A Relaxed Direct-insertion Downscaling Method For Discrete-in-time Data Assimilation

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Abstract

This paper improves the spectrally-filtered direct-insertion downscaling method for discrete-in-time data assimilation by introducing a relaxation parameter that overcomes a constraint on the observation frequency. Numerical simulations demonstrate that taking the relaxation parameter proportional to the time between observations allows one to vary the observation frequency over a wide range while maintaining convergence of the approximating solution to the reference solution. Under the same assumptions we analytically prove that taking the observation frequency to infinity results in the continuous-in-time nudging method.

1 Introduction

Our focus is that of a well-posed dissipative dynamical system

$$\frac{dU}{dt} = \mathcal{F}(U) \qquad \text{with} \qquad U(t_0) = U_0 \tag{1.1}$$

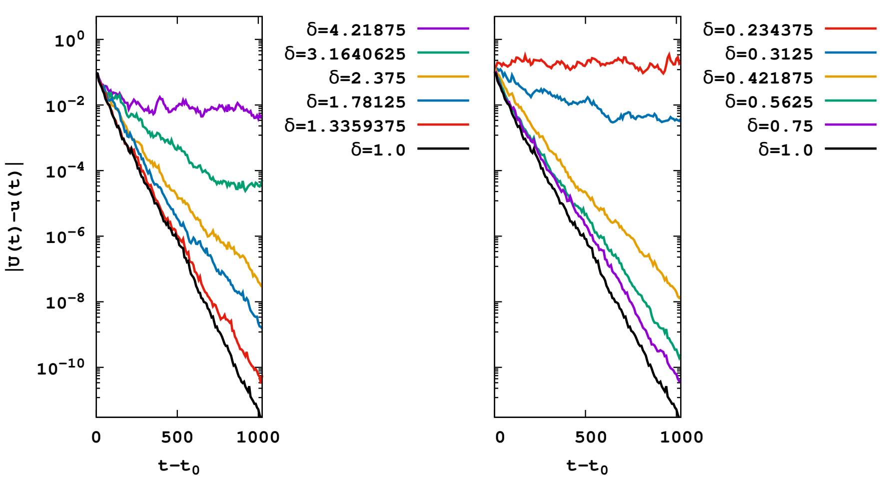
where the initial condition U_0 is unknown but for which a time sequence of partial observations of $U(t)$ are available at times $t = t_n$.

Following [3], [27], related research, the references therein and in particular [6], we consider the concrete setting of the incompressible two-dimensional Navier–Stokes equations. These equations provide an example of a well-posed dissipative dynamical system of the form (1.1) that is simpler than atmospheric or ocean models but with similar nonlinear dynamics. Other systems for which a growing body of analytic and numerical results are available include Rayleigh-Bénard convection studied by Farhat, Lunasin and Titi in [11] (see also Farhat, Glatt-Holtz, Martinez, McQuarrie and Whitehead [12]) as well as the surface quasi-geostrophic equations studied by Jolly, Martinez and Titi in [20] (see also [21]) among others. Although our computations and analysis focus on the two-dimensional Navier–Stokes equations, we hope the resulting intuition and conclusions will apply to operational models with practical applications.

We begin with the spectrally-filtered discrete-in-time downscaling data assimilation algorithm introduced in [6] in the context of the incompressible two-dimensional Navier–Stokes equations. This is a direct-insertion method that recovers unobserved lengthscales by inserting new observational measurements as they are available into the current estimate of the state. The spectral filter helps ensure no high-resolution artifacts are present in the interpolated observational data that might damage the approximation obtained by the numerical model as it is integrated forward in time. From a theoretical point of view, the filtering is needed to obtain suitable estimates in the higher Sobolev norms used to show convergence of the approximating solution to the reference solution over time.

Intuitively more frequent observations should make the predicted state obtained from data assimilation more accurate. For example, in weather forecasting measuring the state of the atmosphere more frequently should—provided a reasonable algorithm is used—allow an approximating solution to track the reference solution more accurately. However, although more frequent observations represent greater knowledge about the reference solution, Figure 1 shows the spectrally-filtered discrete-in-time data assimilation algorithm introduced in [6] actually performs worse when data is inserted more frequently into the model.

Figure 1: The error $|U(t)-u(t)|$ for a reference solution $U(t)$ where the approximating solution $u(t)$ was computed for different observation intervals δ. Large δ is shown on the left; small on the right.



The possibility of this unreasonable behavior was already noticed in the theoretical analysis presented in [6] and commented on as

[Our analysis makes] use of a minimum distance between t_{n+1} and t_n as well as the maximum. Measurements need to be inserted frequently enough to overcome the tendency for two nearby solutions to drift apart, while at the same time the possible lack of orthogonality in our general interpolant observables means measurements should

not be inserted too frequently.

The present research is motivated by numerical evidence that the above constraint concerning the minimum observation frequency is physical and not merely a limitation of our analytic techniques. In particular, given the ideal situation of noise-free observations of exact dynamics in which the approximating solution synchronizes with the reference solution over time, our computations show that inserting measurements more frequently can worsen the quality of the approximation to the point where it subsequently fails to synchronize.

Recall the incompressible two-dimensional Navier-Stokes equations given by

$$\frac{\partial U}{\partial t} + (U\cdot\nabla)U - \nu\Delta U + \nabla p = f, \qquad \nabla\cdot U = 0, \tag{1.2}$$

where U is the velocity of the fluid, ν is the kinematic viscosity, p is the pressure and f is a time-independent body force applied to the fluid. Consider the spatial domain Ω with L-periodic boundary conditions for simplicity.

Recast (1.2) in the functional setting described by Constantin and Foias [8] (see also Robinson [24] or Temam [25]) as follows. Denote by $\mathcal{V}$ the set of all divergence-free L-periodic trigonometric polynomials with zero spatial averages. Let V be the closure of $\mathcal{V}$ in $H^1(\Omega, \mathbf{R}^2)$ and H be the closure of $\mathcal{V}$ in $L^2(\Omega, \mathbf{R}^2)$. Denote the dual of V by V^*.

Let $A\colon V \to V^*$ and $B\colon V\times V \to V^*$ be continuous extensions of the operators given by $Au = -P_H\Delta u$ and $B(u,v) = P_H(u\cdot\nabla v)$ for $u, v \in \mathcal{V}$ where P_H is the orthogonal projection of $L^2(\Omega)$ onto H. Let $\mathcal{D}(A) = \{u \in V : Au \in H\}$ be the domain of A into H. Applying P_H to both sides of (1.2) then expresses the Navier–Stokes equations as

$$\frac{dU}{dt} + \nu AU + B(U,U) = f \qquad \text{where} \qquad U(t_0) = U_0 \in V. \tag{1.3}$$

Note in the periodic case $A = -\Delta$ and that we have assumed $P_H f = f$.

Along with the above functional notation the following *a priori* bounds also appear in [8], [24] and [25] for the incompressible Navier–Stokes equations:

Theorem 1.1. *Let U be a solution to* (1.3) *with $f \in V$ and initial condition $U_0 \in \mathcal{D}(A)$. There exist bounds ρ_α and $\widetilde{\rho}_\alpha$ depending only on $\delta_{\max}$, ν, $\|f\|$ and U_0 such that*

$$|A^{\alpha/2}U| \le \rho_\alpha \quad \textit{for} \quad \alpha = 0,1,2 \tag{1.4}$$

and

$$\Big(\int_{t_n}^{t_{n+1}} |A^{\alpha/2}U|^2 dt\Big)^{1/2} \le \widetilde{\rho}_\alpha \quad \textit{for} \quad \alpha = 1,2,3. \tag{1.5}$$

Furthermore, (1.3) *posses a unique global attractor $\mathcal{A}$ bounded in $\mathcal{D}(A)$. Moreover, if U lies on that global attractor, then the constants ρ_α and $\widetilde{\rho}_\alpha$ may be taken independent of U_0.*

The spaces H, V and $\mathcal{D}(A)$ can be characterized in terms of Fourier series as $H = V_0$, $V = V_1$ and $\mathcal{D}(A) = V_2$ where

$$V_\alpha = \bigg\{ \sum_{k\in\mathcal{J}} u_k e^{ik\cdot x} : \sum_{k\in\mathcal{J}} |k|^{2\alpha}|u_k|^2 < \infty, \quad k\cdot u_k = 0 \quad \text{and} \quad u_{-k} = \overline{u_k} \bigg\}.$$

Here $\mathcal{J} = \{\, 2\pi n/L : n \in \mathbf{Z}^2 \setminus \{0\}\}$ and $u_k \in \mathbf{C}^2$ are the Fourier coefficients for the velocity field u. The zero spatial average along with Parseval's identity implies the norms and inner products for the V_α spaces are equivalent to

$$\|u\|_\alpha = \Big(L^2 \sum_{k\in\mathcal{J}} |k|^{2\alpha} |u_k|^2 \Big)^{1/2} \qquad \text{and} \qquad ((u,v))_\alpha = L^2 \sum_{k\in\mathcal{J}} |k|^{2\alpha} u_k \overline{v_k}.$$

Recall that $V_\alpha \subseteq V_\beta$ for $\beta \le \alpha$ with corresponding Poincaré inequalities

$$\lambda_1^{\alpha-\beta} \|u\|_\beta^2 \le \|u\|_\alpha^2 \qquad \text{for} \qquad u \in V_\alpha, \tag{1.6}$$

where $\lambda_1 = (2\pi/L)^2$. Denote the norms corresponding to H, V and $\mathcal{D}(A)$ by

$$|u| = \|u\|_0, \qquad \|u\| = \|u\|_1 \qquad \text{and} \qquad |Au| = \|u\|_2$$

and the inner products by

$$(u,v) = ((u,v))_0, \qquad ((u,v)) = ((u,v))_1 \qquad \text{and} \qquad (Au,Av) = ((u,v))_2.$$

We pause to remark that a well-posed dissipative dynamical system of the form (1.1) may now be obtained by taking $\mathcal{F}(U) = f - B(U,U) - \nu AU$.

Let S be the solution operator such that $S(t)U_0 = U(t)$, where $U(t)$ is the unique solution to (1.3) with initial condition U_0 at time $t_0 = 0$. Well-posedness along with the fact that the dynamics are autonomous implies $S(t+\delta) = S(\delta)S(t)$ for $t \ge 0$ and $\delta \ge 0$.

The spectral filter appearing in [6] is given for $u \in H$ as

$$P_\lambda u = \sum_{0<|k|^2\le\lambda} u_k e^{ik\cdot x} \quad \text{where} \quad u = \sum_{k\in\mathcal{J}} u_k e^{ik\cdot x}. \tag{1.7}$$

Note the interpolation property

$$|u - P_\lambda u|^2 = L^2 \sum_{|k|^2>\lambda} |u_k|^2 \le L^2 \sum_{|k|^2>\lambda} \frac{|k|^2}{\lambda} |u_k|^2 \le \lambda^{-1} \|u\|^2 \quad \text{for} \quad u \in V, \tag{1.8}$$

and the smoothing properties

$$\|P_\lambda u\|_\alpha^2 = L^2 \sum_{0<|k|^2\le\lambda} |k|^{2\alpha} |u_k|^2 \le \lambda^\alpha L^2 \sum_{0<|k|^2\le\lambda} |u_k|^2 \le \lambda^\alpha |u|^2 \quad \text{for} \quad u \in H. \tag{1.9}$$

In particular note that $\|P_\lambda u\|^2 \le \lambda |u|^2$ and $|AP_\lambda u|^2 \le \lambda^2 |u|^2$. Similarly $|AP_\lambda u|^2 \le \lambda \|u\|^2$.

Assume the observations are interpolated back into the phase space of U by a linear operator $I_h \colon V \to L^2(\Omega)$ that satisfies

$$|U - P_H I_h U|^2 \le c_1 h^2 \|U\|^2 \quad \text{for every} \quad U \in V \quad \text{and} \quad h > 0. \tag{1.10}$$

Here h is a length scale that reflects the resolution of the observations. Note (1.10) provides a bound on the relative error in the spirit of the Bramble–Hilbert Lemma and that there is no requirement on I_h or $P_H I_h$ to be a projection. Linearity, however, is important.

The operators I_h were introduced in [3] as type-I interpolant observables and generally result from spatially-averaged observations, for example, the determining volume elements of Jones and Titi [22]. Note that observations which result from point measurements are classified as type-II and rely on higher norms such as $|Au|$ to obtain inequalities similar to (1.10). Although the treatment of point measurements is out of scope for the present research, we remark that a physically motivated type-I interpolant involving locally averaged approximations of point measurements in space is described in Appendix B of [5].

Since $P_H I_h$ may not be an orthogonal projection, lack of orthogonality complicates the use of $P_H I_h U$ for data assimilation. Another difficulty is the possible roughness of I_h. Such roughness may be characterized as the case when $P_H I_h U \notin V$. This difficulty was mediated in [6] by the use of P_λ as a smoothing filter. To this end set $J = P_\lambda P_H I_h$ where P_λ is the spectral filter defined by (1.7) and consider the smoothed observations given by JU.

