Discrete Haar Transform. Let $N = 2^J$ where $J \in \mathbf{N}$ and suppose that $y_k \in \mathbf{R}$ for $k = 0, 1, \ldots, N-1$ and that

$$f = \sum_{k=0}^{N-1} y_k p_{J,k}.$$

Since $\langle f, h_{j,k} \rangle = 0$ for $j \geq J$ then f may be written as

$$f = \langle f, p_{0,0} \rangle p_{0,0} + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \langle f, h_{j,k} \rangle h_{j,k}.$$

The following C subroutine computes the coefficents

$$\langle f, p_{0,0} \rangle, \langle f, h_{0,0} \rangle, \langle f, h_{1,0} \rangle, \langle f, h_{1,1} \rangle, \ldots, \langle f, h_{J-1,N/2-1} \rangle$$

in terms of y_k using $\mathcal{O}(N)$ operations:

```
1 #include <math.h>
2 void haar1d(double y[],int N){
3     double e[N/2],o[N/2];
4     int n;
5     for(n=N/2;n>=1;n/=2){
6         int k;
7         for(k=0;k<n;k++){
8             e[k]=y[2*k];
9             o[k]=y[2*k+1];
10        }
11        for(k=0;k<n;k++){
12            y[k]=(e[k]+o[k])/sqrt(2.0);
13            y[k+n]=(e[k]-o[k])/sqrt(2.0);
14        }
15    }
16 }
```

Note that to save storage the above subroutine overwrites the array y_k in stages.

Proof. Correctness of the code may be seen by examining the contents of y_k at the end of each iteration through the loop beginning on line 5 and ending on line 15. The first time through the loop $n = N/2$ and we have by the splitting theorem that

$$\begin{aligned} f = \sum_{k=0}^{N-1} y_k p_{J,k} &= \sum_{k=0}^{N/2-1} e_k p_{J,2k} + \sum_{k=0}^{N/2-1} o_k p_{J,2k+1} \\ &= \sum_{k=0}^{N/2-1} \frac{e_k + o_k}{\sqrt{2}} p_{J-1,k} + \sum_{k=0}^{N/2-1} \frac{e_k - o_k}{\sqrt{2}} h_{J-1,k}. \end{aligned}$$

Overwriting the storage used by y_k as

$$y_k \leftarrow \frac{e_k + o_k}{\sqrt{2}} \qquad \text{and} \qquad y_{k+N/2} \leftarrow \frac{e_k - o_k}{\sqrt{2}} \qquad \text{for} \qquad k = 0, 1, \ldots, N/2 - 1$$

yields at the beginning of the second iteration through the loop on line 5 that

$$n = N/4 \qquad \text{and} \qquad f = \sum_{k=0}^{N/2-1} y_k p_{J-1,k} + \sum_{k=N/2}^{N-1} y_k h_{J-1,k-N/2}.$$

Now leaving the values of y_k where $k = N/2, N/2+1, \ldots N-1$ untouched, the splitting theorem is again applied to the coeficients y_k where $k = 0, 1, \ldots, N/2-1$ in the first sum. Thus, at the beginning of the third iteration

$$n = N/8 \qquad \text{and} \qquad f = \sum_{k=0}^{N/4-1} y_k p_{J-2,k} + \sum_{k=N/4}^{N/2-1} y_k h_{J-2,k-N/4} + \sum_{k=N/2}^{N-1} y_k h_{J-1,k-N/2}.$$

The next iteration applies the splitting theorem to y_k where $k = 0, 1, \ldots, N/4-1$ and so forth until at the last iteration we have

$$n = 1 \qquad \text{and} \qquad f = \sum_{k=0}^{2-1} y_k p_{1,k} + \sum_{k=2}^{4-3} y_k h_{1,k-2} + \cdots + \sum_{k=N/2}^{N-1} y_k h_{J-1,k-N/2}.$$

We obtain at exit from the subroutine that

$$f = y_0 p_{0,0} + y_1 h_{0,0} + \sum_{k=2}^{4-3} y_k h_{1,k-2} + \cdots + \sum_{k=N/2}^{N-1} y_k h_{J-1,k-N/2}.$$

The correspondance between coefficents and basis functions may be summarized as

y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	$\cdots$	y_{15}	y_{16}	$\cdots\cdots$	y_{N-1}
$p_{0,0}$	$h_{0,0}$	$h_{1,0}$	$h_{1,1}$	$h_{2,0}$	$h_{2,1}$	$h_{2,2}$	$h_{2,3}$	$h_{3,0}$	$\cdots$	$h_{3,7}$	$h_{4,0}$	$\cdots\cdots$	$h_{J-1,N/2-1}$

To verify the efficiency of the subroutine let $T(N)$ be how many operations are needed to compute the Haar transform of length N. Then $T(N) = T(N/2) + \alpha N$ where $T(1) = 0$ and $\alpha > 0$. Solving the recurrence yields that

$$\begin{aligned} T(2^J) &= T(2^{J-1}) + \alpha 2^J = T(2^{J-2}) + \alpha 2^{J-1} + \alpha 2^J \\ &= T(1) + \alpha 2 + \alpha 2^2 + \cdots + \alpha 2^J = 2\alpha(2^J - 1) = \mathcal{O}(N). \end{aligned}$$