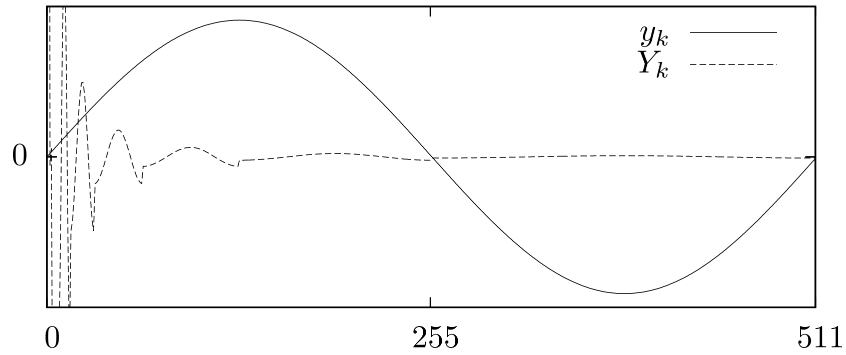
Math 761 Haar Wavelet Transform Project

Digital Image Compression and Analysis

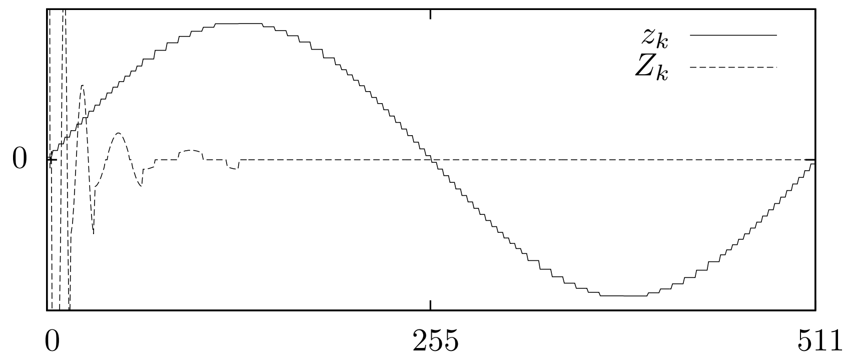
Define $y_n = \sin x_n$ for $n = 0, 1, \ldots, 511$ where $x_n = 2\pi n/512$. Let Y_n be the Haar wavelet transform of y_n. Plotting y_n and Y_n on the same graph we obtain



From the graph it is clear that a large percentage of the Y_n are close to zero. A simple compression scheme is to throw away all Y_n whose absolute value is less than some $\epsilon > 0$. For example, taking $\epsilon = 0.05$ and defining

$$Z_n = \begin{cases} 0 & \text{for } |Y_n| < \epsilon \\ Y_n & \text{otherwise} \end{cases}$$

throws away 82% of the Haar coefficients in this example. Now let z_n be the inverse Haar wavelet transform of Z_n. Plotting z_n and Z_n on the same graph we obtain



Notice that the approximation z_k is made up of tiny stair steps with varying size depending on the slope of the function. The orthogonality of the Haar system implies a certain optimality of the stair steps. This project explores the use of this idea in two dimensions to compress and analyze images.

Part 1. Write a program to compute the two-dimensional Haar transform of a matrix or download the source code `haar2d`. Let $y_{p-1,q-1}$ be the entries of an arbitrary 512×512 matrix and $Y_{m,n}$ be the two-dimensional Haar transform of $y_{m,n}$. Verify numerically that

$$\sum_{m=0}^{511} \sum_{n=0}^{511} |y_{m,n}|^2 = \sum_{m=0}^{511} \sum_{n=0}^{511} |Y_{m,n}|^2.$$

Part 2. The file `image1.dat` contains a black and white image represented as a 512×512 matrix with entries $y_{p-1,q-1}$ where 0 represents black and 1 represents white. Let $Y_{m,n}$ be the two dimensional Haar transform of $y_{m,n}$ and

$$Z_{m,n} = \begin{cases} 0 & \text{for } |Y_{m,n}| < \epsilon \\ Y_{m,n} & \text{otherwise.} \end{cases}$$

Write a program to compute the number of Haar coefficients set to zero

$$\eta = \operatorname{card}\{\, (m,n) : Z_{m,n} = 0 \,\}$$

and the relative error

$$E = \sqrt{\sum_{m=0}^{511} \sum_{n=0}^{511} |Z_{m,n} - Y_{m,n}|^2 \Big/ \sum_{m=0}^{511} \sum_{n=0}^{511} |Y_{m,n}|^2}.$$

for $\epsilon = 0.1$.

Part 3. The file `image2.dat` contains a blurred copy of `image1.dat`. Let $\tilde{y}_{p-1,q-1}$ be the entries for the matrix corresponding to this image and $\tilde{\eta}$ and $\tilde{E}$ depending on $\tilde{\epsilon} > 0$ be defined as above. Find a value of $\tilde{\epsilon}$ such that $\tilde{E} \approx E$. What is $\tilde{\eta}$ for this value of $\tilde{\epsilon}$? Is $\tilde{\eta}$ larger or smaller than the η from part 2? Explain your result.

Part 4. [Extra Credit] Let

$$c_{jkl} = \int_0^1 \int_0^1 f(x,y) h_{j,k}(x) h_{j,l}(y)\, dx\, dy$$

be the coefficients corresponding the two-dimensional Haar basis functions $h_{j,k}(x)h_{j,l}(y)$ where $k, l = 0, 1, \ldots, 2^j - 1$. Use techniques similar to those used in one-dimension to find positive constants α and β such that if f is continuously differentiable then

$$|c_{jkl}| = \mathcal{O}\big(2^{-\alpha j}\big) \qquad \text{as} \qquad j \to \infty$$

and if f has jump discontinuities then

$$|c_{jkl}| = \mathcal{O}\big(2^{-\beta j}\big) \qquad \text{as} \qquad j \to \infty.$$

Compute

$$M_j = \max\{\, |c_{jkl}| : k, l = 0, 1, \ldots, 2^j - 1 \,\} \qquad \text{for} \qquad j = 0, 1, \ldots, 8$$

for `image1.dat` and `image2.dat`. Plot $\log M_j$ versus j and compare the rate of decay in each case to the asymptotic rates $\mathcal{O}(2^{-\alpha j})$ and $\mathcal{O}(2^{-\beta j})$ found analytically. Is there a difference between the sharp image and the blurred one?