This results in an interpolant that satisfies

$$|U - JU|^2 = |U - P_\lambda U|^2 + |P_\lambda(U - P_H I_h U)|^2 \le (\lambda^{-1} + c_1 h^2)\|U\|^2 \tag{1.11}$$

and

$$\|U - JU\|^2 = \|U - P_\lambda U\|^2 + \|P_\lambda(U - P_H I_h U)\|^2 \le \big(1 + \lambda c_1 h^2\big)\|U\|^2, \tag{1.12}$$

which has the continuity properties

$$|JU| = |U - JU| + |U| \le (\lambda^{-1} + c_1 h^2)^{1/2}\|U\| + \lambda_1^{-1}\|U\| \le c_2\|U\| \tag{1.13}$$

and

$$\|JU\| = \|U - JU\| + \|U\| \le (1 + \lambda c_1 h^2)^{1/2}\|U\| + \|U\| \le c_3\|U\| \tag{1.14}$$

where $c_2 = (\lambda^{-1} + c_1 h^2)^{1/2} + \lambda_1^{-1/2}$ and $c_3 = (1 + \lambda c_1 h^2)^{1/2} + 1$.

To focus on the situation when subsequent observations become more frequent, we assume $t_{n+1} - t_n \le \delta_{\max}$ for all n. We now describe main theorem proved in [6] for the spectrally-filtered discrete-in-time data assimilation algorithm whose improvement is the purpose of the present paper.

Definition 1.2. Given an increasing sequence t_n of observation times, the approximating solution $u(t)$ obtained by *spectrally-filtered discrete-in-time data assimilation* is

$$u(t) = S(t - t_n)u_n \qquad \text{for} \qquad t \in [t_n, t_{n+1}),$$

where

$$\begin{cases} u_0 = JU(t_0), \\ u_{n+1} = (I - J)S(t_{n+1} - t_n)u_n + JU(t_{n+1}). \end{cases}$$

We remark taking $J = P_\lambda$ leads to the discrete-in-time data-assimilation method studied in [18] based on observation of the Fourier modes. In this case the quality of the approximation $u(t)$ does not worsen when measurements are inserted more frequently in time. Thus, lack of orthogonality and smoothness in J appear to be important considerations.

When $t_n = t_0 + \delta n$ for some period δ, the main result from [6] may be stated as

Theorem 1.3. *Let U be a solution to the incompressible two-dimensional Navier–Stokes equations and $u(t)$ be the process given by Definition 1.2. Then, for every $\delta > 0$ there exists $h > 0$ and $\lambda > 0$ depending only on c_1, f and ν such that*

$$|u(t) - U(t)| \to 0 \quad \textit{exponentially in time as} \quad t \to \infty.$$

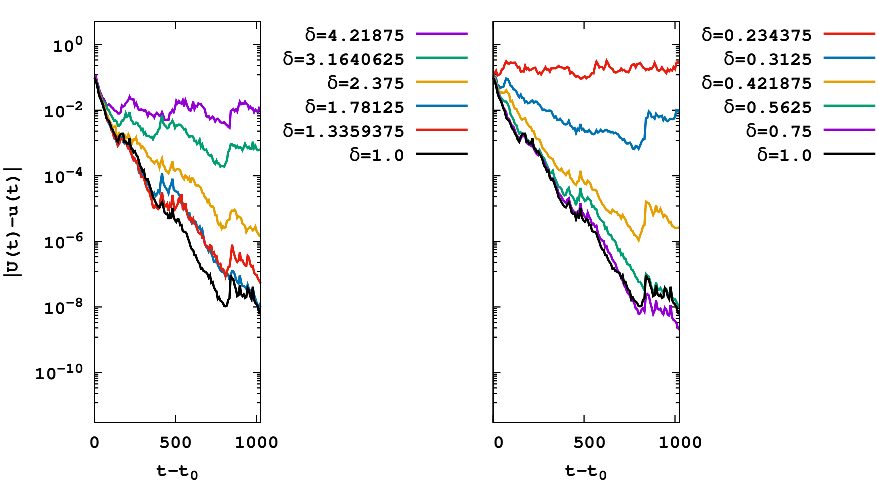
In the presence of model error and noise the synchronization given by Theorem 1.3 as $t \to \infty$ would be only approximate. While it seems reasonable that observing the reference solution less frequently in time with larger δ can be compensated for by requiring higher resolution observations with smaller h, the difficulty we address in this paper is that once the observation period δ is chosen, the theory does not allow more frequent observations without again adjusting the resolution h and filter parameter λ.

For simplicity assume as above that the time between subsequent observations is fixed to be δ. The case where $t_{n+1} - t_n = \delta_n$ with $\max_n \delta_n$ small is interesting and commented on in the conclusions.

Before introducing a relaxation parameter into Definition 1.2 we first present additional numerical evidence to illustrate the constraints on the observation frequency suggested by the proof of Theorem 1.3. Consider a fixed trajectory $U(t)$ on the global attractor of the two-dimensional Navier–Stokes equations obtained by a long-time integration of (1.3) forward in time. Next, apply Definition 1.2 to compute $u(t)$ for different values of δ while holding all other parameters and the trajectory $U(t)$ fixed.

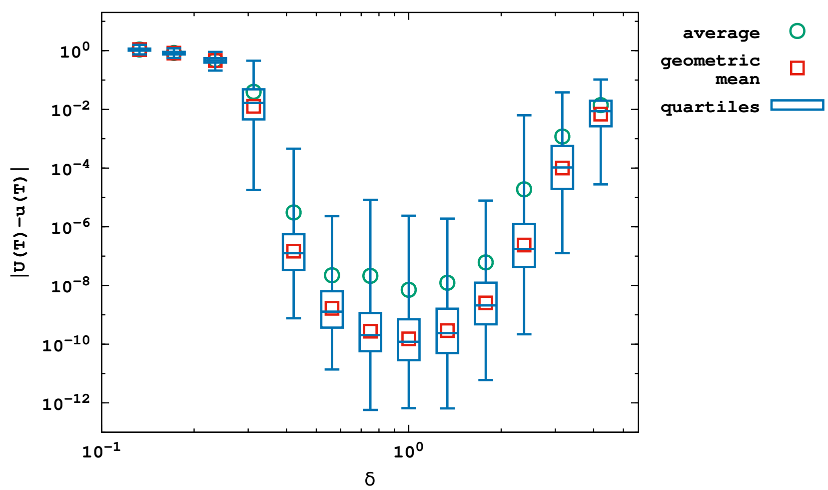
The evolutions of $|U(t) - u(t)|$ for the resulting simulations are depicted in Figure 1. Consider the value of $|U(T) - u(T)|$ at $T = t_0 + 1024$ for the different values of δ and the behavior of each curve leading up to that time. The graph on the left suggests $|U(t) - u(t)|$ does not tend to zero as $t \to \infty$ for values of δ where $\delta > 3$. More surprisingly the graph on the right suggests $|U(t) - u(t)|$ does not tend to zero when $\delta < 0.3$. Thus, taking δ either too small or too large results in failure of the approximating solution u to synchronize with U. This is consistent with the proof of Theorem 1.3.

Figure 2: The same calculation as Figure 1 redone for a different reference solution $U(t)$ lying on the global attractor.



Note the skill by which Definition 1.2 recovers the reference solution $U(t)$ depends on the specific trajectory being observed. Some reference solutions may exhibit dynamical behaviors that are more difficult to recover from the observational data while others are easier to recover. Figure 2 repeats the calculations of Figure 1 for a different reference solution on the global attractor. While the size of $|U(T) - u(T)|$ at $T = t_0 + 1024$ is four decimal orders of magnitude larger in Figure 2 compared to Figure 1, the trends of the curves for large and small values of δ are similar. In particular, taking δ either too small or too large results in failure of the approximating solution u to synchronize with U in both cases while $\delta \approx 1$ is a good choice for the observation interval.

Figure 3: The ensemble average and geometric mean compared to box plots (no outliers removed) of the error $|U(T) - u(T)|$ at time $T = t_0 + 1024$ for 500 simulations varying δ.



To further characterize the rate at which the approximating solution converges to the reference solution for different values of δ, the experiments of Figures 1 and 2 were repeated for an ensemble of 500 reference solutions randomly chosen on the global attractor. Let $\mathcal{E}$ be a fixed ensemble. Compute the ensemble average error for each δ as

$$\langle |U(T) - u(T)| \rangle = \frac{1}{|\mathcal{E}|} \sum_{U \in \mathcal{E}} |U(T) - u(T)|$$

and the geometric mean as

$$\mathrm{GM}\big[|U(T) - u(T)|\big] = \exp\Big(\frac{1}{|\mathcal{E}|} \sum_{U \in \mathcal{E}} \log |U(T) - u(T)|\Big).$$

Here $|\mathcal{E}|$ is the number of solutions in the ensemble.

Figure 3 reports the ensemble averages corresponding to $\delta \in [0.1328125, 4.21875]$ along with the median and some additional statistics. The whiskers in the box plots are long because no outliers have been removed. We remark the ensemble average lies above the median because it is dominated by a the few trajectories in $\mathcal{E}$ that lead to errors at time T

with large magnitudes similar to Figure 2 while the geometric mean and median are much closer. Since the median and geometric mean are more robust in the presence of extreme variations, the δ which minimizes the error shall be identified using these statistics.

We now address the difficulty that more frequent observations leads to lack of synchronization between the approximating solution and reference solution by introducing a relaxation parameter into the data assimilation algorithm given by Definition 1.2. To mitigate the difficulties that occur when inserting observations of the reference solution too frequently into the calculation, it intuitively makes sense to insert those frequent observations less forcefully. We therefore modify the discrete-in-time data assimilation algorithm studied in [6] by introducing a parameter κ which will depend on δ. In details let $\kappa > 0$ and replace the recurrence defining u_{n+1} in Definition 1.2 by

$$u_{n+1} = (I - \kappa J)S(t_{n+1} - t_n)u_n + \kappa JU(t_{n+1}). \tag{1.15}$$

Taking $\kappa = 1$ recovers the original algorithm while $\kappa < 1$ underrelaxes the system. Note that smaller values of κ lead to proportionately smaller updates to the predicted state.

We remark that κ plays a role similar to the term used by the Bayesian update for the ensemble Kalman filter which balances the tradeoff between the accuracy of the predicted state and the quality of the observations (see, for example, Evensen [10] or Grudzien and Bocquet [17]). The context considered here, however, is that of noise-free observations with exactly known dynamics. Thus, it is not noise and model error which lead us to introduce κ but the balance between lack of orthogonality in a rough interpolant and smoothing by the dynamics. Although noise and model error are important for applications, the present research studies only the noise-free case to focus on how κ and δ are related.

The role κ plays in (1.15) may be compared as follows to the role of μ in the continuous nudging approach of [3] given by

$$\frac{dv}{dt} + \nu Av + B(v, v) = f + \mu J(U - v), \quad \text{where} \quad v(t_0) = u_0. \tag{1.16}$$

Note the effect of the nudging over the interval $[t_n, t_{n+1}]$ is approximately

$$\int_{t_n}^{t_{n+1}} \mu J(U - v)dt \approx \delta\mu J\big(U(t_n) - v(t_n)\big). \tag{1.17}$$

Since at the beginning of the same interval (1.15) kicks the approximation by

$$-\kappa JS(t_n - t_{n-1})u_{n-1} + \kappa JU(t_n) = \kappa J\big(U(t_n) - u(t_n^-)\big),$$

it is natural to identify κ with $\delta\mu$. The numerical and theoretical analysis of this identification is the main focus of the present paper.

After setting $\kappa = \mu\delta$ it follows that $\kappa \to 0$ as $\delta \to 0$. Thus, the more frequent the observations the smaller the update based on them. Our research begins by studying how the optimal value of κ depends on δ numerically. These simulations suggest $\kappa = \mu\delta$ is reasonable. To further this connection between κ and δ we then prove analytically that the κ-relaxed discrete-in-time data assimilation algorithm with $\kappa = \delta\mu$ converges to the continuous nudging approach as $\delta \to 0$.

In particular, our main theoretical result is

Theorem 1.4. *Given a reference solution U lying on the global attractor of the incompressible two-dimensional Navier–Stokes equations, let u be the approximating solution obtained by* (1.15) *with $\kappa = \mu\delta$ and let v be obtained by* (1.16)*. Then, for every $T > t_0$ it follows that*

$$\sup\big\{\|u(t) - v(t)\| : t \in [t_0, T]\big\} \to 0 \qquad \textit{as} \qquad \delta \to 0.$$

Moreover, the above result holds for every $J = P_\lambda P_H I_h$ independently of whether h and λ have been chosen so $\|u(t) - U(t)\| \to 0$ or even $\|v(t) - U(t)\| \to 0$ as $t \to \infty$.

The above result allows us to take $\delta \to 0$; however, we were unable to obtain suitable choices of h, λ and μ such that $|u(t) - U(t)| \to 0$ for all δ sufficiently small. Instead we have

Theorem 1.5. *There exists h, λ and μ such that for every $\varepsilon > 0$ there corresponds T large enough and $\delta_0 > 0$ such that*

$$\|u(T) - U(T)\| < \varepsilon \qquad \textit{for all} \qquad 0 < \delta < \delta_0$$

and every reference solution U lying on the global attractor.

