On the Assouad dimension of self-similar sets with overlaps

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Abstract

It is known that, unlike the Hausdorff dimension, the Assouad dimension of a self-similar set can exceed the similarity dimension if there are overlaps in the construction. Our main result is the following precise dichotomy for self-similar sets in the line: either the *weak separation property* is satisfied, in which case the Hausdorff and Assouad dimensions coincide; or the *weak separation property* is not satisfied, in which case the Assouad dimension is maximal (equal to one). In the first case we prove that the self-similar set is Ahlfors regular, and in the second case we use the fact that if the *weak separation property* is not satisfied, one can approximate the identity arbitrarily well in the group generated by the similarity mappings, and this allows us to build a *weak tangent* that contains an interval. We also obtain results in higher dimensions and provide illustrative examples showing that the 'equality/maximal' dichotomy does not extend to this setting.

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1 Introduction

Self-similar sets are arguably the most important examples of fractal sets. This is due to their simple description and the fact that they exhibit many of the properties one expects from fractals. Although they have been studied extensively in the literature over the past half century, there are still many fascinating and challenging open problems available, particularly relating to the overlapping case. That is, self-similar sets F such that every iterated function system with attractor F fails to satisfy the open set condition. Our paper focuses on the Assouad dimension of such sets.

Given a bounded set $E \subseteq \mathbb{R}^d$ where $d \in \mathbb{N}$ and $\rho > 0$, let $\mathcal{N}_E(\rho)$ be the smallest number of open balls of radius ρ required to cover E. Given a (potentially unbounded) set $X \subseteq \mathbb{R}^d$, let

$$\mathcal{N}_X(r, \rho) \;=\; \max\left\{\, \mathcal{N}_{X \cap B_r(x)}(\rho) : x \in X \,\right\},$$

where

$$B_r(x) = \{\, y \in \mathbb{R}^d : \|x - y\| < r \,\}$$

denotes the open ball of radius r centered at x. The *Assouad dimension* $\dim_{\mathrm{A}} X$ is then defined to be the infimum over all s for which there exists K_s such that $\mathcal{N}_X(r, \rho) \le K_s (r/\rho)^s$ for all $0 < \rho < r \le 1$. We note that some authors allow r (and $\rho < r$) to be arbitrarily large, but for bounded sets X this does not change the dimension.

This dimension was introduced by Bouligand [4] to study bi-Lipschitz embeddings and was further studied by Assouad in [1, 2]. Although an interesting notion in its own right, it has found most prominence in the literature due to its relationship with embeddability problems and quasi-conformal mappings. For example, Olson [22] proved that for $X \subseteq \mathbb{R}^d$, $\dim_{\mathrm{A}}(X - X) < s$ implies almost every orthogonal projection of rank s is injective on X and results in an embedding that is bi-Lipschitz except for a logarithmic correction. This result was extended to subsets of infinite-dimensional Hilbert spaces by Olson and Robinson in [23]. Further work relating to Assouad dimension and embeddings may be found in Heinonen [10], Luukkainen [14], Mackay and Tyson [16] and Robinson [25]. Recently, the Assouad dimension has also been gaining substantial attention in the literature on fractal geometry.

Let $\mathcal{I} = \{1, \ldots, |\mathcal{I}|\}$ be a finite index set and $\{S_i\}_{i \in \mathcal{I}}$ be a set of contracting self-maps on some compact subset of $\mathbb{R}^d$. Such a collection of maps is called an *iterated function system* (IFS). It is well known, see for example Edgar [5] or Falconer [6], that there exists a unique non-empty compact set $F \subseteq \mathbb{R}^d$ satisfying

$$F = \bigcup_{i \in \mathcal{I}} S_i(F) \tag{1.1}$$

which is called the *attractor* of the iterated function system. In this paper we shall work exclusively in the setting where the maps are *similarities*. A map $S \colon \mathbb{R}^d \to \mathbb{R}^d$ is called a *similarity* if there exists a ratio $c > 0$ such that

$$\|S(x) - S(y)\| = c\|x - y\| \quad \text{for all} \quad x, y \in \mathbb{R}^d.$$

If $c \in (0, 1)$ then S is called a *contracting similarity*. A *self-similar set* is any set $F \subset \mathbb{R}^d$ for which there exists an iterated function system of contracting similarities $\{S_i\}_{i \in \mathcal{I}}$ that has F as its attractor. Without loss of generality we will assume from now on that $F \subseteq [0, 1]^d$.

Given an iterated function system of contracting similarities, the *similarity dimension* $\dim_{\mathrm{S}}(\{S_i\}_{i \in \mathcal{I}})$ is defined to be the maximum of the spatial dimension d and the unique solution s of the *Hutchinson–Moran formula*

$$\sum_{i \in \mathcal{I}} c_i^s = 1.$$

Note that the similarity dimension is always an upper bound for the Hausdorff dimension of F which we shall denote by $\dim_{\mathrm{H}} F$. The Assouad dimension is also an upper bound on the Hausdorff dimension (see [25], for example).

Most of the commonly used notions of dimension coincide for arbitrary self-similar sets, regardless of separation conditions. Indeed, equality of the box, packing and Hausdorff dimensions has been long known and follows, for example, from the implicit theorems of Falconer [7] and McLaughlin [17]. The additional equality with the lower dimension, which is the natural dual to the Assouad dimension and is *a priori* just a lower bound for the Hausdorff dimension, was proved by Fraser [9, Theorem 2.11]. For this reason we will omit discussion of all notions of dimension apart from the Hausdorff and Assouad dimensions. For a review of the other notions see Falconer [8, Chapters 2-3] or Robinson [25].

In [18] Moran introduced the *open set condition* (OSC) which, if satisfied, guarantees equality of the Hausdorff dimension and the similarity dimension, see for example [8, Section 9.3]. It can occur that the Hausdorff dimension is strictly less than the similarity dimension if different iterates of maps in the iterated function system overlap exactly. An example of this dimension drop is given by the modified Sierpinski triangle of Ngai and Wang [19, Figure 2, page 670]. It is an important open problem to decide if exact overlaps are the only way such a dimension drop can occur, see for example Peres and Solomyak [24, Question 2.6]. Recently an important step towards solving this problem has been made by Hochman [12], where it is shown in $\mathbb{R}$, assuming the defining parameters for the iterated function system are algebraic, that the only cause for a dimension drop is exact overlaps. Hochman's arguments rely on a careful application of ergodic theory and in establishing inverse theorems for the growth of certain entropy functions.

