \infty$ we have that $J_n \to 0$ as $n \to \infty$. Recall that

$$\sin\xi = (-1)^k \sin(\xi - k\pi) = (-1)^k(\xi - k\pi)\operatorname{sinc}(\xi - k\pi).$$

When n is large enough z is inside the contour. In addition the points

$$k\pi \qquad \text{for} \qquad k = -n, -n+1, \ldots, n$$

are also inside the contour. Therefore

$$J_n = \frac{1}{2\pi i}\bigg(\int_{\partial B_\varepsilon(z)} + \sum_{k=-n}^{n}\int_{\partial B_\varepsilon(k\pi)}\bigg)\frac{d\xi}{(\xi - z)^2\sin\xi}.$$

Using Cauchy's derivative formula, we obtain

$$\frac{1}{2\pi i}\int_{\partial B_\varepsilon(z)} \frac{d\xi}{(\xi - z)^2 \sin\xi} = \frac{d}{d\xi}\frac{1}{\sin\xi}\bigg|_{\xi = z} = -(\cot z)(\csc z).$$

Cauchy's formula applied to the other terms yields

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial B_\varepsilon(k\pi)} \frac{d\xi}{(\xi - z)^2 \sin\xi} &= \frac{1}{2\pi i}\int_{\partial B_\varepsilon(k\pi)} \frac{d\xi}{(\xi - z)^2(-1)^k(\xi - k\pi)\operatorname{sinc}(\xi - k\pi)} \\ &= \frac{(-1)^k}{(k\pi - z)^2\operatorname{sinc} 0} = \frac{(-1)^k}{(z - k\pi)^2}\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{k=-n}^{n}\frac{1}{2\pi i}\int_{\partial B_\varepsilon(k\pi)} \frac{d\xi}{(\xi - z)^2\sin\xi} &= \sum_{k=-n}^{n}\frac{(-1)^k}{(z - k\pi)^2} \\ &= \frac{1}{z^2} + \sum_{k=1}^{n}(-1)^k\Big(\frac{1}{(z - k\pi)^2} + \frac{1}{(z + k\pi)^2}\Big) \\ &= \frac{1}{z^2} + 2\sum_{k=1}^{n}(-1)^k\frac{z^2 + k^2\pi^2}{(z^2 - k^2\pi^2)^2}\end{aligned}$$

Taking limits as $n \to \infty$ yields that

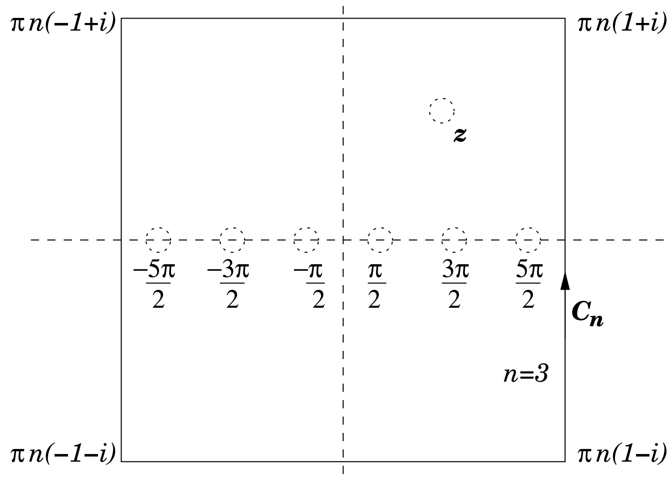
$$(\cot z)(\csc z) = \frac{1}{z^2} + 2\sum_{k=1}^{\infty}(-1)^k\frac{z^2 + k^2\pi^2}{(z^2 - k^2\pi^2)^2}.$$

7. [Carrier, Krook and Pearson, Section 3-2 problem 2.] Show that

$$\frac{1}{\cos z} = 2\pi \sum_0^\infty \frac{(-1)^n (n+1/2)}{(n+1/2)^2\pi^2 - z^2} \tag{3-60}$$