The above result is a consequence of Theorem 1.4 combined with Theorem 2 in [3] using the triangle inequality. We state the proof here to illustrate the limits of our present theory.

Proof of Theorem 1.5. From Theorem 2 in [3] there exists h, λ and μ such that

$$\|v(t) - U(t)\| \to 0 \qquad \text{as} \qquad t \to \infty.$$

Let $\varepsilon > 0$. Therefore, there exists T large enough such that

$$\|v(t) - U(t)\| < \varepsilon/2 \qquad \text{for all} \qquad t \geq T.$$

Now choose $\delta_0 > 0$ in Theorem 1.4 such that

$$\sup\big\{\|u(t) - v(t)\| : t \in [t_0, T]\big\} \leq \varepsilon/2 \qquad \text{for all} \qquad 0 < \delta < \delta_0.$$

It follows that

$$\|u(T) - U(T)\| \leq |u(T) - v(T)| + |v(T) - U(T)| < \varepsilon.$$

This finishes the proof. □

This paper is organized as follows. In Section 2 we study how the optimal value of κ depends on δ numerically and find that $\kappa = \mu\delta$ is reasonable. Section 3 begins with some theoretical preliminaries and ends with the proof of Theorem 1.4. Concluding remarks and directions for future work appear in Section 4. Also included is Appendix A which provides explicit bounds on v in terms of T that are used in Section 3.

2 Numerical Results

This section describes the computations which appear in the introduction. We then present numerical evidence that taking the relaxation parameter κ proportional to δ results in numerical synchronization of the approximating solution with the reference solution over a wide of values for δ where $\delta \le 0.5$ is not too large.

All simulations are performed for a 2π-periodic domain discretized on a 512×512 spatial grid using a dealiased spectral method with 115599 active Fourier modes. Integration in time was by means of the fourth-order exponential time-differencing method introduced by Cox and Matthews [7] with step size $\Delta t = 0.0078125$. To avoid loss of precision the coefficients for the time-stepping method were obtained as in Kassam and Trefethen [23].

The computer program that performs the data assimilation experiments presented in this paper is written in the C programming language by the authors and compiled with GCC [16]. This same code was previously used in [5] to study the effects of noise in time-delay nudging. An overview of how this program works follows. The MPICH MPI library [1] distributes the computation of the reference and corresponding approximating solutions among the computational nodes. Each MPI rank is threaded using OpenMP and the Fourier transforms performed using the FFTW library of Frigo and Johnson [13]. While running, our simulations compute the reference solution using MPI rank 0 and every m time steps where $m\Delta t = \delta$ send the observational measurements of U to the remaining ranks. Those ranks then construct for different values of κ the corresponding approximating solutions.

We remark the one-way flow of observations from rank 0 to the other MPI ranks mimic real-world data gathering. This one-way flow of information also means network communication latency does not significantly affect the performance of our simulations. As a result the data-assimilation experiments could be parallelized across a loosely-coupled cluster of available machines using relatively low-speed gigabit Ethernet.

We now describe further details of the data assimilation experiments. The observational measurements are given by local spatial averages around the points

$$p_i = \big(\lfloor x_{i,1}(256/\pi) + 0.5 \rfloor, \lfloor x_{i,2}(256/\pi) + 0.5 \rfloor \big)$$

on a 512×512 grid where

$$x_i \in \big\{ \big((2n_1 + 1)\pi/9, (2n_2 + 1)\pi/9 \big) : n_1, n_2 = 0, \ldots, 8 \big\}.$$

The spatial averages are computed as

$$U_i \approx \frac{1}{|\mathcal{B}_i|} \sum_{p \in \mathcal{B}_i} U(p\pi/256) \qquad \text{where} \qquad \mathcal{B}_i = \big\{ p \in \mathbf{Z}^2 : |p - p_i|^2 \le 24 \big\}.$$

The condition $|p - p_i|^2 \le 24$ corresponds to an averaging radius of about $\sqrt{24}\pi/256 \approx 0.06$ in the 2π-periodic domain. Note there are 81 local averages and $|\mathcal{B}_i| = 69$ grid points in each average. Since the grid consists of $512^2 = 262144$ points total, the local averages cover about 2.13 percent of the physical domain. This percentage may be interpreted as how well the spatial averages reflect point observations of the velocity fields.

As the MPI ranks used for computing an approximating solution receive the observational data it is interpolated as

$$JU(p) = P_\lambda P_H \sum_{i=1}^{81} U_i \chi_i(p)$$

where χ_i is the characteristic function for the square centered at p_i.

To identify the relationship between κ and δ we performed data-assimilation experiments for sampling intervals of

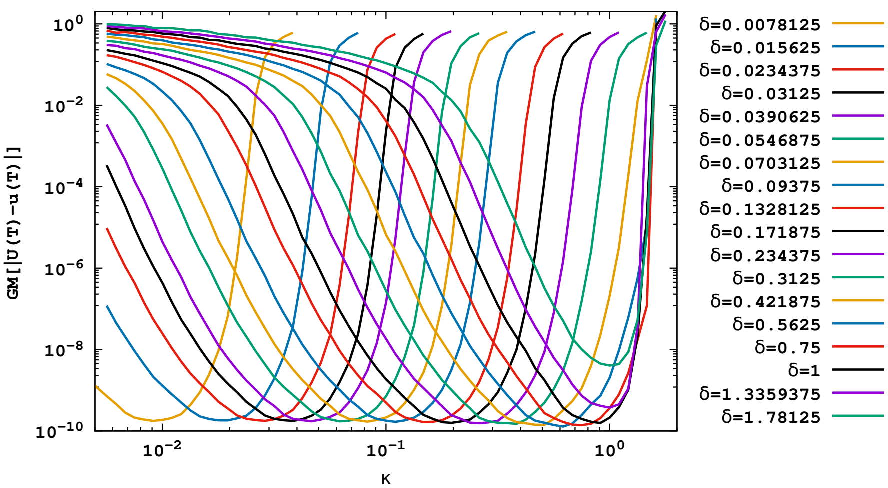
$$\delta = m\Delta t \qquad \text{where} \qquad m = \left\lfloor 228\Big(\frac{3}{4}\Big)^p \right\rfloor \qquad \text{for} \qquad p = 0, 1, \ldots, 17 \tag{2.1}$$

and the relaxation parameters

$$\kappa = \Big(\frac{3}{4}\Big)^{q/3} \qquad \text{for which} \qquad \kappa \in \big[0.0056, 5\delta\big] \qquad \text{and} \qquad q \in \mathbf{Z}. \tag{2.2}$$

In all we consider 873 choices for δ and κ. For each choice multiple simulations were performed corresponding to an ensemble $\mathcal{E}$ of 500 different reference solutions U lying on the global attractor. This resulted in a total of 436500 data assimilation experiments.

Figure 4: The geometric mean of $\|U(T) - u(T)\|_{L^2}$ at $T = t_0 + 1024$ for varying values of δ and κ.



The ensemble $\mathcal{E}$ was obtained from the long-term evolutions of randomly-chosen points in phase space. In particular $U \in \mathcal{E}$ implies $U(t_0) = U_0$ where $U_0 = S(10240)Z_0$ and Z_0 is a random velocity field with energy spectrum similar to a point on the attractor. Although Z_0 is unlikely to be on the global attractor, the evolution time of 10240 is long enough that large scale structures have appeared in the velocity field and further undergone more than 300 large eddy turnovers. We remark that the energetics of each flow appear to enter into a statistically-steady state while continuing to undergo complex fluctuations. This suggests the initial conditions U_0 for each of the trajectories in $\mathcal{E}$ lie very near the global attractor and that this attractor is nontrivial.

After every 3200 time steps the reference and approximating solutions were compared. After 131072 time steps the final value of $|U(T)-u(T)|$ at $T = t_0+1024$ is returned. We note the available hardware—mostly Xeon and Epyc servers—typically performed from 10 to 30 time steps per second depending on the exact processor and memory speed. On average each run about took two hours. To facilitate scheduling the runs were partitioned into batches that computed the reference solution and corresponding approximating solutions for seven parameter choices at a time.

Figure 5: Box plots (no outliers removed) depicting the values of κ that minimize $\|U(T)-u(T)\|_{L^2}$ when $T = t_0+1024$ for each trajectory in the ensemble. Note $\kappa = \mu\delta$ was fitted for $\delta \leq 0.5$.

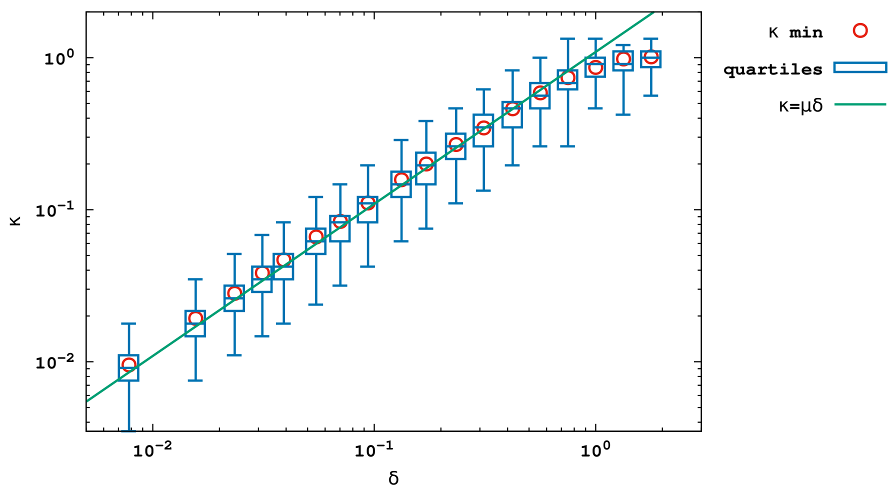


Figure 4 plots the geometric mean of the error at time T for each $\delta \in [0.0078125, 1.78125]$ as a function of κ where $\kappa \in [0.0056, 5\delta]$. Note that additional runs when $\delta = 0.0078125$ were performed with $\kappa \in \big[0.00317, 0.0056\big)$ to ensure that small enough values of κ were considered for this smallest value of δ.

The value $\kappa_{\min}$ of κ that minimizes the geometric mean along the corresponding curve for each δ in Figure 4 was obtained using a least-squares quadratic fit of points near the minimum. To illustrate how $\kappa_{\min}$ depends on δ the points $(\delta, \kappa_{\min})$ are plotted in Figure 5 as circles. Note for $\delta \leq 0.5$ the relationship between $\kappa_{\min}$ and δ appears linear. This linearity is our primary numerical result and the focus of the theory in the following section.

Figure 5 further characterizes the relationship between κ and δ by considering each reference trajectory $U \in \mathcal{E}$ separately to obtain 500 different minimizing values of κ for each δ. The distribution of the minimizing values of κ are then displayed in quartiles for each value of δ considered. Finally a least squares fit of $\kappa = \mu\delta$ for $\delta \leq 0.5$ was performed to obtain $\mu \approx 0.966$. We remark μ is not universal but a tuning parameter that depends on the viscosity ν, body forcing f, spatial domain and the interpolant observable J among others. In particular, the value of μ obtained above is only coincidentally close to one. What is important for the present research is that the relationship between κ and δ appears linear.

3 Theoretical Results

In this section we prove our main theoretical result given as Theorem 1.4 in the introduction. Begin by translating the *a priori* bounds of Theorem 1.1 into bounds on the interpolated observations represented by JU.

Proposition 3.1. *Suppose U lies on the global attractor of* (1.3) *and $f \in V$. There exists constants ρ_J and ρ_K depending on λ, ν, $\|f\|$ and h such that*

$$|JU| \le \rho_J \qquad \textit{and} \qquad \|JU\| \le \rho_K. \tag{3.1}$$

Proof. Estimate using (1.11) followed by (1.4) as

$$|JU| = |U - JU| + |U| \le (\lambda^{-1} + c_1 h^2)^{1/2} \|U\| + |U|$$

to obtain $\rho_J = (\lambda^{-1} + c_1 h^2)^{1/2} \rho_1 + \rho_0$. Similarly use (1.12) to estimate

$$\|JU\| = \|U - JU\| + \|U\| \le (1 + \lambda c_1 h^2)^{1/2} \|U\| + \|U\|.$$

Thus, $\rho_K = (1 + \lambda c_1 h^2)^{1/2} \rho_1 + \rho_1$. □

Our estimates make use of the following version of Theorem 5 from [3] that removes the condition on h and μ but applies only on a finite time interval $[t_0, T]$.

Theorem 3.2. *Suppose $f \in V$ and $v_0 \in \mathcal{D}(A)$. Let v be the solution obtained from* (1.16)*. There exists bounds R_α and $\widetilde{R}_\alpha$ depending on T, t_0, ν, $\|f\|$ and v_0 such that*

$$\|v\|_\alpha \le R_\alpha \qquad \textit{for} \qquad t \in [t_0, T] \qquad \textit{with} \qquad \alpha = 0, 1, 2 \tag{3.2}$$

and

$$\Big(\int_{t_n}^{t_{n+1}} \|v\|_\alpha^2 dt \Big)^{1/2} \le \widetilde{R}_\alpha \quad \textit{for} \quad [t_n, t_{n+1}] \subseteq [t_0, T] \quad \textit{with} \quad \alpha = 1, 2, 3. \tag{3.3}$$

The proof of Theorem 3.2 is provided in Appendix A. We note here that the smoothness of the interpolant J as well as f constrains the regularity of v.