Definition 1.1. *A system* $\{S_i\}_{i\in\mathcal{I}}$ *with attractor* F *satisfies the* open set condition *if there exists a non-empty open set* U *such that*

$$\bigcup_{i\in\mathcal{I}} S_i(U) \subseteq U$$

with the union disjoint.

Note that the open set U need not be unique. Falconer [6, page 122] shows that $F \subseteq \overline{U}$ and a result of Schief [26] is that we may assume that $F \cap U \neq \emptyset$. Notably, if the OSC is satisfied, then the Assouad dimension also coincides with the Hausdorff, and thus similarity, dimension. This result, which has been known in the folklore and essentially dates back to Hutchinson, is due to the fact that self-similar sets are Ahlfors regular, which is a sufficient condition for equality of Hausdorff and Assouad dimension. With $\mathcal{H}^s$ denoting the s-dimensional Hausdorff measure, recall that a set $F \subseteq \mathbb{R}^d$ is called *Ahlfors regular* if there exist constants $a, b > 0$ such that

$$a\, r^s \leq \mathcal{H}^s\big(B_r(x) \cap F\big) \leq b\, r^s$$

for $s = \dim_{\mathrm{H}} F$, all $x \in F$ and all $0 < r < 1$, see Heinonen [10, Chapter 8]. We note that it is sufficient to show that there exists some $r_0 > 0$ such that these bounds hold for all $r \in (0, r_0)$. Without relying on the sophisticated notion of Ahlfors regularity, simple direct proofs are also possible, see for example Olson [22], Olsen [21], Henderson [11], and Fraser [9]. Olsen [21, Question 1.3] asked the natural question of whether $\dim_{\mathrm{H}} F = \dim_{\mathrm{A}} F$ holds for any self-similar set F, regardless of separation properties. This was answered in the negative by Fraser [9, Section 3.1]. Similar techniques were used by Henderson [11] to find examples of self-similar sets such that $\dim_{\mathrm{A}} F$ is arbitrarily small but where $\dim_A(F - F) = 1$. The Assouad dimension of more general attractors has also been considered by, for example, Mackay [15] and Fraser [9] who studied certain planar self-affine constructions.

The weak separation property of Lau and Ngai [13] and Zerner [27] will be our primary tool for studying self-similar sets with overlaps and, indeed, provides a pleasing dichotomy concerning Assouad dimension. Before describing the weak separation property we introduce some notation.

Let $\mathcal{I}^* = \bigcup_{k \geq 1} \mathcal{I}^k$ be the set of all finite sequences with entries in $\mathcal{I}$. For

$$\alpha = \big(i_1, i_2, \ldots, i_k\big) \in \mathcal{I}^*$$

write

$$S_\alpha = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_k}$$

and

$$c_\alpha = c_{i_1} c_{i_2} \cdots c_{i_k}.$$

Note that S_α is a contracting similarity with ratio c_α. Let

$$\overline{\alpha} = (i_1, i_2, \ldots, i_{k-1}) \in \mathcal{I}^* \cup \{\omega\},$$

where ω is the empty sequence of length zero. For notational convenience define $S_\omega = I$ to be the identity map and $c_\omega = 1$. Let

$$\mathcal{E} = \{S_\alpha^{-1} \circ S_\beta : \alpha, \beta \in \mathcal{I}^* \text{ with } \alpha \neq \beta\}$$

and equip the group of all similarities on $\mathbb{R}^d$ with the topology induced by pointwise convergence. We now define the weak separation property.

Definition 1.2 (Lau and Ngai [13], Zerner [27]). *A system $\{S_i\}_{i \in \mathcal{I}}$ with attractor F satisfies the* weak separation property *if*

$$I \notin \overline{\mathcal{E} \setminus \{I\}}.$$

The weak separation property says that we cannot find a sequence of pairs of maps (S_α, S_β) such that $S_\alpha^{-1} \circ S_\beta$ gets arbitrarily close but not equal to the identity. Intuitively, this means that the images $S_\alpha(F)$ may overlap only in a limited number of ways.

It was shown by Zerner [27] that the open set condition is strictly stronger than the weak separation property. In particular using results from Schief [26] it was shown that the open set condition is equivalent to $I \notin \overline{\mathcal{E}}$. Since all maps in $\mathcal{E}$ are similarities, taking the pointwise closure of $\mathcal{E}$ is equivalent to taking the closure in the space of similarities $S \colon [0,1]^d \to \mathbb{R}^d$ equipped with the uniform norm $\| \cdot \|_{L^\infty([0,1]^d)}$ which we denote by $\| \cdot \|_\infty$ throughout, and where the value of d will be clear from the context.

Our main results may be stated as follows. In the case of self-similar sets in the real line we obtain a precise dichotomy.

Theorem 1.3. *Let F be a non-trivial self-similar set in $\mathbb{R}$. If F satisfies the weak separation property, then* $\dim_A F = \dim_H F$*; otherwise,* $\dim_A F = 1$.

In the case of self-similar sets in $\mathbb{R}^d$ we obtain a less precise dichotomy.

Theorem 1.4. *Let F be a self-similar set in $\mathbb{R}^d$ that is not contained in any $(d-1)$-dimensional hyperplane. If F satisfies the weak separation property, then* $\dim_{\mathrm{A}} F = \dim_{\mathrm{H}} F$*; otherwise,* $\dim_{\mathrm{A}} F \geq 1$.

The condition not to be contained in a hyperplane is clearly required because it is easy to construct examples where the WSP is not satisfied, but is satisfied when one restricts to the hyperplane containing the self-similar set. In this situation, clearly the Assouad and Hausdorff dimensions coincide and may be strictly less than one. However, this condition is not restrictive at all, because one can always restrict to a linear space with the appropriate dimension, i.e. where one cannot restrict further to any hyperplane, and then apply Theorem 1.4. Theorems 1.3 and 1.4 may be broken into two parts. The first case covers what happens when the weak separation property holds and the second case covers when the weak separation property fails. Section 2 treats the first case; Section 3 treats second case. After proving our main results we explore some examples which illustrate our theorems and in particular show that Theorem 1.4 is sharp in the following sense: there exist self-similar sets in $\mathbb{R}^d$ not satisfying the WSP, but where the Assouad dimension is strictly less than d. We close with some remarks concerning how we might improve our understanding of the Assouad dimension of a self-similar set in $\mathbb{R}^d$ when the weak separation property is not satisfied.

2 Systems with the Weak Separation Property

Our first result shows that the weak separation property implies that the Hausdorff and Assouad dimensions coincide.

Theorem 2.1. *Let F be a self-similar set in $\mathbb{R}^d$ not contained in any $(d-1)$-dimensional hyperplane. If the weak separation property is satisfied then F is Ahlfors regular and, in particular,* $\dim_{\mathrm{H}} F = \dim_{\mathrm{A}} F$.