Consider the contour $C_n = C_{n,1} + C_{n,2} + C_{n,3} + C_{n,4}$ given by

$$\begin{aligned} C_{n,1} &= \{\, n\pi + it\pi : t \in [-n,n] \,\} \\ C_{n,2} &= \{\, -t\pi + in\pi : t \in [-n,n] \,\} \\ C_{n,3} &= \{\, -n\pi - it\pi : t \in [-n,n] \,\} \\ C_{n,4} &= \{\, t\pi - in\pi : t \in [-n,n] \,\} \end{aligned}$$



Define

$$J_n = \frac{1}{2\pi i} \int_{C_n} \frac{d\xi}{(\xi - z)\cos\xi}.$$

Claim that $J_n \to 0$ and $n \to 0$. Note first that

$$\cos(a+ib) = (\cos a)(\cosh b) - i(\sin a)(\sinh b).$$

Since

$$|\cosh(b)| = \frac{e^b + e^{-b}}{2} \geq \frac{e^{|b|}}{2} \qquad \text{and} \qquad |\sinh(b)| = \frac{|e^b - e^{-b}|}{2} \geq \frac{e^{|b|}-1}{2}$$

it follows that

$$\begin{aligned} |\cos(a+ib)|^2 &= (\cos^2 a)(\cosh^2 b) + (\sin^2 a)(\sinh^2 b) \\ &\geq (\cos^2 a)(\cosh^2 b) \geq 4^{-1}(\cos^2 a)e^{2|b|} \end{aligned} \tag{$*$}$$

and

$$\begin{aligned} |\cos(a+ib)|^2 &= (\cos^2 a)(\cosh^2 b) + (\sin^2 a)(\sinh^2 b) \\ &= (\cos^2 a)(\cosh^2 b - \sinh^2 b) + (\sin^2 a + \cos^2 a)(\sinh^2 b) \\ &= \cos^2 a + \sinh^2 b \geq \sinh^2 b \geq 4^{-1}(e^{|b|}-1)^2. \end{aligned} \tag{$**$}$$

Note also there is n_0 large enough such that $n \geq n_0$ implies

$$|\xi - z| \geq (n - n_0)\pi \qquad \text{for every} \qquad \xi \in C_n.$$

Estimate using $(*)$ as

$$\begin{aligned}
|J_{n,1}| &= \left|\frac{1}{2\pi i}\int_{C_{n,1}} \frac{d\xi}{(\xi - z)\cos\xi}\right| \leq \frac{1}{2\pi}\int_{-n}^{n}\left|\frac{i\pi}{(n\pi + it\pi - z)\cos(n\pi + it\pi)}\right| dt \\
&\leq \frac{1}{2}\int_{-n}^{n}\frac{2dt}{(n-n_0)\pi|\cos(n\pi)|e^{|t\pi|}} = \frac{2}{(n-n_0)\pi}\int_0^n e^{-t\pi}dt \\
&= \frac{2}{(n-n_0)\pi^2}(1-e^{-n\pi}) \to 0 \qquad \text{as} \qquad n \to \infty.
\end{aligned}$$

and using $(**)$ as

$$\begin{aligned}
|J_{n,2}| &= \left|\frac{1}{2\pi i}\int_{C_{n,2}} \frac{d\xi}{(\xi - z)\cos\xi}\right| \leq \frac{1}{2\pi}\int_{-n}^{n}\left|\frac{-\pi}{(-t\pi + in\pi)\cos(-t\pi + in\pi)}\right| dt \\
&\leq \frac{1}{2}\int_{-n}^{n}\frac{2dt}{(n-n_0)\pi(e^{|n\pi|}-1)} = \frac{2}{(n-n_0)\pi}\int_0^n (e^{|n\pi|}-1)dt \\
&= \frac{2n}{(n-n_0)\pi}(e^{|n\pi|}-1) \to 0 \qquad \text{as} \qquad n \to \infty.
\end{aligned}$$