We remark that $v_0 = u_0$ in (1.16) where $u_0 = JU(t_0)$. Since $J = P_\lambda P_H I_h$ then $P_\lambda v_0 = v_0$. Consequently (1.9) followed by Proposition 3.1 yields the bound

$$|Av_0| = \|P_\lambda v_0\|_2 \le \lambda |v_0| = \lambda |JU(t_0)| \le \lambda \rho_J.$$

In particular, we may assume $v_0 \in \mathcal{D}(A)$ in the proof of Theorem 1.4. Moreover, since the reference solution U lies on the global attractor $\mathcal{A}$ we may further assume the bounds R_α and $\widetilde{R}_\alpha$ to be independent of U_0.

Now use (1.11) and (1.12) as in the proof of Proposition 3.1 to obtain bounds on $|Jv|$ and $\|Jv\|$. Thus, there exists R_J and R_K further depending on h and λ such that

$$|Jv| \le R_J \qquad \text{and} \qquad \|Jv\| \le R_K. \tag{3.4}$$

A number of inequalities involving the non-linear term are summarized in Chapter II Appendix A of Foias, Manley, Rosa and Temam [14]. We recall here for reference

$$|(B(u,v),w)| \le c|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w| \tag{3.5}$$

for $u \in V$, $v \in \mathcal{D}(A)$ and $w \in H$ as well as

$$|(B(u,v),w)| \le c|u|^{1/2}|Au|^{1/2}\|v\||w| \tag{3.6}$$

for $u \in \mathcal{D}(A)$, $v \in V$ and $w \in H$. Note that the above inequalities hold for a suitable constant c depending on the domain Ω which in our case is L-periodic.

It follows from (3.5) that

$$|B(u,v)| \le c|u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2} \qquad \text{for} \qquad u \in V \text{ and } v \in \mathcal{D}(A) \tag{3.7}$$

and from (3.6) that

$$|B(u,v)| \le c|u|^{1/2}|Au|^{1/2}\|v\| \qquad \text{for} \qquad u \in \mathcal{D}(A) \text{ and } v \in V. \tag{3.8}$$

Again from (3.7) it follows that

$$\|B(u,v)\| \le c\|u\|^{1/2}|Au|^{1/2}\|v\|^{1/2}|Av|^{1/2} + c|u|^{1/2}\|u\|^{1/2}|Av|^{1/2}|A^{3/2}v|^{1/2} \tag{3.9}$$

for $u \in \mathcal{D}(A)$ and $v \in \mathcal{D}(A^{3/2})$. Setting $u = v$ and interpolating then yields

$$\|B(u,u)\| \le c|u|^{1/2}\|u\|^{1/2}|Au|^{1/2}|A^{3/2}u|^{1/2} \qquad \text{for} \qquad u \in \mathcal{D}(A^{3/2}). \tag{3.10}$$

For simplicity the c appearing above and in (3.11) below will denote a single constant chosen suitably large so all of the inequalities hold.

We shall also employ an estimate appearing in [5], Foias, Mondaini and Titi [15] and Titi [26] based on the Brezis–Gallouët inequality which we state here as

Proposition 3.3. *If v and w are in $\mathcal{D}(A)$ then*

$$\left|\big(B(w,v)+B(v,w),Aw\big)\right| \le c\|w\|\|v\|\left(1+\log\frac{|Av|}{\lambda_1^{1/2}\|v\|}\right)^{1/2}|Aw|, \tag{3.11}$$

where c is a non-dimensional constant depending only on the domain.

Fix $T > t_0$. To study the limit on the interval $[t_0,T]$ as the time between observations $\delta \to 0$ define $t_n = t_0 + \delta n$ for $n = 0,\dots,N$ where N is the greatest integer such that $t_N \le T$. Assume $\delta \le \delta_{\max}$ where $\delta_{\max} \le T - t_0$; otherwise, if $\delta > T - t_0$, then no observational measurements would occur during the interval $[t_0,T]$ and there would be no data to assimilate. Upon substituting $\kappa = \delta\mu$ into (1.15) and using the fact that $t_{n+1} - t_n = \delta$, the relaxed version of Definition 1.2 becomes

Definition 3.4. Assume $\delta \leq T - t_0$ and define $t_n = t_0 + \delta n$ for $n = 0, \dots, N$ where N is the greatest integer such that $t_N \leq T$. The relaxed direct-insertion method is defined as the approximating solution $u(t)$ for $t \in [t_0, T]$ given by

$$u(t) = \begin{cases} S(t - t_n)u_n & \text{for} \quad t \in [t_n, t_{n+1}) \quad \text{with} \quad n = 0, \dots, N-1, \\ S(t - t_N)u_N & \text{for} \quad t \in [t_N, T] \end{cases}$$

where

$$\begin{cases} u_0 = JU(t_0), \\ u_{n+1} = S(\delta)u_n + \delta\mu J\big\{U(t_{n+1}) - S(\delta)u_n\big\}. \end{cases}$$

In preparation to prove Theorem 1.4 let $w_n = v(t_n) - u_n$ and $w(t) = v(t) - u(t)$. Here $v(t)$ is the approximating solution obtained by (1.16). Definition 3.4 implies

$$u(t_{n+1}^-) = \lim_{t \nearrow t_{n+1}} u(t) = S(\delta)u_n.$$

Therefore

$$\begin{aligned} w_{n+1} &= v(t_{n+1}) - (1 - \delta\mu J)u(t_{n+1}^-) - \delta\mu JU(t_{n+1}) \\ &= w(t_{n+1}^-) - \delta\mu J(U(t_{n+1}) - v(t_{n+1}) + w(t_{n+1}^-)) \\ &= (I - \delta\mu J)w(t_{n+1}^-) - \delta\mu J(U(t_{n+1}) - v(t_{n+1})). \end{aligned}$$

Consequently w satisfies

$$\frac{dw}{dt} + \nu Aw + B(v, w) + B(w, v) - B(w, w) = \mu J(U - v) \quad \text{on} \quad t \in [t_n, t_{n+1}) \tag{3.12}$$

with initial condition

$$w_n = (I - \delta\mu J)w(t_n^-) - \delta\mu J(U(t_n) - v(t_n)) \qquad \text{for} \qquad n > 0 \tag{3.13}$$

and $w_0 = 0$ since we use the same initial state for both data assimilation methods. For notational convenience we define $w(t_0^-) = 0$ so that (3.13) also holds when $n = 0$.

Continue with the following estimate on the integral of L^2 norm of Aw.

Theorem 3.5. *There exists M_0, M_1 and $\widetilde{M}_2$ depending only on T, t_0, ν, $\|f\|$ and the domain Ω such that*

$$|w| \leq M_0 \qquad \text{and} \qquad \|w\| \leq M_1 \qquad \text{for all} \qquad t \in [t_0, T]$$

and

$$\Big(\int_{t_n}^{t_{n+1}} |Aw|^2 dt\Big)^{1/2} \leq \widetilde{M}_2 \qquad \text{for} \qquad [t_n, t_{n+1}] \subseteq [t_0, T]. \tag{3.14}$$

Proof. Let N be the greatest integer such that $t_N \leq T$. By Theorem 3.2 there exists R_α such that $\|v\|_\alpha \leq R_\alpha$ for $t \in [t_0, T]$ and $\alpha = 0, 1, 2$. Since the reference trajectory U lies on the global attractor and $v_0 = JU(t_0)$ than these bounds can be taken independent of v_0.

Multiply (3.12) by Aw and use the fact that $(B(w,w),Aw)=0$ in two-dimensional periodic domains to obtain

$$\frac{1}{2}\frac{d\|w\|^2}{dt}+\nu|Aw|^2+(B(v,w),Aw)+(B(w,v),Aw)=\mu(J(U-v),Aw). \tag{3.15}$$

From (3.11) it follows that

$$\begin{aligned}|(B(v,w),Aw)+(B(w,v),Aw)|&\le c\|w\|\|v\|\bigg(1+\log\frac{|Av|}{\lambda_1^{1/2}\|v\|}\bigg)^{1/2}|Aw|\\ &\le cR_1^{1/2}R_L^{1/2}\|w\||Aw|\le\frac{c^2R_1R_L}{\nu}\|w\|^2+\frac{\nu}{4}|Aw|^2,\end{aligned} \tag{3.16}$$

where

$$R_L=R_1\bigg(1+\log\frac{R_2}{\lambda_1^{1/2}R_1}\bigg).$$

Also

$$\mu(J(U-v),Aw)\le\frac{\mu^2}{\nu}|J(U-v)|^2+\frac{\nu}{4}|Aw|^2\le\frac{\mu^2}{\nu}C_1^2+\frac{\nu}{4}|Aw|^2 \tag{3.17}$$

where $C_1=\rho_J+R_J$.

Combining (3.16) and (3.17) with (3.15) yields

$$\frac{d\|w\|^2}{dt}+\nu|Aw|^2\le\frac{2c^2R_1R_L}{\nu}\|w\|^2+\frac{2\mu^2}{\nu}C_1^2\le C_2\|w\|^2+C_3. \tag{3.18}$$

Here $C_2=2c^2R_1R_L/\nu$ and $C_3=2\mu^2C_1^2/\nu$. Now multiply (3.18) by e^{-C_2t} and integrate from t_n to t where $t\in[t_n,t_{n+1})$ for $n=0,1,\ldots,N-1$ and $t\in[t_N,T]$ for $n=N$. Thus,

$$\|w(t)\|^2+\nu\int_{t_n}^t e^{C_2(t-s)}|Aw(s)|^2ds\le e^{C_2(t-t_n)}\|w_n\|^2+\frac{C_3}{C_2}\big(e^{C_2(t-t_n)}-1\big). \tag{3.19}$$

We obtain after applying (1.12) that

$$\begin{aligned}\|w_n\|&=\|(I-\delta\mu J)w(t_n^-)-\delta\mu J(U(t_n)-v(t_n))\|\\ &\le|1-\delta\mu|\|w(t_n^-)\|+\delta\mu\|(I-J)w(t_n^-)\|+\delta\mu\|J(U(t_n)-v(t_n))\|.\end{aligned}$$

Therefore

$$\|w_n\|\le(1+\delta\mu C_4)\|w(t_n^-)\|+\delta\mu C_5 \tag{3.20}$$

where $C_4=1+\sqrt{1+\lambda c_1h^2}$ and $C_5=\rho_K+R_K$. Consequently, under the assumption $\delta\le\delta_{\max}$ it follows that

$$\begin{aligned}\|w_n\|^2&\le(1+\delta\mu C_4)^2\|w(t_n^-)\|^2+2\delta\mu(1+\delta\mu C_4)C_5\|w(t_n^-)\|+\delta^2\mu^2C_5^2\\ &\le(1+\delta\mu C_4)^2\|w(t_n^-)\|^2+\delta\mu\big\{(1+\delta\mu C_4)^2\|w(t_n^-)\|^2+C_5^2\big\}+\delta^2\mu^2C_5^2\\ &=(1+\delta\mu)(1+\delta\mu C_4)^2\|w(t_n^-)\|^2+\delta\mu(1+\delta\mu)C_5^2\\ &\le(1+\delta\mu C_6)\|w(t_n^-)\|^2+\delta\mu C_7\end{aligned}$$

where $C_6 = 2C_4 + \delta_{\max}\mu C_4^2 + (1+\delta_{\max}\mu C_4)^2$ and $C_7 = (1+\delta_{\max}\mu)C_5^2$.