Before beginning our proof we introduce the notations

$$\mathcal{I}_r = \left\{ \alpha \in \mathcal{I}^* : c_\alpha \leq r < c_{\overline{\alpha}} \right\} \tag{2.1}$$

and

$$\mathcal{I}_r(x) = \left\{ \alpha \in \mathcal{I}_r : B_r(x) \cap S_\alpha(F) \neq \emptyset \right\}.$$

Note that for $r \in (0,1)$ we have

$$F = \bigcup_{\alpha \in \mathcal{I}_r} S_\alpha(F) \quad \text{and} \quad B_r(x) \cap F \subseteq \bigcup_{\alpha \in \mathcal{I}_r(x)} S_\alpha(F).$$

We remark that the common value for the Hausdorff and Assouad dimensions given by Theorem 2.1 may be strictly less than the similarity dimension. This happens when exact overlaps allow certain compositions of maps to be deleted. In fact, if one takes the infimum over all such 'reduced IFSs', then Zerner [27, Theorem 2] shows that this gives the common value. In particular, if the WSP is satisfied and F is not contained in a hyperplane, then

$$\dim_{\mathrm{H}} F = \dim_{\mathrm{A}} F = \lim_{r \to 0} \dim_{\mathrm{S}}(\{S_\alpha : \alpha \in \mathcal{I}_r\}) = \inf_{r \in (0,1)} \dim_{\mathrm{S}}(\{S_\alpha : \alpha \in \mathcal{I}_r\}).$$

To see why this number drops from the original similarity dimension if there are exact overlaps, observe that $\{S_\alpha : \alpha \in \mathcal{I}_r\}$ is a *set* of maps rather than a *multiset*, i.e. repeated maps are included only once. Zerner [27, Proposition 2] also shows that the number on the right of the above equation is equal to the original similarity dimension if and only if the OSC is satisfied. The problem is that this value may be very difficult to compute. Ngai and Wang [19] introduced the *finite type condition* which is strictly stronger than the WSP (see Nguyen [20]), but allows the above value to be computed explicitly.

Proof of Theorem 2.1. Let F be a self-similar set in $\mathbb{R}^d$ not contained in any $(d-1)$-dimensional hyperplane and assume that the weak separation property is satisfied. Let $c_{\min} = \min\{\, c_i : i \in \mathcal{I}\,\}$ and write $|F|$ for the diameter of F. To prove the result, it suffices to show that F is Ahlfors regular. Let $s = \dim_{\mathrm{H}} F$ denote the Hausdorff dimension of F. First observe that the weak separation property implies that F is an s-set, i.e.

$$0 < \mathcal{H}^s(F) < \infty,$$

which is a necessary condition for Ahlfors regularity. Indeed, any self-similar set has finite Hausdorff measure in the critical dimension and Zerner shows that, together with F not being contained in a hyperplane, the weak separation property implies that this measure is also positive [27, Corollary to Proposition 2]. Let $x \in F$ and $r \in (0, \min\{1, |F|\})$ be arbitrary. The lower bound in the definition of Ahlfors regularity is easy to establish. Choose $\alpha \in \mathcal{I}^*$ such that

$$x \in S_\alpha(F) \quad \text{and} \quad c_\alpha \le r/|F| < c_{\overline{\alpha}}.$$

It is clear that such an $\alpha \in \mathcal{I}^*$ exists, that it satisfies $c_\alpha > c_{\min} r/|F|$ and that $S_\alpha(F) \subseteq B_r(x)$. As such, using the scaling property for Hausdorff measure,

$$\mathcal{H}^s\big(B_r(x) \cap F\big) \ge \mathcal{H}^s\big(S_\alpha(F)\big) \ge c_\alpha^s\, \mathcal{H}^s(F) \ge \frac{c_{\min}^s\, \mathcal{H}^s(F)}{|F|^s}\, r^s.$$

The upper bound is more awkward to establish and relies on the weak separation property. Observe that

$$\begin{aligned}
\mathcal{H}^s\big(B_r(x) \cap F\big) &\le \mathcal{H}^s\Big(\bigcup_{\alpha \in \mathcal{I}_r(x)} S_\alpha(F)\Big) \\
&\le \sum_{\alpha \in \mathcal{I}_r(x)} c_\alpha^s\, \mathcal{H}^s(F) \\
&\le \mathcal{H}^s(F)\, |\mathcal{I}_r(x)|\, r^s.
\end{aligned}$$

Thus in order to finish the proof, it suffices to bound $|\mathcal{I}_r(x)|$ independently of x and r. Of course, if the defining system has complicated overlaps, this is impossible, but we can prove it in our case by applying one of the equivalent formulations of the weak separation property given by Zerner. In particular, [27, Theorem 1 (3a), (5a)] shows the definition we use in this paper is equivalent to the following (stated in our notation): for all $y \in \mathbb{R}^d$, there exists $l(y) \in \mathbb{N}$ such that for any $\beta \in \mathcal{I}^*$ and $r > 0$, every ball with radius r contains at most $l(y)$ elements of the set

$$\{S_\alpha(S_\beta(y)) : \alpha \in \mathcal{I}_r\}.$$

Set $y = 0$ and choose $\beta \in \mathcal{I}$ arbitrarily and observe that if $\alpha \in \mathcal{I}_r(x)$, then

$$S_\alpha(S_\beta(0)) \in S_\alpha(F) \subseteq B(x, r + r|F|).$$

Since $\mathbb{R}^d$ is a doubling metric space we can find a cover of $B(x, r+r|F|)$ by fewer than L balls of radius r, where L is a uniform constant independent of x and r. Each of these r-balls can contain no more than $l(0)$ of the points $\{S_\alpha(S_\beta(0)) : \alpha \in \mathcal{I}_r\}$ and so we deduce that $|\mathcal{I}_r(x)| \leq l(0)\, L$, completing the proof. □

3 Systems without the Weak Separation Property

Our second result shows that if the weak separation property does not hold, then the Assouad dimension is bounded below by one. On the real line this gives a set of maximal dimension.

Theorem 3.1. *Let F be a non-trivial self-similar set in $\mathbb{R}$. If the weak separation property is not satisfied, then* $\dim_{\mathrm{A}} F = 1$.

In $\mathbb{R}^d$ we can only show that the dimension is at least one.

Theorem 3.2. *Let F be a self-similar set in $\mathbb{R}^d$ not contained in any $(d-1)$-dimensional hyperplane. If the weak separation property is not satisfied, then* $\dim_{\mathrm{A}} F \geq 1$.