Similar arguments show that

$$\left|\frac{1}{2\pi i}\int_{C_{n,3}} \frac{d\xi}{(\xi - z)\cos\xi}\right| \to 0 \qquad \text{and} \qquad \left|\frac{1}{2\pi i}\int_{C_{n,4}} \frac{d\xi}{(\xi - z)\cos\xi}\right| \to 0$$

as $n \to \infty$. Therefore

$$J_n = J_{n,1} + J_{n,2} + J_{n,3} + J_{n,4} \to 0 \qquad \text{as} \qquad n \to \infty.$$

Consider the contour

$$\Gamma = C_n - \partial B_\varepsilon(z) - \sum_{k=1}^{n} \partial B_\varepsilon\big((k-\tfrac{1}{2})\pi\big) - \sum_{k=1}^{n} \partial B_\varepsilon\big(-(k-\tfrac{1}{2})\pi\big)$$

where $n \geq n_0$. Since the function

$$\xi \to \frac{1}{(\xi - z)\cos\xi}$$

is analytic on a neighborhood of the area enclosed by this contour, Cauchy's theorem implies

$$\int_\Gamma \frac{d\xi}{(\xi - z)\cos\xi} = 0.$$

Therefore

$$J_n = \frac{1}{2\pi i}\bigg(\int_{\partial B_\varepsilon(z)} + \sum_{k=1}^{n}\Big\{\int_{\partial B_\varepsilon((k-\frac{1}{2})\pi)} + \int_{\partial B_\varepsilon(-(k-\frac{1}{2})\pi)}\Big\}\bigg)\frac{d\xi}{(\xi - z)\cos\xi}$$

Now Cauchy's formula implies

$$\frac{1}{2\pi i}\int_{\partial B_\varepsilon(z)}\frac{d\xi}{(\xi-z)\cos\xi}=\frac{1}{\cos z}.$$

Moreover, since

$$\begin{aligned}\cos\xi &= (-1)^k\cos(\xi-k\pi)=(-1)^k\sin\big(\xi-(k-\tfrac{1}{2})\pi\big)\\ &=(-1)^k\big(\xi-(k-\tfrac{1}{2})\pi\big)\operatorname{sinc}\big(\xi-(k-\tfrac{1}{2})\pi\big)\end{aligned}$$

we obtain that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\partial B_\varepsilon((k-\frac{1}{2})\pi)}\frac{d\xi}{(\xi-z)\cos\xi}\\ &\quad=\frac{1}{2\pi i}\int_{\partial B_\varepsilon((k-\frac{1}{2})\pi)}\frac{d\xi}{(\xi-z)(-1)^k\big(\xi-(k-\frac{1}{2})\pi\big)\operatorname{sinc}\big(\xi-(k-\frac{1}{2})\pi\big)}\\ &\quad=\frac{(-1)^k}{\big((k-\frac{1}{2})\pi-z\big)\operatorname{sinc}0}=\frac{(-1)^k}{(k-\frac{1}{2})\pi-z}.\end{aligned}$$

Similarly, since

$$\cos\xi=-(-1)^k\big(\xi+(k-\tfrac{1}{2})\pi\big)\operatorname{sinc}\big(\xi+(k-\tfrac{1}{2})\pi\big)$$

we obtain that

$$\frac{1}{2\pi i}\int_{\partial B_\varepsilon(-(k+\frac{1}{2})\pi)}\frac{d\xi}{(\xi-z)\cos\xi}=-\frac{(-1)^k}{\big(-(k-\frac{1}{2})\pi-z\big)}=\frac{(-1)^k}{(k-\frac{1}{2})\pi+z}.$$

It follows that

$$\begin{aligned}J_n&=\frac{1}{\cos z}+\sum_{k=1}^{n}(-1)^k\Big\{\frac{1}{(k-\frac{1}{2})\pi-z}+\frac{1}{(k-\frac{1}{2})\pi+z}\Big\}\\ &=\frac{1}{\cos z}+2\pi\sum_{k=1}^{n}\frac{(-1)^k(k-\frac{1}{2})}{(k-\frac{1}{2})^2\pi^2-z^2}.\end{aligned}$$

Taking the limit as $n\to\infty$ yields

$$\frac{1}{\cos z}=-2\pi\sum_{k=1}^{\infty}\frac{(-1)^k(k-\frac{1}{2})}{(k-\frac{1}{2})^2\pi^2-z^2}=2\pi\sum_{k=0}^{\infty}\frac{(-1)^k(k+\frac{1}{2})}{(k+\frac{1}{2})^2\pi^2-z^2}$$

which is the first formula.