Substituting into (3.19) and dropping the integral on the left results in

$$\|w(t)\|^2 \le e^{C_2(t-t_n)}\big\{(1+\delta\mu C_6)\|w(t_n^-)\|^2 + \delta\mu C_7\big\} + \frac{C_3}{C_2}\big(e^{C_2(t-t_n)} - 1\big). \tag{3.21}$$

Subsequently for $n = 0, 1, \ldots, N-1$ taking the limit as $t \to t_{n+1}$ obtains

$$\|w(t_{n+1}^-)\|^2 \le e^{\delta C_2}(1+\delta\mu C_6)\|w(t_n^-)\|^2 + \delta\mu e^{\delta C_2}C_7 + \frac{C_3}{C_2}\big(e^{\delta C_2} - 1\big).$$

Setting $y_n = \|w(t_n^-)\|^2$ gives the inequality $y_{n+1} \le \alpha y_n + \beta$ where $\alpha = e^{\delta C_2}(1+\delta\mu C_6)$ and $\beta = \delta\mu e^{\delta C_2}C_7 + (C_3/C_2)(e^{\delta C_2} - 1)$. By induction

$$\begin{aligned}
y_1 &\le \alpha y_0 + \beta\\
y_2 &\le \alpha y_1 + \beta \le \alpha^2 y_0 + (1+\alpha)\beta\\
&\vdots\\
y_n &\le \alpha^n y_0 + \textstyle\sum_{j=0}^{n-1}\alpha^j\beta \le \alpha^n y_0 + \frac{\alpha^n - 1}{\alpha - 1}\beta.
\end{aligned}$$

Using the fact that $x \le e^x - 1 \le xe^x$ for $x \ge 0$ we obtain

$$\alpha - 1 = e^{\delta C_2} - 1 + \delta\mu e^{\delta C_2}C_6 \ge \delta(C_2 + \mu e^{\delta C_2}C_6),$$

$$\alpha^n = e^{n\delta C_2}(1+\delta\mu C_6)^n \le e^{(t_n - t_0)(C_2+\mu C_6)}$$

and

$$\beta = \delta\mu e^{\delta C_2}C_7 + (C_3/C_2)(e^{\delta C_2} - 1) \le \delta e^{\delta C_2}(\mu C_7 + C_3).$$

Consequently, for $n = 1, 2, \ldots, N$ it follows from $\delta \le \delta_{\max}$ that

$$\begin{aligned}
\|w(t_n^-)\|^2 &\le e^{(t_n-t_0)(C_2+\mu C_6)}\|w(t_0^-)\|^2 + \frac{e^{(t_n-t_0)(C_2+\mu C_6)} - 1}{C_2 + \mu e^{\delta C_2}C_6}e^{\delta C_2}(\mu C_7 + C_3)\\
&\le e^{2\delta_{\max}(C_2+\mu C_6)}\|w(t_0^-)\|^2 + \frac{e^{2\delta_{\max}(C_2+\mu C_6)} - 1}{(C_2 + \mu C_6)}e^{\delta_{\max}C_2}(\mu C_7 + C_3).
\end{aligned}$$

Since $\|w(t_0^-)\| = 0$, there is a constant C_8 independent of δ such that

$$\|w(t_n^-)\| \le C_8 \qquad \text{for} \qquad n = 1, \ldots, N.$$

It follows from (3.20) that

$$\|w_n\| \le (1+\delta\mu C_4)\|w(t_n^-)\| + \delta\mu C_5 \le (1+\delta_{\max}\mu C_4)C_8 + \delta_{\max}\mu C_5$$

and so there is also a constant C_9 independent of δ such that

$$\|w_n\| \le C_9 \qquad \text{for} \qquad n = 1, \ldots, N.$$

Now use (3.19) to show there is M_1 such that $\|w(t)\| \le M_1$ for $t \in [t_0, T]$. Note that

$$M_1^2 = e^{C_2\delta_{\max}}C_9^2 + \frac{C_3}{C_2}\big(e^{C_2\delta_{\max}} - 1\big).$$

The Poincaré inequality (1.6) stated as $\lambda_1|w|^2 \le \|w\|^2$ immediately implies

$$|w(t)| \le \lambda_1^{-1/2}\|w(t)\| \le M_0 \qquad \text{for} \qquad t \in [t_0, T]$$

where $M_0 = \lambda_1^{-1/2} M_1$.

To obtain (3.14) suppose $n = 0, 1, \ldots, N-1$ and integrate (3.18) from t_n to t. Then taking the limit as $t \to t_{n+1}$ yields

$$\|w(t_{n+1}^-)\|^2 - \|w_n\|^2 + \nu \int_{t_n}^{t_{n+1}} |Aw|^2 dt \le C_2 \int_{t_n}^{t_{n+1}} \|w\|^2 dt + \delta C_3.$$

Thus

$$\nu \int_{t_n}^{t_{n+1}} |Aw|^2 dt \le \|w_n\|^2 + \delta_{\max}(C_2 M_1^2 + C_3)$$

or equivalently

$$\Big(\int_{t_n}^{t_{n+1}} |Aw|^2 dt\Big)^{1/2} \le \widetilde{M}_2 \qquad \text{where} \qquad \widetilde{M}_2^2 = \frac{C_9^2}{\nu} + \delta_{\max}\frac{C_2 M_1^2 + C_3}{\nu}.$$

Finally, noting that M_0, M_1 and $\widetilde{M}_2$ depend only on T, t_0, ν, $\|f\|$ and Ω finishes the proof. □

Corollary 3.6. *There exists constants M_J and M_K such that*

$$|Jw| \le M_J \qquad \text{and} \qquad \|Jw\| \le M_K \qquad \text{for all} \qquad t \in [t_0, T].$$

Proof. This follows at once from Theorem 3.5 applied to (1.11) and (1.12) exactly analogous to the way (3.1) and (3.4) were obtained. □

The next step is to bootstrap Theorem 3.5 to obtain bounds which tend to zero as $\delta \to 0$. In order to do this we first prove the following two lemmas.

Lemma 3.7. *There exists M_D independent of δ such that*

$$\int_{t_n}^{t_{n+1}} |w(t) - w_n| dt \le \delta^{3/2} M_D \qquad \text{for} \qquad [t_n, t_{n+1}] \subseteq [t_0, T].$$

Proof. Estimate as

$$\begin{aligned}\int_{t_n}^{t_{n+1}} |w(t) - w_n| dt &\le \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \Big|\frac{dw(t)}{dt}\Big| dt\, ds \le \delta \int_{t_n}^{t_{n+1}} \Big|\frac{dw(t)}{dt}\Big| dt \\ &\le \delta \int_{t_n}^{t_{n+1}} \big\{\nu|Aw| + |B(v,w)| + |B(w,v)| + |B(w,w)| + \mu|J(U-v)|\big\}.\end{aligned}$$

By the Theorem 3.5 we have

$$\int_{t_n}^{t_{n+1}} \nu|Aw| \le \nu\delta^{1/2}\Big(\int_{t_n}^{t_{n+1}} |Aw|^2\Big)^{1/2} \le \delta^{1/2}\nu\widetilde{M}_2. \tag{3.22}$$

Using (3.7) we have

$$\int_{t_n}^{t_{n+1}} |B(v,w)| \le c\int_{t_n}^{t_{n+1}} |v|^{1/2}\|v\|^{1/2}\|w\|^{1/2}|Aw|^{1/2} \le cR_0^{1/2}R_1^{1/2}M_1^{1/2}\int_{t_n}^{t_{n+1}} |Aw|^{1/2}$$
$$\le cR_0^{1/2}R_1^{1/2}M_1^{1/2}\delta^{3/4}\Big(\int_{t_n}^{t_{n+1}} |Aw|^2\Big)^{1/4} \le \delta^{3/4}cR_0^{1/2}R_1^{1/2}M_1^{1/2}\widetilde{M}_2^{1/2}$$

and from (3.8) it follows

$$\int_{t_n}^{t_{n+1}} |B(w,v)| \le c\int_{t_n}^{t_{n+1}} |w|^{1/2}|Aw|^{1/2}\|v\| \le \delta^{3/4}cM_0^{1/2}R_1\widetilde{M}_2^{1/2}.$$

as well as

$$\int_{t_n}^{t_{n+1}} |B(w,w)| \le c\int_{t_n}^{t_{n+1}} |w|^{1/2}|Aw|^{1/2}\|w\| \le \delta^{3/4}cM_0^{1/2}M_1\widetilde{M}_2^{1/2}.$$

Finally,

$$\int_{t_n}^{t_{n+1}} \mu|J(U-v)| \le \delta\mu(\rho_J+R_J).$$

Upon collecting the above estimates together we obtain

$$\int_{t_n}^{t_{n+1}} |w(t)-w_n|dt \le \delta^{3/2}M_D$$

where

$$M_D = \nu\widetilde{M}_2 + \delta_{\max}^{1/4}c\big\{R_0^{1/2}R_1^{1/2}M_1^{1/2} + M_0^{1/2}R_1 + M_0^{1/2}M_1\big\}\widetilde{M}_2^{1/2} + \delta_{\max}^{1/2}\mu(\rho_J+R_J).$$

Noting that M_D is independent of δ finishes the proof. □

Corollary 3.8. *There exists M_E independent of δ such that*

$$\int_{t_n}^{t_{n+1}} |w(t)-w(t_n^-)|dt \le \delta^{3/2}M_E \qquad \textit{for} \qquad [t_n,t_{n+1}]\subseteq[t_0,T].$$

Proof. Since $w_n - w(t_n^-) = -\delta\mu J\big(w(t_n^-)+U(t_n)-v(t_n)\big)$ then

$$\int_{t_n}^{t_{n+1}} |w_n-w(t_n^-)|dt \le \delta^2\mu(M_J+\rho_J+R_J).$$

It follows from Lemma 3.7 that

$$\int_{t_n}^{t_{n+1}} |w(t)-w(t_n^-)|dt \le \delta^{3/2}M_D + \delta^2\mu(M_J+\rho_J+R_J) \le \delta^{3/2}M_E$$

where $M_E = M_D + \delta_{\max}^{1/2}\mu(M_J+\rho_J+R_J)$. □

Lemma 3.9. *There exists* ρ_D *and* R_D *independent of* δ *such that for* $[t_n, t_{n+1}] \subseteq [t_0, T]$ *holds*

$$\int_{t_n}^{t_{n+1}} \|J(U(t) - U(t_n))\| dt \le \delta^{3/2} \rho_D \quad \textit{and} \quad \int_{t_n}^{t_{n+1}} \|J(v(t) - v(t_n))\| dt \le \delta^{3/2} R_D.$$

Proof. Estimate using (1.14)

$$\int_{t_n}^{t_{n+1}} \|J(U(t) - U(t_n))\| dt \le c_3 \int_{t_n}^{t_{n+1}} \|U(t) - U(t_n)\| dt \le c_3 \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \Big\| \frac{dU(t)}{dt} \Big\| dt\, ds$$
$$\le c_3 \delta \int_{t_n}^{t_{n+1}} \big\{ \nu \|AU\| + \|B(U,U)\| + \|f\| \big\}$$

By the Cauchy–Schwartz inequality and (1.5)

$$\nu \int_{t_n}^{t_{n+1}} \|AU\| \le \delta^{1/2} \nu \Big(\int_{t_n}^{t_{n+1}} \|AU\|^2 \Big)^{1/2} \le \delta^{1/2} \nu \widetilde{\rho}_3.$$

Similarly (3.10) implies

$$\int_{t_n}^{t_{n+1}} \|B(U,U)\| \le c \int_{t_n}^{t_{n+1}} |U|^{1/2} \|U\|^{1/2} |AU|^{1/2} |A^{3/2}U|^{1/2}$$
$$\le c \int_{t_n}^{t_{n+1}} \rho_0^{1/2} \rho_1^{1/2} |AU|^{1/2} |A^{3/2}U|^{1/2} \le \delta^{1/2} c (\rho_0 \rho_1 \widetilde{\rho}_2 \widetilde{\rho}_3)^{1/2}.$$

Therefore,

$$\rho_D = c_3 \big(\nu \widetilde{\rho}_3 + c(\rho_0 \rho_1 \widetilde{\rho}_2 \widetilde{\rho}_3)^{1/2} + \delta_{\max}^{1/2} \|f\| \big).$$

The proof of the second inequality is similar except there is one more term which we estimate using (3.1) and (3.4) as

$$\int_{t_n}^{t_{n+1}} \|J(v(t) - v(t_n))\| dt \le c_3 \int_{t_n}^{t_{n+1}} \|v(t) - v(t_n)\| dt \le c_3 \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \Big\| \frac{dv(t)}{dt} \Big\| dt\, ds$$
$$\le c_3 \delta \int_{t_n}^{t_{n+1}} \big\{ \nu \|Av\| + \|B(v,v)\| + \|f\| + \mu \|J(U - v)\| \big\} \le \delta^{3/2} R_D$$

where $R_D = c_3 \big(\nu \widetilde{R}_3 + c(R_0 R_1 \widetilde{R}_2 \widetilde{R}_3)^{1/2} + \delta_{\max}^{1/2} (\|f\| + \mu \rho_K + \mu R_K) \big)$. □

We are now ready to prove our main theoretical result stated as Theorem 1.4 in the introduction. Namely, under the above hypotheses we prove

Theorem 3.10. *For every* $T > t_0$ *that*

$$\sup \big\{ \|w(t)\| : t \in [t_0, T] \big\} \to 0 \qquad \textit{as} \qquad \delta \to 0.$$

Proof. Fix $T > t_0$. Let $t_n = t_0 + \delta n$ where $\delta \le \delta_{\max}$ and $\delta_{\max} \le T - t_0$. Let N be the greatest integer such that $t_N \le T$ and consider the interval $[t_0, T]$. Taking the inner product of (3.12) with Aw as in the proof of Theorem 3.5 we obtain

$$\frac{1}{2} \frac{d\|w\|^2}{dt} + \frac{1}{2} \nu |Aw|^2 \le \frac{c^2 R_1 R_L}{2\nu} \|w\|^2 + \mu (J(U - v), Aw). \tag{3.23}$$

Note the factor-of-two improvement in the coefficient of $\|w\|^2$ on the right comes from a different application of Young's inequality. In particular, rather than (3.16) we estimate

$$|(B(v,w),Aw)+(B(w,v),Aw)| \le \frac{c^2R_1R_L}{2\nu}\|w\|^2+\frac{\nu}{2}|Aw|^2.$$

Upon applying the Poincaré inequality (1.6) written as $\lambda_1\|w\|^2\le |Aw|^2$ we obtain

$$\frac{d\|w\|^2}{dt}\le C_{10}\|w\|^2+2\mu(w,AJ(U-v)). \tag{3.24}$$

where $C_{10}=c^2R_1R_L/\nu-\nu\lambda_1$.