Theorem 3.1 is the special case of Theorem 3.2 when $d = 1$, however, for clarity of exposition we give a separate proof of Theorem 3.1 first. Another reason for doing this is that Theorem 1.3 is the main result of the paper as it provides a precise result in $\mathbb{R}$ and so it is expedient to give a clear proof of Theorem 3.1 without the extra technical details required in $\mathbb{R}^d$. Note that combining Theorem 2.1 with Theorem 3.1 yields Theorem 1.3 which provides a precise dichotomy for self-similar sets in the line. Combining Theorem 2.1 with Theorem 3.2 yields Theorem 1.4.

One of the most powerful techniques for proving lower bounds for Assouad dimension is to construct weak tangents to the set. This approach has been pioneered by Mackay and Tyson [16]. We first recall the definition of the Hausdorff distance and then give the definition of weak tangent from [16], see also [15].

Definition 3.3. *Given compact subsets X and Y of $\mathbb{R}^d$, the Hausdorff distance between X and Y is defined as*

$$\mathrm{d}_{\mathcal{H}}(X, Y) = \max\left\{\, \mathrm{p}_{\mathcal{H}}(X, Y), \mathrm{p}_{\mathcal{H}}(Y, X)\right\}$$

where

$$\mathrm{p}_{\mathcal{H}}(X, Y) = \sup_{x\in X} \inf_{y \in Y} \|x - y\|.$$

Definition 3.4. *Let F and $\hat{F}$ be compact subsets of $\mathbb{R}^d$. We say that $\hat{F}$ is a weak tangent of F if there exists a compact set $X \subseteq \mathbb{R}^d$ and a sequence of similarity maps $T_k \colon \mathbb{R}^d \to \mathbb{R}^d$ such that $d_{\mathcal{H}}(T_k(F) \cap X, \hat{F}) \to 0$ as $k \to \infty$.*

Proposition 3.5. *If $\hat{F}$ is a weak tangent of F then* $\dim_{\mathrm{A}} \hat{F} \leq \dim_{\mathrm{A}} F$.

The proof of Proposition 3.5 can be found in [16, Proposition 6.1.5] and [15, Proposition 2.1]. Taking $X = F$ and $T_k = I$ shows that F is a weak tangent of itself. Alternatively taking $X = \overline{B_1(0)}$ and $T_k(x) = x/k$ shows that $\{0\}$ is a weak tangent of any non-empty set F. In either of these cases Proposition 3.5 is clearly true. More interesting examples can occur when the ratios c_k corresponding to the similarities T_k satisfy $c_k \to \infty$ as $k \to \infty$.

In our work, we shall employ the following slightly weaker notion of a tangent than the weak tangent of Definition 3.4.

Definition 3.6. *Let F and $\hat{F}$ be compact subsets of $\mathbb{R}^d$. We say $\hat{F}$ is a weak pseudo-tangent of F if there exists a sequence of similarity maps $T_k \colon \mathbb{R}^d \to \mathbb{R}^d$ such that $\mathrm{p}_{\mathcal{H}}(\hat{F}, T_k(F)) \to 0$ as $k \to \infty$.*

We choose the name weak pseudo-tangent because the main difference between Definition 3.6 and Definition 3.4 is the replacement of the Hausdorff metric $\mathrm{d}_{\mathcal{H}}$ with the one-sided Hausdorff pseudo-metric $\mathrm{p}_{\mathcal{H}}$. Note that we also do not need to take the intersection with some explicitly chosen set X in the definition of the weak pseudo-tangent. Fortunately for our analysis the Assouad dimension of a pseudo-tangent still forms a lower bound on the Assouad dimension of the original set.

Proposition 3.7. *If $\hat{F}$ is a weak pseudo-tangent of F then $\dim_{\mathrm{A}} \hat{F} \leq \dim_{\mathrm{A}} F$.*

The proof of Proposition 3.7 follows the proofs in the weak-tangent setting given by Mackay and Tyson, but we include the details for completeness. In fact our proof is slightly simpler since we do not have to take intersections with some previously chosen X and we need only control 'one side' of the convergence. Moreover, Proposition 3.5 follows from Proposition 3.7 since every weak tangent is a restriction of a weak pseudo-tangent.

Proof. Let $s > \dim_{\mathrm{A}} F$. By definition there exists K_s such that

$$\mathcal{N}_F(r, \rho) \leq K_s (r/\rho)^s \quad \text{for all} \quad 0 < \rho < r \leq 1.$$

Since $\hat{F}$ is a weak pseudo-tangent there exist similarities T_k such that

$$\mathrm{p}_{\mathcal{H}}(\hat{F}, T_k(F)) \to 0 \quad \text{as} \quad k \to \infty.$$

Since any cover of an r-ball in $T_k(F)$ by ρ-balls gives rise to a cover of a corresponding r/c_k-ball in F by the same number of ρ/c_k-balls, and vice-versa, it follows that

$$\mathcal{N}_{T_k(F)}(r, \rho) = \mathcal{N}_F(r/c_k, \rho/c_k) \leq K_s (r/\rho)^s \quad \text{for all} \quad 0 < \rho < r \leq 1.$$

Note that K_s is independent of k. Given $r \in (0, 1/2)$ and $\rho \in (0, r)$ choose k so large that

$$\mathrm{p}_{\mathcal{H}}(\hat{F}, T_k(F)) \leq \rho/4.$$

Thus, every point of $\hat{F}$ is within $\rho/2$ distance from some point of $T_k(F)$ and in particular

$$\hat{F} \subseteq \bigcup_{\eta \in T_k(F)} B_{\rho/2}(\eta). \tag{3.1}$$

Given $x \in \hat{F}$ choose $y \in T_k(F)$ such that $\|x - y\| < \rho/2$. Then

$$B_r(x) \subseteq B_{r+\rho/2}(y) \subseteq B_{2r}(y).$$

Let $\{ B_{\rho/2}(y_i) : i = 1, \ldots, N \}$ be a cover of $T_k(F) \cap B_{2r}(y)$ with

$$N \le K_s \Big(\frac{2r}{\rho/2} \Big)^s.$$

It follows that

$$\hat{F} \cap B_r(x) \subseteq \bigcup_{\eta \in T_k(F) \cap B_{2r}(y)} B_{\rho/2}(\eta) \subseteq \bigcup_{i=1}^{N} \bigcup_{\eta \in B_{\rho/2}(y_i)} B_{\rho/2}(\eta) = \bigcup_{i=1}^{N} B_\rho(y_i).$$

Therefore

$$\mathcal{N}_{\hat{F} \cap B_r(x)}(\rho) \le N \le K_s 4^s (r/\rho)^s$$

for all $x \in \hat{F}$ and $0 < \rho < r < 1/2$ which is sufficient to prove $\dim_{\mathrm{A}} \hat{F} \le s$. It follows that $\dim_{\mathrm{A}} \hat{F} \le \dim_{\mathrm{A}} F$. □

3.1 Proof of Theorem 3.1

Similarities S_i on the real line with ratio $c_i > 0$ come in two types: with reflection and without. Namely,

$$S_i(x) = -c_i x + b_i \quad \text{and} \quad S_i(x) = c_i x + b_i.$$

In the first case the derivative $S_i' = -c_i < 0$. In order to treat iterated function systems that contain both types of similarities we first prove the following result.