We now derive the second formula

$$\tan z=2z\sum_{n=0}^{\infty}\frac{1}{(n+1/2)^2\pi^2-z^2}\tag{3-61}$$

using a slight modification of the previous work. Define

$$K_n = \frac{1}{2\pi i}\int_{C_n} \frac{\sin\xi\, d\xi}{(\xi - z)\cos\xi}.$$

where C_n is as in the previous problem, the square with corners $\pi n(\pm 1 \pm i)$. To show that $K_n \to 0$ as $n \to \infty$ we estimate as follows. First note that

$$\sin(a + ib) = (\sin a)(\cosh b) + i(\cos a)(\sinh b).$$

Therefore

$$\begin{aligned} |\sin(a+ib)|^2 &= (\sin^2 a)(\cosh^2 b) + (\cos^2 a)(\sinh^2 b) \\ &= (\sin^2 a)(\cosh^2 b) \pm (\cos^2 a)(\cosh^2 b) + (\cos^2 a)(\sinh^2 b) \qquad (***) \\ &= \cosh^2 b - \cos^2 a \le \cosh^2 b \le e^{2|b|}. \end{aligned}$$

When $z = n\pi + it\pi$ we obtain

$$|\tan(n\pi + it)|^2 = \frac{\sinh^2 t}{\cosh^2 t}.$$

Therefore

$$\begin{aligned} K_{n,1} &= \frac{1}{2\pi i}\int_{C_{n,1}} \frac{\sin\xi\, d\xi}{(\xi - z)\cos\xi} = \frac{1}{2\pi i}\int_{-n}^{n} \frac{1}{(n\pi + it\pi - z)}\frac{i(-1)^n \sinh t\pi}{(-1)^n \cosh t\pi} i\pi dt \\ &= \frac{i}{2}\int_{-n}^{n} \frac{1}{(n\pi + it\pi - z)}\frac{\sinh t\pi}{\cosh t\pi}dt \end{aligned}$$

and similarly

$$\begin{aligned} K_{n,3} &= \frac{1}{2\pi i}\int_{C_{n,3}} \frac{\sin\xi\, d\xi}{(\xi - z)\cos\xi} = \frac{1}{2\pi i}\int_{-n}^{n} \frac{1}{(-n\pi - it\pi - z)}\frac{i(-1)^n \sinh(-t\pi)}{(-1)^n \cosh(-t\pi)}(-i\pi)dt \\ &= \frac{i}{2}\int_{-n}^{n} \frac{1}{(-n\pi - it\pi - z)}\frac{\sinh t\pi}{\cosh t\pi}dt. \end{aligned}$$

Therefore

$$\begin{aligned} \left|K_{n,1} + K_{n,3}\right| &= \left|\frac{i}{2}\int_{-n}^{n} \left(\frac{1}{n\pi + it\pi - z} - \frac{1}{n\pi + it\pi + z}\right)\frac{\sinh t\pi}{\cosh t\pi}dt\right| \\ &= \left|\frac{i}{2}\int_{-n}^{n} \frac{2z}{(n\pi + it\pi)^2 - z^2}\frac{\sinh t\pi}{\cosh t\pi}dt\right| \\ &\le \int_{-n}^{n} \frac{|z|}{|n\pi + it\pi|^2 - |z|^2}\left|\frac{\sinh t\pi}{\cosh t\pi}\right|dt \\ &\le \frac{|z|}{n^2\pi^2 - |z|^2}\int_{-n}^{n}\left|\frac{\sinh t\pi}{\cosh t\pi}\right|dt = \frac{2|z|}{n^2\pi^2 - |z|^2}\int_{0}^{n}\frac{\sinh t\pi}{\cosh t\pi}dt \\ &= \frac{2|z|}{n^2\pi^2 - |z|^2}\frac{\log\cosh n\pi}{\pi} \le \frac{2|z|}{n^2\pi^2 - |z|^2}\frac{\log(e^{n\pi}/2)}{\pi} \\ &= \frac{2|z|}{n^2\pi^2 - |z|^2}\frac{n\pi - \log 2}{\pi} \to 0 \qquad \text{as} \qquad n \to \infty. \end{aligned}$$