Recall that $w(t_n)=w_n$. The definition of w_n given in (3.13) implies

$$w_n=(1-\delta\mu)w(t_n^-)+\delta\mu(I-J)w(t_n^-)-\delta\mu J(U(t_n)-v(t_n)).$$

Therefore,

$$\begin{aligned}\|w_n\|^2 &= (1-\delta\mu)^2\|w(t_n^-)\|^2+\delta^2\mu^2\|(I-J)w(t_n^-)-J(U(t_n)-v(t_n))\|^2\\ &\quad+2\delta\mu\big(A^{1/2}(1-\delta\mu)w(t_n^-),A^{1/2}(I-J)w(t_n^-)\big)\\ &\quad-2\delta\mu(1-\delta\mu)\big(A^{1/2}w(t_n^-),A^{1/2}J(U(t_n)-v(t_n))\big)\\ &\le (1+\delta\mu C_{11})\|w(t_n^-)\|^2-2\delta\mu\big(w(t_n^-),AJ(U(t_n)-v(t_n))\big)+\delta^2\mu^2C_{12}\end{aligned}$$

where

$$C_{11}=2+\delta_{\max}\mu+2c_3(1+\delta_{\max}\mu)$$

and

$$C_{12}=(M_1+M_K+\rho_K+R_K)^2+2M_1(\rho_K+R_K).$$

Note for the above estimates we have used

$$(1-\delta\mu)^2\le 1+\delta\mu(2+\delta_{\max}\mu),$$

$$\delta^2\mu^2\|(I-J)w(t_n^-)-J(U(t_n)-v(t_n))\|^2\le\delta^2\mu^2(M_1+M_K+\rho_K+R_K)^2,$$

$$\begin{aligned}2\big(A^{1/2}(1-\delta\mu)w(t_n^-),A^{1/2}(I-J)w(t_n^-)\big)&\le 2(1+\delta_{\max}\mu)\|w(t_n^-)\|\|(I-J)w(t_n^-)\|\\ &\le 2(1+\delta_{\max}\mu)(1+\lambda c_1h^2)^{1/2}\|w(t_n^-)\|^2\le 2c_3(1+\delta_{\max}\mu)\|w(t_n^-)\|^2\end{aligned}$$

and

$$2\delta^2\mu^2\big(A^{1/2}w(t_n^-),A^{1/2}J(U(t_n)-v(t_n))\big)\le 2\delta^2\mu^2M_1(\rho_K+R_K).$$

Multiply (3.24) by the integrating factor $e^{-C_{10}t}$ to obtain

$$\frac{d}{dt}\big(e^{-C_{10}t/}\|w\|^2\big)\le 2e^{-C_{10}t}\mu(w,AJ(U-v)). \tag{3.25}$$

Now, integrate over $[t_n, t_{n+1})$ to obtain

$$
\begin{aligned}
\|w(t_{n+1}^-)\|^2 &\le e^{C_{10}\delta}\|w_n\|^2 + 2\mu\int_{t_n}^{t_{n+1}} e^{C_{10}(t_{n+1}-t)}\big(w(t), AJ(U(t)-v(t))\big)dt\\
&\le e^{C_{10}\delta}\big\{(1+\delta\mu C_{11})\|w(t_n^-)\|^2 - 2\delta\mu\big(w(t_n^-), AJ(U(t_n)-v(t_n))\big) + \delta^2\mu^2 C_{12}\big\}\\
&\quad + 2\mu\int_{t_n}^{t_{n+1}} e^{C_{10}(t_{n+1}-t)}(w(t), AJ(U(t)-v(t)))dt\\
&\le e^{C_{10}\delta}\big\{(1+C_{11}\delta\mu)\|w(t_n^-)\|^2 + \delta^2\mu^2 C_{12}\big\}\\
&\quad + 2\mu\int_{t_n}^{t_{n+1}} \Big\{e^{C_{10}(t_{n+1}-t)}\big(AJ(U(t)-v(t)), w(t)\big)\\
&\qquad\qquad - e^{C_{10}\delta}\big(AJ(U(t_n)-v(t_n)), w(t_n^-)\big)\Big\}dt\\
&\le e^{C_{10}\delta}\big\{(1+C_{11}\delta\mu)\|w(t_n^-)\|^2 + \delta^2\mu^2 C_{12} + 2\mu(I_1+I_2+I_3+I_4)\big\}
\end{aligned}
$$

where

$$I_1 = \int_{t_n}^{t_{n+1}} \big(e^{C_{10}(t_n-t)}-1\big)\big(A^{1/2}J(U(t)-v(t)), A^{1/2}w(t)\big)dt,$$

$$I_2 = \int_{t_n}^{t_{n+1}} (AJ(U(t)-v(t)), w(t)-w(t_n^-))dt,$$

$$I_3 = \int_{t_n}^{t_{n+1}} \big(A^{1/2}J(U(t)-U(t_n)), A^{1/2}w(t_n^-)\big)dt$$

and

$$I_4 = -\int_{t_n}^{t_{n+1}} \big(A^{1/2}J(v(t)-v(t_n)), A^{1/2}w(t_n^-))dt.$$

Let us estimate I_1 first. Using (3.1) and (3.4) followed by $(e^{C_{10}\delta}-1) \le C_{10}\delta e^{C_{10}\delta}$ yields

$$
\begin{aligned}
I_1 &\le \int_{t_n}^{t_{n+1}} \big(1-e^{C_{10}(t_n-t)}\big)\big\|J(U(t)-v(t))\big\|\|w(t)\|dt\\
&\le (\rho_K+R_K)M_1 e^{-C_{10}\delta}\int_{t_n}^{t_{n+1}} \big(e^{C_{10}\delta}-1\big)dt \le \delta^2 C_{13}
\end{aligned}
$$

where $C_{13} = (\rho_K+R_K)M_1C_{10}$.

Second estimate I_2. Using $J = P_\lambda J$ and (1.9) with $\alpha = 2$ followed by Corollary 3.7 yields

$$
\begin{aligned}
I_2 &\le \int_{t_n}^{t_{n+1}} \big|AJ(U(t)-v(t))\big||w(t)-w(t_n^-)|dt\\
&\le \int_{t_n}^{t_{n+1}} \big|AP_\lambda J(U(t)-v(t))\big||w(t)-w(t_n^-)|dt\\
&\le \int_{t_n}^{t_{n+1}} \lambda^{1/2}\big\|J(U(t)-v(t))\big\||w(t)-w(t_n^-)|dt\\
&\le \lambda^{1/2}(\rho_K+R_K)\int_{t_n}^{t_{n+1}} |w(t)-w(t_n^-)|dt \le \delta^{3/2}C_{14}
\end{aligned}
$$

where $C_{14} = \lambda^{1/2}(\rho_K + R_K)M_E$.

Finally estimate I_3 and I_4 using Lemma 3.9 as

$$\begin{aligned} I_3 &\le \int_{t_n}^{t_{n+1}} \big\|J(U(t) - U(t_n))\big\| \|w(t_n^-)\| dt \\ &\le M_1 \int_{t_n}^{t_{n+1}} \big\|J(U(t) - U(t_n))\big\| dt \le \delta^{3/2} C_{15} \end{aligned}$$

where $C_{15} = M_1 \rho_D$ and similarly

$$\begin{aligned} I_4 &\le \int_{t_n}^{t_{n+1}} \big\|J(v(t) - v(t_n))\big\| \|w(t_n^-)\| dt \\ &\le M_1 \int_{t_n}^{t_{n+1}} \big\|J(v(t) - v(t_n))\big\| dt \le \delta^{3/2} C_{16} \end{aligned}$$

where $C_{16} = M_1 R_D$.

It follows that

$$\|w(t_{n+1}^-)\|^2 \le e^{\delta C_{10}} \big\{ (1 + \delta \mu C_{11}) \|w(t_n^-)\|^2 + \delta^{3/2} C_{17} \big\}$$

where $C_{17} = \delta_{\max}^{1/2}(\mu^2 C_{12} + 2\mu C_{13}) + 2\mu(C_{14} + C_{15} + C_{16})$.

Setting $y_n = \|w(t_n^-)\|^2$ gives the inequality $y_{n+1} \le \alpha y_n + \beta$ where $\alpha = e^{\delta C_{10}}(1 + \delta \mu C_{11})$ and $\beta = \delta^{3/2} e^{\delta C_{10}} C_{17}$. Note that

$$\alpha - 1 \ge \delta(C_{10} + \mu e^{\delta C_{10}} C_{11}) \qquad \text{and} \qquad \alpha^n \le e^{(t_n - t_0)(C_{10} + \mu C_{11})}.$$

Therefore, by induction

$$\begin{aligned} \|w(t_n^-)\| &\le e^{(t_n - t_0)(C_{10} + \mu C_{11})} \|w(t_0^-)\| + \frac{e^{(t_n - t_0)(C_{10} + \mu C_{11})} - 1}{C_{10} + \mu e^{\delta C_{10}} C_{11}} \delta^{1/2} e^{\delta C_{10}} C_{17} \\ &\le e^{2\delta_{\max}(C_{10} + \mu C_{11})} \|w(t_0^-)\| + \frac{e^{2\delta_{\max}(C_{10} + \mu C_{11})} - 1}{C_{10} + \mu C_{11}} \delta^{1/2} e^{\delta_{\max} C_{10}} C_{17}. \end{aligned}$$

Since $\|w(t_0^-)\| = 0$ it follows there is C_{18} independent of δ such that

$$\|w(t_n^-)\| \le \delta^{1/2} C_{18} \qquad \text{for} \qquad n = 1, \ldots, N.$$

Again from (3.20) we have

$$\|w_n\| \le (1 + \delta \mu C_4) \delta^{1/2} C_{18} + \delta \mu C_5 \le \delta^{1/2} C_{19}$$

where $C_{19} = (1 + \delta_{\max} \mu C_4) C_{18} + \delta_{\max}^{1/2} \mu C_5$. Finally, (3.19) implies

$$\|w(t)\|^2 \le e^{C_2 \delta} \delta C_{19}^2 + \frac{C_3}{C_2}\big(e^{C_2 \delta} - 1\big) \le e^{C_2 \delta} \delta C_{19}^2 + C_3 \delta e^{C_2 \delta} \le \delta C_{20}$$

where $C_{20} = e^{C_2 \delta_{\max}}(C_{19}^2 + C_3)$. Therefore $\|w(t)\| \to 0$ as $\delta \to 0$ uniformly on $[t_0, T]$. □

4 Concluding Remarks

In the context of the incompressible two-dimensional Navier–Stokes equations we have introduced a relaxation parameter κ depending on the time δ between successive observations into the update step of the spectrally-filtered direct-insertion method. This overcomes the lack of synchronization when the observations are inserted too frequently. The relationship between κ and δ may be obtained numerically by varying these parameters independently and then minimizing the average error about an ensemble of reference solutions. It turns out that the relation $\kappa = \mu\delta$ for a particular fixed μ works well over a wide range.

Analytically we have shown as $\delta \to 0$ that the approximation corresponding to the κ-relaxed discrete-in-time method converges to the solution obtained by continuous nudging on any finite time interval. Although this does not imply $\|u(t) - U(t)\| \to 0$ as $t \to \infty$ for all sufficiently small δ, the numerics suggest for suitable h, λ and μ that u does indeed synchronize with U. A complete analysis proving such a result would be of great interest.

Data assimilation for the two-dimensional magnetohydrodynamic equations studied by Biswas, Hudson, Larios and Pie [4] and Hudson and Jolly [19] provide an opportunity to further test the scaling $\kappa = \delta\mu$. Let $\mathbf{S}$ be the semigroup corresponding to the evolution of the two-dimensional magnetohydrodynamic equations and $\mathbf{X} = (U_1, U_2, B_1, B_2)$ be the state of the velocity and magnetic fields. Given the free-running solution $\mathbf{X}(t) = \mathbf{S}(t)\mathbf{X}_0$ consider interpolated observations $\mathbf{JX}(t_n)$ at times t_n taken on one component of each field. Thus,

$$\mathbf{J}(u_1, u_2, b_1, b_2) = (Ju_1, 0, Jb_1, 0)$$

where Ju_1 and Jb_1 could be projections onto the N lowest Fourier modes or interpolants of the form $P_\lambda P_H I_h$ as in (1.10). The analysis step corresponding to (1.15) then becomes

$$\mathbf{x}_{n+1} = (I - \delta\mu\mathbf{J})\mathbf{S}(t_{n+1} - t_n)\mathbf{x}_n + \delta\mu\mathbf{JX}(t_n).$$

Our conjecture is that $\mathbf{x}_n$ will as $\delta \to 0$ converge to the approximating solutions obtained by the continuous-in-time nudging method given as Algorithm 2.3 in [19].

