Maximum Modulus Theorem. Suppose f is analytic on a connected open set Ω and that $|f(z)|$ attains a maximum at some point $z_0 \in \Omega$. Then f is identically constant.

Proof. Suppose $R > 0$ is chosen so that a neighborhood of the disk of radius R centered at z_0 is contained in Ω. Let $A = \{\,\zeta : |\zeta - z_0| < R\,\}$ be the disk. The proof now procedes in four steps.

Step 1. Claim that

$$\frac{1}{\pi R^2}\int_A |f(\zeta)|\,dA = |f(z_0)|.$$

Proof of Step 1. Let $\gamma(t) = z_0 + re^{2\pi it}$ where $0 < r < R$. By the Cauchy formula

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_{[\gamma]} \frac{f(\zeta)}{\zeta - z_0}\,d\zeta = \frac{1}{2\pi i}\int_0^1 \frac{f(z_0 + re^{2\pi it})}{re^{2\pi it}} r2\pi i e^{2\pi it}\,dt \\ &= \int_0^1 f(z_0 + re^{2\pi it})\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta. \end{aligned}$$

Therefore

$$\frac{R^2}{2} f(z_0) = \int_0^R f(z_0)\,rdr = \frac{1}{2\pi}\int_0^R\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta\,rdr = \frac{1}{2\pi}\int_A f(\zeta)\,dA.$$

Since $|f(\zeta)| \le |f(z_0)|$ for every $\zeta \in A$ then

$$|f(z_0)| \le \frac{1}{\pi R^2}\int_A |f(\zeta)|\,dA \le \frac{1}{\pi R^2}\int_A |f(z_0)|\,dA = |f(z_0)|.$$

Therefore, this inequality must in fact be an equality.

Step 2. Claim that $|f(\zeta)| = |f(z_0)|$ for every $\zeta \in A$.

Proof of Step 2. For contradiction suppose there was $\zeta_0 \in A$ such that $|f(\zeta_0)| < |f(z_0)|$. Since f is continuous there exists $h > 0$ and $\delta > 0$ such that $\delta < |z_0 - \zeta_0|$ and

$$|f(\zeta)| + h < |f(z_0)| \qquad \text{for every} \qquad |\zeta - \zeta_0| < \delta.$$

Define $B = \{\,\zeta : |\zeta - \zeta_0| < \delta\,\}$. Then $B \subseteq A$ and

$$\begin{aligned} |f(z_0)| &= \frac{1}{\pi R^2}\int_A |f(\zeta)|\,dA = \frac{1}{\pi R^2}\int_{A\setminus B} |f(\zeta)|\,dA + \frac{1}{\pi R^2}\int_B |f(\zeta)|\,dA \\ &= \frac{1}{\pi R^2}\int_{A\setminus B} |f(z_0)|\,dA + \frac{1}{\pi R^2}\int_B (|f(z_0)| - h)\,dA \\ &= \frac{1}{\pi R^2}\int_A |f(z_0)|\,dA - \frac{1}{\pi R^2}\int_B h\,dA = |f(z_0)| - h\frac{\delta^2}{R^2} < |f(z_0)| \end{aligned}$$

which is a contradiction.

Step 3. Claim that $f(\zeta) = f(z_0)$ for every $\zeta \in A$.

Proof of Step 3. If $|f(z_0)| = 0$ then $|f(\zeta)| = 0$ for every $\zeta \in A$. Thus $f(\zeta) = 0 = f(z_0)$ for every $\zeta \in A$ and we are done. Otherwise, Let $f(\zeta) = u(x,y)+iv(x,y)$ where $\zeta = x+iy$. Since $|f(\zeta)|$ is constant for $\zeta \in A$ we have that

$$\frac{d}{dx}|f(\zeta)|^2 = 2uu_x + 2vv_x = 0 \qquad \text{and} \qquad \frac{d}{dy}|f(\zeta)|^2 = 2uu_y + 2vv_y = 0.$$

Applying the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ we obtain

$$uu_y + vu_x = 0 \qquad \text{and} \qquad uu_x - vu_y = 0$$

After a little algebra one obtains

$$(u^2 + v^2)u_y = 0 \qquad \text{and} \qquad (u^2 + v^2)u_x = 0.$$

Since $u^2 + v^2 = |f(z_0)|^2 > 0$ then the above equations imply that $u_y = 0$ and $u_x = 0$. Therefore u is constant. Since $u_y = 0$ and $u_x = 0$ the Cauchy-Riemann equations imply $v_x = 0$ and $v_y = 0$. Therefore v is also constant. If follows that $f(\zeta) = f(z_0)$ for all $\zeta \in A$.

Step 4. Claim that f is constant on all of Ω.

Proof of Step 4. In class we drew some intersecting circles and said since Ω was connected then f must be constant on all of Ω. For those who have taken a course in point-set topology here are the full details.

Define $W = \{\, z \in \Omega : f(z) = f(z_0)\,\}$. Since f is continuous then W is closed in the topology relative to Ω. We shall show W is also open. For each $z \in W$ choose $R_z > 0$ so that a neighborhood of the disk $A_z = \{\, \zeta : |\zeta - z| < R_z\}$ is contained in Ω. The previous steps applied to A_z implies $f(\zeta) = f(z)$ for every $\zeta \in A_z$. Since $f(z) = f(z_0)$ for every $z \in W$ then $A_z \subseteq W$ for every $z \in W$. Therefore

$$W \subseteq \bigcup_{z\in W} A_z \subseteq W \qquad \text{implies} \qquad W = \bigcup_{z\in W} A_z.$$

Since the A_z are open, then their union W is open. Therefore W is both open and closed in the topology relative to Ω. If follows that $\Omega \setminus W$ is also both open and closed in the relative topology. Since Ω is connected it can not be equal any nontrivial disjoint union of open sets. Since $z_0 \in W$ is follows that $W = \Omega$. Therefore f is constant on all of Ω.