On the other hand

$$K_{n,2} = \frac{1}{2\pi i}\int_{C_{n,2}} \frac{\sin\xi\, d\xi}{(\xi - z)\cos\xi} = \frac{1}{2\pi i}\int_{-n}^{n} \frac{1}{(-t\pi + in\pi - z)}\frac{\sin(-t\pi + in\pi)}{\cos(-t\pi + in\pi)}(-\pi)dt$$

and

$$K_{n,4} = \frac{1}{2\pi i}\int_{C_{n,4}} \frac{\sin\xi\, d\xi}{(\xi - z)\cos\xi} = \frac{1}{2\pi i}\int_{-n}^{n} \frac{1}{(t\pi - in\pi - z)}\frac{\sin(t\pi - in\pi)}{\cos(t\pi - in\pi)}\pi dt.$$

Therefore using $(**)$ and $(***)$ we obtain

$$\begin{aligned}
|K_{n,2} + K_{n,4}| &= \frac{1}{2}\Big|\int_{-n}^{n}\Big(\frac{-1}{t\pi - in\pi + z} + \frac{1}{t\pi - in\pi - z}\Big)\frac{\sin(t\pi - in\pi)}{\cos(t\pi - in\pi)}dt\Big| \\
&= \frac{1}{2}\Big|\int_{-n}^{n}\frac{2z}{(t\pi - in\pi)^2 - z^2}\frac{\sin(t\pi - in\pi)}{\cos(t\pi - in\pi)}dt\Big| \\
&\le \frac{|z|}{n^2\pi^2 - |z|^2}\int_{-n}^{n}\Big|\frac{\sin(t\pi - in\pi)}{\cos(t\pi - in\pi)}\Big|dt \\
&\le \frac{|z|}{n^2\pi^2 - |z|^2}\int_{-n}^{n}\Big|\frac{e^{|n\pi|}}{2^{-1}(e^{|n\pi|} - 1)}\Big|dt \\
&= \frac{|z|}{n^2\pi^2 - |z|^2}\frac{4ne^{|n\pi|}}{(e^{|n\pi|} - 1)} \to 0 \qquad \text{as} \qquad n \to \infty.
\end{aligned}$$

It follows that $K_n \to 0$ as $n \to \infty$. On the other hand,

$$K_n = \frac{1}{2\pi i}\Big(\int_{\partial B_\varepsilon(z)} + \sum_{k=1}^{n}\Big\{\int_{\partial B_\varepsilon((k-\frac{1}{2})\pi)} + \int_{\partial B_\varepsilon(-(k-\frac{1}{2})\pi)}\Big\}\Big)\frac{\sin\xi\, d\xi}{(\xi - z)\cos\xi}.$$

Now Cauchy's formula implies

$$\frac{1}{2\pi i}\int_{\partial B_\varepsilon(z)}\frac{\sin\xi\, d\xi}{(\xi - z)\cos\xi} = \frac{\sin z}{\cos z} = \tan z.$$

Also

$$\frac{1}{2\pi i}\int_{\partial B_\varepsilon((k-\frac{1}{2})\pi)}\frac{\sin\xi\, d\xi}{(\xi - z)\cos\xi} = \frac{(-1)^k\sin(k-\frac{1}{2})\pi}{\big((k-\frac{1}{2})\pi - z\big)\operatorname{sinc} 0} = \frac{-1}{(k-\frac{1}{2})\pi - z}$$

and

$$\frac{1}{2\pi i}\int_{\partial B_\varepsilon(-(k-\frac{1}{2})\pi)}\frac{\sin\xi\, d\xi}{(\xi - z)\cos\xi} = -\frac{(-1)^k\sin\big(-(k-\frac{1}{2})\pi\big)}{\big(-(k-\frac{1}{2})\pi - z\big)\operatorname{sinc} 0} = \frac{1}{(k-\frac{1}{2})\pi + z}$$