In the more general case where $t_{n+1} - t_n = \delta_n$ and $\max_n \delta_n$ is small (1.17) reads as

$$\int_{t_n}^{t_{n+1}} \mu J(U - v)dt \approx \delta_n \mu J\big(U(t_n) - v(t_n)\big)$$

which immediately suggests the time-dependent relaxation parameter $\kappa_n = \delta_n\mu$. We expect similar results to those obtained here also hold when (1.15) is replaced by

$$u_{n+1} = (I - \delta_n\mu J)S(t_{n+1} - t_n)u_n + \delta_n\mu JU(t_{n+1}).$$

We remark that the time between successive observations may not be uniform when there are missing observations. Thus, it may happen that $\max_n \delta_n$ is not small but for most of the time δ_n is small. In this case taking $\kappa_n = \min(1, \delta_n\mu)$ would avoid overrelaxation where $\kappa_n > 1$. This suggests another direction for future work: Find conditions on sequences of δ_n which for some values of n are large and for which the approximation u numerically synchronizes with the reference solution U.

A Appendix

In this appendix we follow the methods used to prove Theorem 5 in [3] to obtain explicit bounds on v given by equation (1.16) that avoid imposing additional conditions on h and μ but hold only for a finite time interval $[t_0, T]$. These bounds depend on T and are valid for the case when v does not synchronize with U.

We begin with the following existence theorem:

Theorem A.1. *Suppose $J = P_\lambda P_H I_h$ where I_h satisfies* (1.10)*. Then for any $h > 0$ and $\mu > 0$ the equations* (1.16) *with $f \in V$ and $v(t_0) \in V$ have unique strong solutions that satisfy*

$$v \in C\big([t_0, T]; V\big) \cap L^2\big((t_0, T); \mathcal{D}(A)\big) \tag{A.1}$$

and

$$\frac{dv}{dt} \in L^2\big((t_0, T); H\big) \tag{A.2}$$

for any $T > t_0$.

The following analysis makes use of the Aubin Compactness Theorem which we state here for reference as

Theorem A.2. *Consider three separable reflexive Banach spaces $X_1 \subset X_0 \subset X_{-1}$ in which the inclusion $X_1 \subset X_0$ is compact and the inclusion $X_0 \subset X_{-1}$ is continuous. If v_m is a sequence such that*

$$\big\|v_m\big\|_{L^2((t_0,T),X_1)} \qquad \textit{and} \qquad \Big\|\frac{dv_m}{dt}\Big\|_{L^2((t_0,T),X_{-1})}$$

are uniformly bounded in m. There there is a subsequence m_j and $v \in L^2\big((t_0, T), X_0\big)$ such that

$$\int_{t_0}^{T} \big\|v(\tau) - v_{m_j}(\tau)\big\|^2_{X_0} d\tau \to 0 \qquad \textit{as} \qquad j \to \infty.$$

Details are in Aubin [2], the proof of Lemma 8.2 in [8] or page 224 of [14].

Our analysis also makes use of a particular case of the Lions–Magenes theorem stated as Theorem 3 in Section 5.9.2 of [9] or Lemma 1.2 in Chapter III of [25]. We state this result here for reference as

Theorem A.3. *Suppose $X_1 \subset X_0 \subset X_{-1}$ is a Gelfand triple. Thus, X_1 is densely and continuously embedded in X_0 and X_{-1} is the dual of X_1 with a pairing that extends the inner product on X_0. Suppose $u \in L^2\big((t_0, T); X_1\big)$ with $du/dt \in L^2\big((t_0, T); X_{-1}\big)$. Then, after redefinition on a set of measure zero, the mapping*

$$t \to \|u(t)\|^2_{X_0}$$

is absolutely continuous with

$$\frac{d}{dt}\|u(t)\|^2_{X_0} = 2\Big\langle \frac{du(t)}{dt}, u(t)\Big\rangle \tag{A.3}$$

for almost every $t_0 \leq t \leq T$.

We turn now to the

Proof of Theorem A.1. Let $g = f + \mu JU$. By Proposition 3.1

$$|g| \leq |f| + \mu |JU| \leq G_0 \qquad \text{where} \qquad G_0 = |f| + \mu \rho_J. \tag{A.4}$$

Now proceed by the Galerkin method.

Let P_m be the projection onto the m lowest Fourier modes. Note the projection P_m employed in this proof is related to the spectral filter P_λ introduced in (1.7) such that $P_m = P_\lambda$ for $m = \text{card}\{\, k \in \mathcal{J} : 0 < |k|^2 \leq \lambda \,\}$. Although the Galerkin method could be carried out using P_λ, for consistency with [25], [8], [14] and [24] we shall employ P_m.

Choose m large enough that $P_m P_\lambda = P_\lambda$. Thus, $P_m J = J$. Now, consider the truncation of (1.16) given by

$$\frac{dv_m}{dt} + \nu A v_m + P_m B(v_m, v_m) = P_m g - \mu J v_m \tag{A.5}$$

where $v_m(t_0) = P_m v_0$ and $t > t_0$. Note $v_m \in P_m H$ is finite dimensional with dimension m.

Our goal is to find bounds on v_m which are uniform in m and hence to let $m \to \infty$ to obtain a solution for (1.16). Taking inner product of (A.5) with v_m we have

$$\frac{1}{2}\frac{d}{dt}|v_m|^2 + \nu \|v_m\|^2 = (g, v) - \mu (J v_m, v_m) \leq \frac{1}{2\mu}|g|^2 + \frac{\mu}{2}|v_m|^2 - \mu (J v_m, v_m). \tag{A.6}$$

Estimate the last term of (A.6) using (1.11) as

$$\begin{aligned} -\mu(Jv_m, v_m) &= \mu(v_m - Jv_m, v_m) - \mu|v_m|^2 \leq \mu|v_m - Jv_m||v_m| - \mu|v_m|^2 \\ &\leq \frac{\mu}{2}\frac{\nu}{(\lambda^{-1} + c_1 h^2)\mu}|v_m - Jv_m|^2 + \frac{\mu}{2}\frac{(\lambda^{-1} + c_1 h^2)\mu}{\nu}|v_m|^2 - \mu|v_m|^2 \\ &= \frac{\nu}{2}\|v_m\|^2 + \left(\frac{(\lambda^{-1} + c_1 h^2)\mu^2}{2\nu} - \mu\right)|v_m|^2. \end{aligned} \tag{A.7}$$

Substituting (A.7) back into (A.6) and multiplying by two then yields

$$\frac{d}{dt}|v_m|^2 + \nu \|v_m\|^2 \leq \frac{1}{\mu}|g|^2 + \gamma |v_m|^2 \tag{A.8}$$

where $\gamma \geq (\lambda^{-1} + c_1 h^2)\mu^2/\nu - \mu$. We further assume $\gamma > 0$. Note the case when γ may be chosen negative was treated in [3] and leads to the synchronization of v with U.

Now, drop the viscosity term $\nu\|v_m\|^2$ from (A.8) and apply (A.4) to obtain

$$\frac{d}{dt}|v_m|^2 - \gamma |v_m|^2 \leq \frac{G_0^2}{\mu} \qquad \text{or equivalently} \qquad \frac{d}{dt}\big(|v_m|^2 e^{-\gamma t}\big) \leq \frac{G_0^2}{\mu} e^{-\gamma t}.$$

Since $|v_m(t_0)| \leq |v(t_0)|$, integrating from t_0 to t yields

$$|v_m(t)|^2 \leq |v(t_0)|^2 e^{\gamma(t - t_0)} + \frac{G_0^2}{\gamma\mu}(e^{\gamma(t-t_0)} - 1). \tag{A.9}$$

Moreover, $\gamma > 0$ implies $e^{\gamma t} \le e^{\gamma T}$ for $t \in [t_0, T]$. It follows that

$$|v_m(t)| \le R_0 \qquad \text{where} \qquad R_0^2 = |v(t_0)|^2 e^{\gamma(T-t_0)} + \frac{G_0^2}{\gamma\mu}(e^{\gamma(T-t_0)} - 1). \tag{A.10}$$

We emphasize that R_0 depends on T but holds uniformly in m.

Next, integrate (A.8) directly as

$$|v_m(T)|^2 - |v(t_0)|^2 + \nu \int_{t_0}^T \|v_m(\tau)\|^2 d\tau \le \frac{1}{\mu} \int_{t_0}^T |g(\tau)|^2 d\tau + \gamma \int_{t_0}^T |v_m|^2 d\tau.$$

From this (A.4) and (A.10) imply

$$\nu \int_{t_0}^T \|v_m(\tau)\|^2 d\tau \le \left(\frac{G_0^2}{\mu} + \gamma R_0^2 \right) (T - t_0) + |v(t_0)|^2. \tag{A.11}$$

It follows that

$$\Big(\int_{t_0}^T \|v_m(\tau)\|^2 d\tau \Big)^{1/2} \le \widetilde{R}_1 \quad \text{where} \quad \widetilde{R}_1^2 = \frac{1}{\nu} \left(\frac{G_0^2}{\mu} + \gamma R_0^2 \right) (T - t_0) + \frac{1}{\nu} R_0^2. \tag{A.12}$$

To estimate $\|v_m(t)\|^2$ and $\int_{t_0}^T |Av_m(\tau)|^2 d\tau$ take inner product of (A.5) with Av_m. Thus,

$$\frac{1}{2}\frac{d}{dt}\|v_m\|^2 + \nu|Av_m|^2 + \big(B(v_m, v_m), Av_m\big) = (g, Av_m) - \mu(Jv_m, Av_m). \tag{A.13}$$

From (3.5) followed by Young's inequality with $p = 4$ and $q = 4/3$ we have

$$\begin{aligned} |(B(v_m, v_m), Av_m)| &\le c|v_m|^{1/2}\|v_m\|^{1/2}\|v_m\|^{1/2}|Av_m|^{1/2}|Av_m| \\ &= \Big(\frac{6^{3/4}}{\nu^{3/4}} c|v_m|^{1/2}\|v_m\| \Big) \Big(\frac{\nu^{3/4}}{6^{3/4}} |Av_m|^{3/2} \Big) \le \frac{54}{\nu^3} c^4 |v_m|^2 \|v_m\|^4 + \frac{\nu}{8}|Av_m|^2. \end{aligned} \tag{A.14}$$

We also have

$$|(g, Av_m)| \le |g||Av_m| \le \frac{2}{\nu} G_0^2 + \frac{\nu}{8}|Av_m|^2 \tag{A.15}$$

and recalling the definition of γ that

$$\begin{aligned} -\mu(Jv_m, Av_m) &= \mu(v_m - Jv_m, Av_m) - \mu\|v_m\|^2 \\ &\le \frac{\mu^2}{\nu}|v_m - Jv_m|^2 + \frac{\nu}{4}|Av_m|^2 - \mu\|v_m\|^2 \le \gamma\|v_m\|^2 + \frac{\nu}{4}|Av_m|^2. \end{aligned} \tag{A.16}$$

Substituting (A.14), (A.15) and (A.16) into (A.13) yields

$$\frac{d}{dt}\|v_m\|^2 + \nu|Av_m|^2 \le \frac{108}{\nu^3} c^4 |v_m|^2 \|v_m\|^4 + \frac{4G_0^2}{\nu} + 2\gamma\|v_m\|^2 \tag{A.17}$$

and consequently

$$\frac{d}{dt}\|v_m\|^2 - \Big(\frac{108}{\nu^3} c^4 |v_m|^2 \|v_m\|^2 + 2\gamma \Big) \|v_m\|^2 \le \frac{4G_0^2}{\nu} \tag{A.18}$$

for all $t \in [t_0, T]$. Define

$$\psi_m(t) = \exp\left\{ -\int_{t_0}^{t} \Big(\frac{108}{\nu^3} c^4 |v_m(\tau)|^2 \|v(\tau)\|^2 + 2\gamma \Big) d\tau \right\}.$$

Note that

$$\begin{aligned} &\int_{t_0}^{t} \Big(\frac{108}{\nu^3} c^4 |v_m(\tau)|^2 \|v_m(\tau)\|^2 + 2\gamma \Big) d\tau \\ &\qquad \leq \frac{108}{\nu^3} c^4 R_0^2 \int_{t_0}^{T} \|v_m(\tau)\|^2 d\tau + 2(T - t_0)\gamma \leq \frac{108}{\nu^3} c^4 R_0^2 \widetilde{R}_1^2 + 2\gamma(T - t_0) < \infty \end{aligned}$$

and therefore

$$\psi_m(t) \geq \exp\left\{ -\Big(\frac{108}{\nu^3} c^4 R_0^2 \widetilde{R}_1^2 + 2\gamma(T - t_0) \Big) \right\} \qquad \text{for all} \qquad t \in [0, T].$$

Multiplying (A.18) by $\psi_m(t)$, integrating and using that $\psi_m(t_0) = 1$ and $\psi_m(t_0) \leq 1$ yields

$$\psi_m(t) \|v_m(t)\|^2 - \|v_m(t_0)\|^2 \leq \frac{4G_0^2}{\nu} \int_0^t \psi_m(\tau) d\tau \leq \frac{4G_0^2}{\nu}(T - t_0). \tag{A.19}$$