Lemma 3.8. *Suppose that there exist $\alpha_k, \beta_k \in \mathcal{I}^*$ such that*

$$\| S_{\alpha_k}^{-1} \circ S_{\beta_k} - I \|_\infty \to 0 \quad \textit{as} \quad k \to \infty.$$

Then there exist $\alpha_k', \beta_k' \in \mathcal{I}^*$ such that

$$\| S_{\alpha_k'}^{-1} \circ S_{\beta_k'} - I \|_\infty \to 0 \quad \textit{as} \quad k \to \infty$$

and $S_{\alpha_k'}' = c_{\alpha_k'} > 0$ for all k.

Proof. If there are an infinite number of k such that $S_{\alpha_k}' > 0$ then let α_k' be this subsequence and we are done. Otherwise, there must be an infinite number of k such that $S_{\alpha_k}' < 0$. By taking a subsequence we may assume, in fact, that $S_{\alpha_k}' < 0$ for all k.

Now, since S_{α_k} contains a reflection for every k, there must be at least one similarity in the iterated function system that contains a reflection. Without loss of generality, assume that $S_1' = -c_1 < 0$. Define α_k' and β_k' so that

$$S_{\alpha_k'}^{-1} = S_1^{-1} \circ S_{\alpha_k}^{-1} \quad \text{and} \quad S_{\beta_k'} = S_{\beta_k} \circ S_1.$$

It follows that $S_{\alpha_k'}' > 0$ for all k and furthermore that

$$\begin{aligned} |(S_{\alpha_k'}^{-1} \circ S_{\beta_k'} - I)(x)| &= |S_1^{-1}\big(S_{\alpha_k}^{-1} \circ S_{\beta_k} \circ S_1(x)\big) - S_1^{-1}\big(S_1(x)\big)| \\ &= c_1^{-1} |S_{\alpha_k}^{-1} \circ S_{\beta_k} \circ S_1(x) - S_1(x)| \\ &= c_1^{-1} |(S_{\alpha_k}^{-1} \circ S_{\beta_k} - I)(S_1(x))|. \end{aligned}$$

Since $F \subseteq [0,1]$ it follows that $S_1(x) \in [0,1]$ for $x \in [0,1]$. Therefore

$$\| S_{\alpha_k'}^{-1} \circ S_{\beta_k'} - I \|_\infty \le c_1^{-1} \| S_{\alpha_k}^{-1} \circ S_{\beta_k} - I \|_\infty \to 0,$$

which finishes the proof of the lemma. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Since F is non-trivial there must be two distinct points $a, b \in F$ and similarities S_1 and S_2 such that

$$S_1(a) = a \quad \text{and} \quad S_2(b) = b. \tag{3.2}$$

Choose $\rho > 0$ small enough that $B_{2\rho}(a) \cap B_{2\rho}(b) = \emptyset$.

Since the Weak Separation Property does not hold, $I \in \overline{\mathcal{E} \setminus \{I\}}$ and we can choose $\alpha_k, \beta_k \in \mathcal{I}^*$ such that

$$0 < \|S_{\alpha_k}^{-1} \circ S_{\beta_k} - I\|_\infty \to 0 \quad \text{as} \quad k \to \infty.$$

By Lemma 3.8 we may assume that

$$S'_{\alpha_k} = c_{\alpha_k} > 0 \quad \text{for all} \quad k.$$

Define $\varphi_k = S_{\alpha_k}^{-1} \circ S_{\beta_k} - I$. Note that $\varphi_k \neq 0$ for all k. Therefore either φ_k is a similarity or a non-zero constant function. Since similarities are injective, images of the disjoint 2ρ balls $B_{2\rho}(a)$ and $B_{2\rho}(b)$ are disjoint. In particular, the origin can lie in at most one of the these images. On the other-hand, if φ_k is a non-zero constant function, then the image of both of these balls consists of the same non-zero point and in particular does not contain the origin. It follows that each $\varphi_k \colon \mathbb{R} \to \mathbb{R}$ must satisfy at least one of

$$\varphi_k(B_{2\rho}(a)) \subseteq (0, \infty), \qquad \varphi_k(B_{2\rho}(b)) \subseteq (0, \infty),$$
$$\varphi_k(B_{2\rho}(a)) \subseteq (-\infty, 0) \quad \text{or} \quad \varphi_k(B_{2\rho}(b)) \subseteq (-\infty, 0).$$

Moreover, by taking a subsequence we may assume that all φ_k satisfy the same inclusion. For definiteness assume $\varphi_k(B_{2\rho}(a)) \subseteq (0, \infty)$ for all k. The other cases may be obtained from the first by relabelling a and b and making a change of coordinates to reverse the direction of the real line if necessary.

Let $f = S_1 \circ S_1$. Then $f' = c = c_1^2 > 0$. This avoids the changes in sign that might occur when S_1 contains a reflection. Fix M large enough that $f^M(F) \subseteq B_\rho(a)$ (see (3.2)). Define

$$\begin{aligned} \zeta_k &= \inf \big\{ \varphi_k(x) : x \in B_{2\rho}(a) \big\}, \\ \delta_k &= \inf \big\{ \varphi_k(x) : x \in B_\rho(a) \big\}, \\ \text{and} \qquad \Delta_k &= \sup \big\{ \varphi_k(x) : x \in B_\rho(a) \big\}. \end{aligned}$$

Then

$$\delta_k - \zeta_k = \rho|\varphi'_k| \quad \text{and} \quad \Delta_k - \delta_k = 2\rho|\varphi'_k|.$$

Therefore (since $\zeta_k \geq 0$)

$$\rho|\varphi'_k| \leq \delta_k \quad \text{and} \quad \Delta_k \leq 3\delta_k.$$

While the inequality $\Delta_k \leq 3\delta_k$ holds whether φ_k is a similarity or a non-zero constant function, note that in the latter case $\varphi'_k = 0$ and then $\Delta_k = \delta_k$. In either case, since we know $\|\varphi_k\|_\infty \to 0$ then $\delta_k \to 0$ as $k \to \infty$.