It follows that

$$\begin{aligned}
K_n &= \tan z + \sum_{k=1}^{n}\Big\{\frac{-1}{(k-\frac{1}{2})\pi - z} + \frac{1}{(k-\frac{1}{2})\pi + z}\Big\} \\
&= \tan z - 2z\sum_{k=1}^{n}\frac{1}{(k-\frac{1}{2})^2\pi^2 - z^2}.
\end{aligned}$$

Taking the limit as $n \to \infty$ yields

$$\tan z = 2z \sum_{k=1}^{\infty} \frac{1}{(k-\frac{1}{2})^2\pi^2 - z^2} = 2z \sum_{k=0}^{\infty} \frac{1}{(k+\frac{1}{2})^2\pi^2 - z^2}$$

which is the second formula.

We now derive the third formula

$$\cot z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2}.$$

Using a modification of the the work for the formula for $1/\sin z$ in the book, define

$$L_n = \frac{1}{2\pi i} \int_{C_n} \frac{\cos\xi \, d\xi}{\xi(\xi - z)\sin\xi}$$

where C_n is the square with corners $(n+1/2)(\pm 1 \pm i)\pi$. In particular,

$$C_n = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$$

where

$$\begin{aligned}
\Gamma_1 &= \{\, (n+1/2)\pi + it\pi : t \in [-n-1/2), n+1/2] \,\} \\
\Gamma_2 &= \{\, -t\pi + i(n+1/2)\pi : t \in [-n-1/2), n+1/2] \,\} \\
\Gamma_3 &= \{\, -(n+1/2)\pi - it\pi : t \in [-n-1/2), n+1/2] \,\} \\
\Gamma_4 &= \{\, t\pi - i(n+1/2)\pi : t \in [-n-1/2), n+1/2] \,\}.
\end{aligned}$$

Combining the identity

$$\begin{aligned}
|\sin(a+ib)|^2 &= (\sin^2 a)(\cosh^2 b) + (\cos^2 a)(\sinh^2 b) \\
&= (\sin^2 a)(\cosh^2 b) \mp (\sin^2 a)(\sinh^2 b) + (\cos^2 a)(\sinh^2 b) \\
&= \sin^2 a + \sinh^2 b
\end{aligned}$$

with $(**)$ yields

$$\left|\cot(a+ib)\right|^2 = \frac{(\cos^2 a)(\cosh^2 b) + (\sin^2 a)(\sinh^2 b)}{(\sin^2 a)(\cosh^2 b) + (\cos^2 a)(\sinh^2 b)} = \frac{\cos^2 a + \sinh^2 b}{\sin^2 a + \sinh^2 b}.$$

It follows for points on Γ_1 that

$$\left|\cot((n+1/2)\pi + it\pi)\right|^2 = \frac{\sinh^2 t\pi}{1 + \sinh^2 t\pi} \le 1.$$

For points on Γ_2 note that

$$\begin{aligned}
\left|\cot(-t\pi + i(n+1/2)\pi)\right|^2 &= \frac{\cos^2 t\pi + \sinh^2(n+1/2)\pi}{\sin^2 t\pi + \sinh^2(n+1/2)\pi} \\
&\le \frac{1 + \sinh^2(n+1/2)}{\sinh^2(n+1/2)} \to 1 \qquad \text{as} \qquad n \to \infty.
\end{aligned}$$