Since ψ_m is decreasing on $[t_0, T]$, it follows that

$$\|v_m(t)\| \leq R_1 \qquad \text{where} \qquad R_1^2 = \frac{1}{\psi_m(T)} \left\{ \|v(t_0)\|^2 + \frac{4G_0^2}{\nu}(T - t_0) \right\}.$$

On the other hand, directly integrating inequality (A.17) gives us

$$\begin{aligned} &\|v_m(T)\|^2 - \|v_m(t_0)\|^2 + \nu \int_{t_0}^{T} |Av_m(\tau)|^2 d\tau \\ &\qquad \leq \frac{108}{\nu^3} c^4 \int_0^T \Big(|v_m(\tau)|^2 \|v_m(\tau)\|^2 + 2\gamma \Big) \|v_m(\tau)\|^2 d\tau + \frac{4G_0^2}{\nu}(T - t_0). \end{aligned} \tag{A.20}$$

Estimate

$$\begin{aligned} &\int_{t_0}^{T} \Big(|v_m(\tau)|^2 \|v_m(\tau)\|^2 + 2\gamma \Big) \|v_m(\tau)\|^2 d\tau \\ &\qquad\qquad \leq (R_0^2 R_1^2 + 2\gamma) \int_{t_0}^{T} \|v_m(\tau)\|^2 d\tau \leq (R_0^2 R_1^2 + 2\gamma) \widetilde{R}_1^2. \end{aligned}$$

Substituting this into (A.20) yields

$$\Big(\int_{t_0}^{T} |Av_m(\tau)|^2 d\tau \Big)^{1/2} \leq \widetilde{R}_2$$

where

$$\widetilde{R}_2^2 = \frac{108}{\nu^4} c^4 (R_0^2 R_1^2 + 2\gamma) \widetilde{R}_1^2 + \frac{4G_0^2}{\nu^2}(T - t_0) + \frac{1}{\nu} R_1^2.$$

Next we obtain uniform bounds on dv_m/dt. Setting $R_J = (\lambda^{-1} + c_1 h^2)^{1/2} R_1 + R_0$ as in the proof of Proposition 3.1 we obtain that $|Jv_m| \le R_J$. From (A.5) and (3.7) holds

$$\Big|\frac{dv_m}{dt}\Big| \le \nu |Av_m| + c|v_m|^{1/2}\|v_m\||Av_m|^{1/2} + G_0 + \mu R_J$$
$$\le 2\nu|Av_m| + \frac{c^2}{4\nu} R_0 R_1^2 + G_0 + \mu R_J.$$

Therefore,

$$\Big|\frac{dv_m}{dt}\Big|^2 \le 8\nu^2 |Av_m|^2 + 2\Big(\frac{c^2}{4\nu} R_0 R_1^2 + G_0 + \mu R_J\Big)^2$$

and we obtain

$$\Big(\int_{t_0}^{T} \Big|\frac{dv_m}{dt}\Big|^2\Big)^{1/2} \le R_T \quad \text{where} \quad R_T^2 = 8\nu^2 \widetilde{R}_2 + 2\Big(\frac{c^2}{4\nu} R_0 R_1^2 + G_0 + \mu R_J\Big)^2 (T - t_0).$$

Hence dv_m/dt is bounded in $L^2((t_0, T), H)$ uniformly in m.

At this point we have shown that the solutions u_m satisfy the bounds

$$\big\|v_m\big\|_{L^2((t_0,T),\mathcal{D}(A))} \le \widetilde{R}_2 \qquad \text{and} \qquad \Big\|\frac{dv_m}{dt}\Big\|_{L^2((t_0,T),H)} \le R_T$$

uniformly in m. Since the inclusion $\mathcal{D}(A) \subseteq V$ is compact and $V \subseteq H$ is continuous, then by Theorem A.2 there is a subsequence m_j and $v \in L^2((t_0, T), V)$ such that

$$\int_{t_0}^{T} \big\|v(\tau) - v_{m_j}(\tau)\big\|^2 d\tau \to 0 \qquad \text{as} \qquad j \to \infty.$$

Taking additional subsequences if necessary we may further suppose $v \in L^\infty((t_0, T); V)$ and that the weak time derivative $dv/dt \in L^2((t_0, T); H)$.

It follows that

$$v \in L^\infty((t_0, T); V) \cap L^2((t_0, T); \mathcal{D}(A)) \qquad \text{and} \qquad dv/dt \in L^2((t_0, T); H). \tag{A.21}$$

Moreover, since $J\colon V \to V$ is continuous then for every $\phi \in H$ holds

$$\Big(\frac{dv}{dt}, \phi\Big) + \nu(Av, \phi) + (B(v, v), \phi) = (g, \phi) - \mu(Jv, \phi) \tag{A.22}$$

for almost every $t \in [t_0, T]$.

It remains to show such solutions are unique, contained in $C([t_0, T]; V)$ and depend continuously on the initial data. Let v_1 and v_2 be solutions satisfying (A.21) and (A.22). Choose K large enough such that $\|v_1\| \le K$ and $\|v_2\| \le K$ for almost every $t \in [t_0, T]$. Let $\tilde{v} = v_1 - v_2$. Then $\tilde{v}$ satisfies

$$\frac{d\tilde{v}}{dt} + \nu A\tilde{v} + B(v_1, \tilde{v}) + B(\tilde{v}, v_2) = -\mu J\tilde{v}. \tag{A.23}$$

By the Riesz representation theorem every element $\varphi^* \in V_2^*$ can be represented by $\varphi \in V_2$ such that $\langle \varphi^*, w\rangle = (\!(w,\varphi)\!)_2$ for all $w \in V_2$. Setting $\psi_k = |k|^2\varphi_k$ we obtain $\psi \in H$ where

$$(\!(w,\psi)\!) = L^2 \sum_{k\in\mathcal{J}} |k|^2 w_k \overline{\psi_k} = L^2 \sum_{k\in\mathcal{J}} |k|^4 w_k \overline{\varphi_k} = (\!(w,\varphi)\!)_2 = \langle \varphi^*, w\rangle.$$

This identifies the dual of V_2 with H in such a way that extends the inner product of V. Thus, $\mathcal{D}(A) \subset V \subset H$ is a Gelfand triple.

Now take inner product of (A.23) with $A\tilde v$ and apply Theorem A.3 to obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde v\|^2 + \nu|A\tilde v|^2 + (B(v_1,\tilde v), A\tilde v) + (B(\tilde v, v_2), A\tilde v) = -\mu(J\tilde v, A\tilde v). \tag{A.24}$$

Estimate

$$|(B(v_1,\tilde v), A\tilde v)| \le c|v_1|^{1/2}\|v_1\|^{1/2}\|\tilde v\|^{1/2}|A\tilde v|^{3/2} \le \frac{27c^4}{4\nu^3\lambda_1}\|v_1\|^4\|\tilde v\|^2 + \frac{\nu}{4}|A\tilde v|^2,$$

$$|(B(\tilde v, v_2), A\tilde v)| \le c|\tilde v|^{1/2}\|v_2\||A\tilde v|^{3/2} \le \frac{27c^4}{4\nu^3\lambda_1}\|v_2\|^4\|\tilde v\|^2 + \frac{\nu}{4}|A\tilde v|^2$$

and

$$-\mu(J\tilde v, A\tilde v) \le \mu|J\tilde v||A\tilde v| \le \frac{\mu^2}{2\nu}|J\tilde v|^2 + \frac{\nu}{2}|A\tilde v|^2 \le \frac{\mu^2 c_2^2}{2\nu}\|\tilde v\|^2 + \frac{\nu}{2}|A\tilde v|^2.$$

Hence,

$$\frac{d}{dt}\|\tilde v\|^2 \le \Big(\frac{27c^4}{\nu^3\lambda_1}K^4 + \frac{\mu^2c_2^2}{\nu}\Big)\|\tilde v\|^2 \qquad \text{for all} \qquad t\in[t_0,T]. \tag{A.25}$$

Integrating (A.25) yields

$$\|\tilde v(t)\|^2 \le \|\tilde v(t_0)\|^2 \exp\Big\{\Big(\frac{27c^4}{\nu^3\lambda_1}K^4 + \frac{\mu^2c_2^2}{\nu}\Big)(t-t_0)\Big\}.$$

Since $\tilde v(t_0) = v_1(t_0) - v_2(t_0)$ this shows strong solutions are unique and depend continuously on the initial data. □

Note that the proof of Lemma A.1 has already provided the bounds

$$\|v\|_\alpha \le R_\alpha \quad \text{for} \quad \alpha = 0,1 \qquad \text{and} \qquad \Big(\int_{t_0}^T \|v\|_\alpha^2\Big)^{1/2} \le \widetilde R_\alpha \quad \text{for} \quad \alpha = 1,2.$$

To finish the proof of Theorem 3.2 we need explicit formulas for R_2 and $\widetilde R_3$. To this end we further assume $v_0 \in \mathcal{D}(A)$ and present the following formal estimates that if desired may be made rigorous using the same techniques as above.

Provided $f \in V$ and $U_0 \in V$, it follows that $g = f + \mu JU \in V$ for all $t \ge 0$. In particular,

$$\|g\| \le \|f\| + \mu\|JU\| \le G_1 \qquad \text{where} \qquad G_1 = \|f\| + \mu\rho_K.$$

Now take the inner product of equation (A.5) with A^2v to obtain

$$\frac{1}{2}\frac{d}{dt}|Av|^2 + \nu|A^{3/2}v|^2 + \big(A^{1/2}B(v,v), A^{3/2}v\big) = \big(A^{1/2}g, A^{3/2}v\big) - \mu\big(A^{1/2}Jv, A^{3/2}v\big).$$

Estimate using (3.10) as

$$\begin{aligned}\big|\big(A^{1/2}B(v,v),A^{3/2}v\big)\big| &\le \|B(v,v)\||A^{3/2}v| \le c|v|^{1/2}\|v\|^{1/2}|Av|^{1/2}|A^{3/2}v|^{3/2}\\ &\le \frac{9^3}{2^5}\frac{c^4}{\nu^3}|v|^2\|v\|^2|Av|^2+\frac{\nu}{6}|A^{3/2}v|^2 \le \frac{9^3}{2^5}\frac{c^4}{\nu^3}R_0^2R_1^2|Av|^2+\frac{\nu}{6}|A^{3/2}v|^2.\end{aligned}$$

Also

$$\big|\big(A^{1/2}g,A^{3/2}v\big)\big| \le \|g\||A^{3/2}v| \le \frac{3}{2\nu}G_1^2+\frac{\nu}{6}|A^{3/2}v|^2$$

and

$$\mu\big|\big(A^{1/2}Jv,A^{3/2}v\big)\big| \le \mu\|Jv\||A^{3/2}v| \le \frac{3\mu^2}{2\nu}R_K^2+\frac{\nu}{6}|A^{3/2}v|^2.$$

It follows that

$$\frac{d}{dt}|Av|^2+\nu|A^{3/2}v|^2 \le c_4|Av|^2+c_5 \tag{A.26}$$

where

$$c_4=\frac{9^3}{2^4}\frac{c^4}{\nu^3}R_0^2R_1^2 \qquad \text{and} \qquad c_5=\frac{3}{\nu}\big(G_1^2+\mu^2R_K^2\big).$$

Dropping the second term of the left, multiplying by e^{-c_4t} and integrating over $[t_0,t)$ yields

$$|Av|^2 \le e^{c_4(t-t_0)}|Av_0|^2+\frac{c_5}{c_4}\big(e^{c_4(t-t_0)}-1\big).$$

Therefore, for $t\in[t_0,T]$ holds

$$|Av|^2 \le R_2 \qquad \text{where} \qquad R_2=e^{c_4(T-t_0)}|Av_0|^2+\frac{c_5}{c_4}\big(e^{c_4(T-t_0)}-1\big).$$

This is the $\alpha=2$ bound needed for (3.2).

Next, directly integrate (A.26) over $[t_0,T]$ to obtain

$$|Av(T)|^2-|Av_0|^2+\nu\int_{t_0}^T|A^{3/2}v|^2 \le c_4\int_{t_0}^T|Av|^2+(T-t_0)c_5$$

so that

$$\Big(\int_{t_0}^T|A^{3/2}v|^2\Big)^{1/2} \le \widetilde{R}_3 \qquad \text{where} \qquad \widetilde{R}_3^2=\frac{1}{\nu}\big(R_2^2+c_4\widetilde{R}_2^2+(T-t_0)c_5\big).$$

Finally, note that $[t_n,t_{n+1}]\subseteq[t_0,T]$ implies

$$\Big(\int_{t_n}^{t_{n+1}}\|v\|_\alpha^2\Big)^{1/2} \le \Big(\int_{t_0}^T\|v\|_\alpha^2\Big)^{1/2} \le \widetilde{R}_\alpha.$$

Therefore, the bounds $\widetilde{R}_\alpha$ for $\alpha=1,2,3$ obtained in this appendix satisfy (3.3).

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