We now derive an estimate that will be used as the basis for our inductive construction in the next paragraph. First note that

$$\begin{aligned} &(f^{-m} \circ S_{\alpha_k}^{-1} \circ S_{\beta_k} \circ f^m - I)(x) \\ &\qquad = f^{-m}\big(S_{\alpha_k}^{-1} \circ S_{\beta_k} \circ f^m(x)\big) - f^{-m}\big(f^m(x)\big) \\ &\qquad = c^{-m}\big(S_{\alpha_k}^{-1} \circ S_{\beta_k} \circ f^m(x) - f^m(x)\big) \\ &\qquad = c^{-m}\varphi_k\big(f^m(x)\big). \end{aligned}$$

Since $f^m(x) \in B_\rho(a)$ for every $x \in F$ and $m \geq M$ we obtain

$$c^{-m}\delta_k \leq (f^{-m} \circ S_{\alpha_k}^{-1} \circ S_{\beta_k} \circ f^m - I)(x) \leq 3c^{-m}\delta_k \tag{3.3}$$

for every $x \in F$ and $m \geq M$.

Given $\epsilon > 0$ we now define natural numbers k_j, m_j, and maps g_j and h_j by induction on j. First, choose k_1 and $m_1 \geq M$ so large that

$$c^{-m_1}\delta_{k_1} < \epsilon \leq c^{-m_1-1}\delta_{k_1}$$

(this can be done by first choosing k_1 large enough that $\delta_{k_1} < \epsilon c^{-M}$ and then choosing m_1 appropriately) and define

$$g_1 = f^{-m_1} \circ S_{\alpha_{k_1}}^{-1} \quad \text{and} \quad h_1 = S_{\beta_{k_1}} \circ f^{m_1}.$$

From the estimate (3.3) and our choice of k_1 and m_1 it follows that

$$c\epsilon \leq (g_1 \circ h_1 - I)(x) \leq 3\epsilon$$

holds for every $x \in F$. Let $d_j = g_j'$ be the derivative of g_j. Since both f and $S_{\alpha_{k_1}}$ have positive derivatives, we know that $d_1 > 0$. Thus d_1 is the ratio corresponding to the similarity g_1 such that $|g_1(x) - g_1(y)| = d_1|x - y|$ for every $x, y \in \mathbb{R}$. By the construction below, it will also follow that $d_j > 0$ for every $j \in \mathbb{N}$.

For $j \geq 2$ choose k_j and $m_j \geq M$ so large that

$$d_{j-1}c^{-m_j}\delta_{k_j} < \epsilon \leq d_{j-1}c^{-m_j-1}\delta_{k_j}$$

(this can be done in a similar way to the choice of k_1 and m_1, above) and define

$$g_j = g_{j-1} \circ f^{-m_j} \circ S_{\alpha_{k_j}}^{-1} \quad \text{and} \quad h_j = S_{\beta_{k_j}} \circ f^{m_j} \circ h_{j-1}.$$

Since $d_j = d_{j-1}c^{-m_j}c_{\alpha_{k_j}}^{-1}$ and $d_1 > 0$, it follows by induction that $d_j > 0$. Since $h_{j-1}(x) \in F$ we have $f^{m_j}(h_{j-1}(x)) \in B_\rho(a)$ and so (recalling that $\varphi_k = S_{\alpha_k}^{-1} \circ S_{\beta_k} - I$)

$$\begin{aligned}(g_j \circ h_j - g_{j-1} \circ h_{j-1})(x) &= g_{j-1} \circ f^{-m_j}(S_{\alpha_{k_j}}^{-1} \circ S_{\beta_{k_j}} \circ f^{m_j} \circ h_{j-1}(x)) \\ &\qquad - g_{j-1} \circ f^{-m_j}(f^{m_j} \circ h_{j-1}(x)) \\ &= d_{j-1}c^{-m_j}\varphi_{k_j}(f^{m_j} \circ h_{j-1}(x))\end{aligned}$$

implies for every $x \in F$ that

$$c\epsilon \leq (g_j \circ h_j - g_{j-1} \circ h_{j-1})(x) \leq 3\epsilon.$$

We next claim for all $n \in \mathbb{N}$ that

$$\{a\} \cup \{\, g_j \circ h_j(a) : j = 1, \ldots, n \,\} \subseteq \{\, g_n \circ S_\beta(a) : \beta \in \mathcal{I}^* \,\}. \tag{3.4}$$

This follows from induction on n. For $n = 1$ we readily see this is true by taking $\beta \in \mathcal{I}^*$ so that $S_\beta = h_1$ and then taking β so that $S_\beta = g_1^{-1}$. Suppose the claim holds true for $n - 1$, then

$$\{a\} \cup \{\, g_j \circ h_j(a) : j = 1, \ldots, n-1 \,\} \subseteq \{\, g_{n-1} \circ S_\beta(a) : \beta \in \mathcal{I}^* \,\}.$$

Choose $\gamma \in \mathcal{I}^*$ so that $S_\gamma = S_{\alpha_{k_n}} \circ f^{m_n}$. Since $g_n = g_{n-1} \circ f^{-m_n} \circ S_{\alpha_{k_n}}^{-1}$, it follows that

$$\begin{aligned}\{\, g_n \circ S_\beta(a) : \beta \in \mathcal{I}^* \,\} &\supseteq \{\, g_n \circ S_\gamma \circ S_\beta(a) : \beta \in \mathcal{I}^* \,\} \\ &= \{\, g_{n-1} \circ S_\beta(a) : \beta \in \mathcal{I}^* \,\} \\ &\supseteq \{a\} \cup \{\, g_j \circ h_j(a) : j = 1, \ldots, n-1 \,\}.\end{aligned}$$

Additionally taking $\beta \in \mathcal{I}^*$ so that $S_\beta = h_n$ completes the induction.