Therefore, there exists n_0 large enough that $n \geq n_0$ implies

$$|\cot \xi|^2 \leq 2 \qquad \text{for every} \qquad \xi \in C_n.$$

It follows for $n \geq n_0$ that

$$\begin{aligned}|L_n| &= \left|\frac{1}{2\pi i}\int_{C_n} \frac{\cos\xi\, d\xi}{\xi(\xi - z)\sin\xi}\right| \leq \frac{1}{2\pi}\int_{C_n} \frac{2|d\xi|}{|\xi|\big(|\xi| - |z|\big)} \\ &\leq \frac{1}{\pi}\,\frac{4(2n+1)}{(n+1/2)\pi\big((n+1/2)\pi - |z|\big)} \to 0 \qquad \text{as} \qquad n \to \infty.\end{aligned}$$

Now

$$L_n = \frac{1}{2\pi i}\bigg(\int_{\partial B_\varepsilon(z)} + \sum_{k=-n}^{n}\int_{\partial B_\varepsilon(k\pi)}\bigg)\frac{\cos\xi\, d\xi}{\xi(\xi - z)\sin\xi}.$$

By Cauchy's formula

$$\frac{1}{2\pi i}\int_{\partial B_\varepsilon(z)} \frac{\cos\xi\, d\xi}{\xi(\xi - z)\sin\xi} = \frac{\cos z}{z\sin z} = \frac{\cot z}{z}.$$

When $k \neq 0$ we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial B_\varepsilon(k\pi)} \frac{\cos\xi\, d\xi}{\xi(\xi - z)\sin\xi} &= \frac{1}{2\pi i}\int_{\partial B_\varepsilon(k\pi)} \frac{\cos\xi\, d\xi}{\xi(\xi - z)(-1)^k(\xi - k\pi)\operatorname{sinc}(\xi - k\pi)} \\ &= \frac{\cos k\pi}{k\pi(k\pi - z)(-1)^k\operatorname{sinc} 0} = \frac{1}{k\pi(k\pi - z)}\end{aligned}$$

Since $\operatorname{sinc}\xi$ is an even function which is differentiable, then

$$\frac{d\operatorname{sinc}\xi}{d\xi}\bigg|_{\xi=0} = \operatorname{sinc}'(0) = 0.$$

Therefore, when $k = 0$ Cauchy's derivative formula yields

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\partial B_\varepsilon(0)} \frac{\cos\xi\, d\xi}{\xi(\xi - z)\sin\xi} \\ &\quad= \frac{1}{2\pi i}\int_{\partial B_\varepsilon(0)} \frac{\cos\xi\, d\xi}{\xi^2(\xi - z)\operatorname{sinc}\xi} = \frac{d}{d\xi}\,\frac{\cos\xi}{(\xi - z)\operatorname{sinc}\xi}\bigg|_{\xi=0} \\ &\quad= \frac{-\sin\xi(\xi - z)\operatorname{sinc}\xi - \cos\xi\big(\operatorname{sinc}\xi + (\xi - z)\operatorname{sinc}'\xi\big)}{(\xi - z)^2\operatorname{sinc}^2\xi}\bigg|_{\xi=0} = \frac{-1}{z^2}\end{aligned}$$

It follows taking $n \to \infty$ that

$$\frac{\cot z}{z} = \frac{1}{z^2} - \sum_{k=1}^{\infty}\bigg(\frac{1}{k\pi(k\pi - z)} + \frac{1}{k\pi(k\pi + z)}\bigg) = \frac{1}{z^2} - \sum_{k=1}^{\infty}\frac{2k\pi}{k\pi(k^2\pi^2 - z^2)}.$$

Consequently

$$\cot z = \frac{1}{z} - 2z \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2 - z^2} = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{1}{z^2 - k^2\pi^2}.$$

Next prove the fourth formula

$$\frac{1}{\sin^2 z} = \sum_{n=-\infty}^{n=\infty} \frac{1}{(z - n\pi)^2}. \tag{3-63}$$