We finish the proof by constructing a weak pseudo-tangent to F that has Assouad dimension 1. Let $g_{j,n}$ be the functions g_j defined above when $\epsilon = (cn)^{-1}$ and define $T_n = g_{n,n}$. It follows from (3.4) that

$$\begin{aligned}T_n F &\supseteq \{\, g_{n,n} \circ S_\beta(a) : \beta \in \mathcal{I}^* \,\} \\ &\supseteq \{a\} \cup \{\, g_{j,n} \circ h_{j,n}(a) : j = 1, \ldots, n \,\}.\end{aligned}$$

Since $n^{-1} \leq g_{j,n} \circ h_{j,n}(a) - g_{j-1,n} \circ h_{j-1,n}(a) \leq 3(cn)^{-1}$ we obtain

$$\mathrm{p}_{\mathcal{H}}([a, a+1], T_n(F)) \leq 3(cn)^{-1} \to 0.$$

Therefore $[a, a+1]$ is a weak pseudo-tangent of F and

$$\dim_{\mathrm{A}} F \geq \dim_{\mathrm{A}} [a, a+1] = 1.$$

This finishes the proof of our lower bound for self-similar sets on the real line that do not satisfy the weak separation condition. $\square$

3.2 Proof of Theorem 3.2

In this section we prove the more complicated case concerning self-similar sets F in $\mathbb{R}^d$. Thus, F is the attractor of an iterated function system $\{S_i\}_{i \in \mathcal{I}}$ given by

$$S_i(x) = c_i O_i x + b_i$$

where $O_i \in O(d)$ is a $d \times d$ orthogonal matrix, $c_i \in (0,1)$ and $b_i \in \mathbb{R}^d$ for each $i \in \mathcal{I}$. In the case of the real line $O_i = \pm 1$, and so O_i^2 always yielded the identity. However, in higher dimensions there are orthogonal matrices such that $O^m \neq I$ for all $m \geq 0$. To deal with this we make use of Lemma 3.9 below. Given any $\gamma \in \mathcal{I}^*$, define $O_\gamma \in O(d)$ and $b_\gamma \in \mathbb{R}^d$ such that

$$S_\gamma(x) = c_\gamma O_\gamma x + b_\gamma.$$

Note that S_γ is a similarity with ratio c_γ such that

$$\|S_\gamma(x) - S_\gamma(y)\| = c_\gamma \|x - y\| \quad \text{for all} \quad x, y \in \mathbb{R}^d.$$

We now define

$$\mathcal{G} = \{\, O_\alpha : \alpha \in \mathcal{I}^* \,\} \quad \text{and} \quad G = \overline{\mathcal{G}}.$$

and show that for any $\epsilon > 0$ there exists a finite collection of elements of the form O_α with $\alpha \in I^*$ which can be used to approximate every element of G to within ϵ.

Lemma 3.9. *The set G is a compact group. Consequently, given $\epsilon > 0$ there exists a finite set $\mathcal{J} \subseteq \mathcal{I}^*$ such that for every $U \in G$ there is $\alpha \in \mathcal{J}$ such that $\|U - O_\alpha\| < \epsilon$.*

Proof. Let $O(d)$ be the group of all $d \times d$ orthogonal matrices. We claim that G is a subgroup of $O(d)$. Indeed, since $G \subseteq O(d)$ and $\mathcal{G}$ is a semigroup, then G is at least a semigroup. Let $U \in G$ and consider the sequence of iterates U^n where $n \in \mathbb{N}$. Since $O(d)$ is compact and G is closed, there exists a subsequence $U^{n_k} \to V$ where $V \in G$. It follows that $U^{n_{k+1}-n_k} \to I$ and consequently $I \in G$. We may further assume that $n_{k+1} - n_k \geq 2$ for every k, from which it follows that $U^{n_{k+1}-n_k-1} \to U^{-1}$ and consequently $U^{-1} \in G$. This proves the claim.

Now, since G is compact,

$$G \subseteq \bigcup_{\alpha \in \mathcal{I}^*} B_\epsilon(O_\alpha) \quad \text{implies that} \quad G \subseteq \bigcup_{\alpha \in \mathcal{J}} B_\epsilon(O_\alpha)$$

for some finite subset $\mathcal{J} \subseteq \mathcal{I}^*$. Let $U \in G$. Then $U \in B_\epsilon(O_\alpha)$ for some $\alpha \in \mathcal{J}$ and consequently $\|U - O_\alpha\| < \epsilon$. □

For self-similar sets in $\mathbb{R}^d$ we insist that F not lie in any $(d-1)$-dimensional hyperplane of $\mathbb{R}^d$. To make use of this hypothesis we first show that we can find a collection of fixed points of the similarities which span $\mathbb{R}^d$, after a suitable translation.

Lemma 3.10. *Suppose that F is a self-similar set in $\mathbb{R}^d$ that is not contained in any hyperplane. Then there exist $\gamma_n \in \mathcal{I}^*$, where $n = 1, \ldots, d+1$, such that the fixed points a_n of the maps S_{γ_n} satisfy* $\operatorname{span}\{\, a_n - a_{d+1} : n = 1, \ldots, d \,\} = \mathbb{R}^d$.

Proof. Given $a_n \in \mathbb{R}^d$ let A be the $d \times d$ matrix given by the vectors $a_n - a_{d+1}$ as

$$A = \Big[a_1 - a_{d+1} \Big| a_2 - a_{d+1} \Big| \cdots \Big| a_d - a_{d+1} \Big].$$

We are looking for a_n such that $\operatorname{rank} A = d$ or equivalently such that $\det A \neq 0$. Since $\det A$ depends continuously on the a_n then

$$\mathcal{R} = \{\, (a_1, \ldots, a_{d+1}) : \det A \neq 0 \,\} \subseteq [\mathbb{R}^d]^{d+1}$$

is open. Since F is not contained in any hyperplane, there exist $b_n \in F$, $n = 1, \ldots, d+1$, such that $(b_1, \ldots, b_{d+1}) \in \mathcal{R}$. Moreover, since $\mathcal{R}$ is open, there is $\epsilon > 0$ such that $|a_n - b_n| < \epsilon$ for $n = 1, \ldots, d+1$ implies that $(a_1, \ldots, a_{d+1}) \in \mathcal{R}$.

Choose $r < c_{\min}$ such that $r \operatorname{diam} F < \epsilon$. Then

$$b_n \in F = \bigcup_{\alpha \in \mathcal{I}_r} S_\alpha(F) \quad \text{implies that} \quad b_n \in S_{\gamma_n}(F)$$

for some $\gamma_n \in I_r$ (see (2.1) for the definition of I_r). Let a_n be the fixed point of S_{γ_n}. Then

$$a_n \in F \quad \text{implies that} \quad a_n = S_{\gamma_n}(a_n) \in S_{\gamma_n}(F).$$

Therefore

$$|a_n - b_n| \leq \operatorname{diam} S_{\gamma_n} F = c_{\gamma_n} \operatorname{diam} F \leq r \operatorname{diam} F < \epsilon$$

and consequently $(a_1, \ldots, a_{d+1}) \in \mathcal{R}$. □

As the geometry in $\mathbb{R}^d$ is more complicated than on the real line we will need to prove one more lemma before commencing with the proof of Theorem 3.2 itself.