Define

$$M_n = \frac{1}{2\pi i} \int_{C_n} \frac{d\xi}{(\xi - z)\sin^2 \xi}$$

where C_n is the contour in the previous problem with corners at $(n+1/2)(\pm 1 \pm i)\pi$. Upon using the estimate

$$|\sin(a+ib)|^2 \geq (\sin^2 a)(\cosh^2 b) \geq 4^{-1}(\sin^2 a)e^{|b|}$$

on the vertical boundaries given by Γ_1 and Γ_3 and

$$|\sin(a+ib)|^2 = \cosh^2 b - \cos^2 a \geq 4^{-1}e^{2|b|} - 1$$

on the horizontal boundaries given by Γ_2 and Γ_4, that the same arguments as in the proof of the formula for $1/\cos z$ again show that $M_n \to 0$ as $n \to \infty$. Cauchy's derivative formula yields that

$$\begin{aligned}
&\int_{\partial B_\varepsilon(k\pi)} \frac{d\xi}{(\xi - z)\sin^2 \xi} \\
&\qquad = \int_{\partial B_\varepsilon(k\pi)} \frac{d\xi}{(\xi - z)(\xi - k\pi)^2 \operatorname{sinc}^2(\xi - k\pi)} \\
&\qquad = \frac{d}{d\xi} \frac{1}{(\xi - z)\operatorname{sinc}^2(\xi - k\pi)}\Big|_{\xi = k\pi} \\
&\qquad = \frac{-\operatorname{sinc}^2(\xi - k\pi) - 2(\xi - z)\operatorname{sinc}(\xi - k\pi)\operatorname{sinc}'(\xi - k\pi)}{(\xi - z)^2 \operatorname{sinc}^4(\xi - k\pi)}\Big|_{\xi = k\pi} \\
&\qquad = \frac{-1}{(k\pi - z)^2}.
\end{aligned}$$

By Cauchy's formula we obtain

$$M_n = \frac{1}{\sin^2 z} + \sum_{k=-n}^{n} \frac{-1}{(k\pi - z)^2}.$$

Noting that the series is absolutely convergence and taking the limit $n \to \infty$ yields

$$\frac{1}{\sin^2 z} = \sum_{k=-\infty}^{\infty} \frac{1}{(k\pi - z)^2} = \sum_{k=-\infty}^{\infty} \frac{1}{(z - k\pi)^2}.$$

Now prove the fifth formula

$$\frac{1}{\cos^2 z} = \sum_{n=-\infty}^{\infty} \frac{1}{\big(z-(n+1/2)\pi\big)^2} \tag{3-64}$$

Define

$$P_n = \frac{1}{2\pi i}\int_{C_n} \frac{d\xi}{(\xi - z)\cos^2\xi}$$

where C_n is the contour with corners $n(\pm 1 \pm i)\pi$. Again following the proof for the formula $1/\cos z$ it follows that $P_n \to 0$ as $n \to \infty$. Cauchy's derivative formula then implies

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\partial B((k-\frac{1}{2})\pi)} \frac{d\xi}{(\xi - z)\cos^2\xi} \\
&\quad = \frac{1}{2\pi i}\int_{\partial B((k-\frac{1}{2})\pi)} \frac{d\xi}{(\xi - z)\big(\xi-(k-\frac{1}{2})\pi\big)^2 \operatorname{sinc}^2\big(\xi-(k-\frac{1}{2})\pi\big)} \\
&\quad = \frac{d}{d\xi}\frac{1}{(\xi - z)\operatorname{sinc}^2\big(\xi-(k-\frac{1}{2})\pi\big)}\Big|_{\xi=(k-\frac{1}{2})\pi} \\
&\quad = \frac{-1}{\big((k-\frac{1}{2})\pi - z\big)^2}
\end{aligned}$$

It follows that

$$P_n = \frac{1}{\cos^2 z} + \sum_{k=-n+1}^{n} \frac{-1}{\big((k-\frac{1}{2})\pi - z\big)^2}.$$

Noting that the series is absolutely convergence and taking the limit $n \to \infty$ yields

$$\frac{1}{\cos^2 z} = \sum_{k=-\infty}^{\infty} \frac{1}{\big((k-\frac{1}{2})\pi - z\big)^2} = \sum_{k=-\infty}^{\infty} \frac{1}{\big(z-(k+\frac{1}{2})\pi\big)^2}.$$