Lemma 3.11. *Let F be a self-similar set in $\mathbb{R}^d$ not contained in any $(d-1)$-dimensional hyperplane. Let Φ_k be a sequence of affine linear maps on $\mathbb{R}^d$ such that $0 < \|\Phi_k\|_\infty \to 0$ as $k \to \infty$. Then there exist $\rho > 0$, $\gamma \in \mathcal{I}^*$ and $a \in \mathbb{R}^d$ such that $S_\gamma(a) = a$ and a subsequence k_j such that for every j we have*

$$\sup\{ \|\Phi_{k_j}(x)\| : x \in B_\rho(a) \} \leq 3 \inf\{ \|\Phi_{k_j}(x)\| : x \in B_\rho(a) \}$$

and $\rho\|D\Phi_{k_j}\| \leq \|\Phi_{k_j}(a)\|$.

Proof. Note that the derivative $D\Phi_k$ is a $d \times d$ matrix. We separate our proof into two cases: the first case is when there are an infinite number of k such that the derivative $D\Phi_k = 0$ and we may assume by taking subsequences that $D\Phi_{k_j} = 0$ for all k_j; the second case is when there are only a finite number of k such that $D\Phi_k = 0$ and we may assume $D\Phi_{k_j} \neq 0$ for all j.

If $D\Phi_{k_j} = 0$ for all j, then $\Phi_{k_j} = B_j$ where $B_j \in \mathbb{R}^d$ and $B_j \neq 0$. In this case any $\rho > 0$ and $\gamma \in \mathcal{I}^*$ yields a similarity S_γ with fixed point a such that $0 \notin \Phi_{k_j}(B_{2\rho}(a))$.

If $D\Phi_{k_j} \neq 0$ for all j, write the singular value decomposition of $D\Phi_{k_j}$ as

$$D\Phi_{k_j} = U_j \Sigma_j V_j^{-1}$$

where $U_j, V_j \in O(d)$ are $d \times d$ orthogonal matrices and Σ_j is a diagonal matrix with singular values ordered such that $\sigma_{j,1} \geq \cdots \geq \sigma_{j,d} \geq 0$. Since $O(d)$ is compact, we may assume by taking further subsequences, that $V_j \to V$ for some $V \in O(d)$. Define

$$s_j = \sigma_{j,1} = \|D\Phi_{k_j}\|, \quad u_j = U_j e_1, \quad v_j = V_j e_1 \quad \text{and} \quad v_* = V e_1,$$

where $e_1 = (1, 0, \ldots, 0)$. Then $0 < \|D\Phi_{k_j}\| \to 0$ implies that $0 < s_j \to 0$ as $j \to \infty$.

Since F is not contained in any hyperplane, Lemma 3.10 implies that there exist $\gamma_n \in \mathcal{I}^*$ and $a_n \in F$ such that

$$S_{\gamma_n}(a_n) = a_n \quad \text{and} \quad \operatorname{span}\{ a_n - a_{d+1} : n = 1, \ldots d \} = \mathbb{R}^d. \tag{3.5}$$

By (3.5) there exists n such that

$$(a_n - a_{d+1}) \cdot v_* \neq 0.$$

Define $\varphi_j(x) = u_j \cdot \Phi_{k_j}(x)$. Note that

$$\varphi_j(x) = u_j \cdot (D\Phi_{k_j})x + z_j = U_j e_1 \cdot U_j \Sigma_j V_j^{-1} x + z_j = s_j v_j \cdot x + z_j$$

for the scalars $z_j = u_j \cdot \Phi_{k_j}(0)$. Therefore

$$\varphi_j(a_n) - \varphi_j(a_{d+1}) = s_j v_j \cdot (a_n - a_{d+1}).$$

Since $v_j \to v_*$ and $s_j > 0$ it follows for j large enough that $\varphi_j(a_n) - \varphi_j(a_{d+1}) \neq 0$. By choosing $\rho > 0$ small enough and taking another subsequence, we may conclude that

$$\varphi_j(B_{2\rho}(a_n)) \cap \varphi_j(B_{2\rho}(a_{d+1})) = \emptyset \quad \text{for every} \quad j.$$

For each j at least one of the statements $0 \notin \varphi_j(B_{2\rho}(a_n))$ or $0 \notin \varphi_j(B_{2\rho}(a_{d+1}))$ is true. By taking yet one more subsequence we may fix a to be either a_n or a_{d+1} and γ to be

γ_n or γ_{d+1}, respectively, so that $0 \notin \varphi_j(B_{2\rho}(a))$ for every j. It follows that $S_\gamma(a) = a$ and $0 \notin \Phi_{k_j}(B_{2\rho}(a))$ for every j.

Define

$$\begin{aligned}\delta_k &= \inf\left\{ \|\Phi_k(x)\| : x \in B_\rho(a) \right\} \\ \Delta_k &= \sup\left\{ \|\Phi_k(x)\| : x \in B_\rho(a) \right\}.\end{aligned} \tag{3.6}$$

If $D\Phi_k = 0$ then $\delta_k = \Delta_k$. Otherwise $0 \notin \varphi_k(B_{2\rho}(a))$ implies that for $x \in B_\rho(a)$

$$\|\Phi_k(x)\| \geq |u_k \cdot \Phi_k(x)| = |s_k v_k \cdot x + z_k| \geq s_k \rho,$$

and therefore $\delta_k \geq s_k\rho$. On the other hand $\|\Phi_k(x) - \Phi_k(y)\| \leq s_k\|x - y\|$ implies that

$$\Delta_k \leq \delta_k + 2s_k\rho \leq 3\delta_k.$$

This completes the proof of the lemma. □

We are now ready begin the proof of Theorem 3.2. The proof will construct a weak tangent to F that has dimension greater than one; we recover a 'one-dimensional' structure by constructing a set all of whose points lie in a particular cone in $\mathbb{R}^n$. For a set $U \in \mathbb{R}^d$ we use

$$\mathcal{C}(U) = \{ \lambda u : \lambda > 0 \text{ and } u \in U \}$$

for the cone generated by U.

Proof of Theorem 3.2. Since the weak separation condition is not satisfied, there exist $\alpha_k, \beta_k \in \mathcal{I}^*$ such that $0 < \|S_{\alpha_k}^{-1} \circ S_{\beta_k} - I\|_\infty \to 0$. Define $\Phi_k = S_{\alpha_k}^{-1} \circ S_{\beta_k} - I$. After taking subsequences we may assume by Lemma 3.11 that there exist $\gamma \in \mathcal{I}^*$ and $a \in \mathbb{R}^d$ such that $S_\gamma(a) = a$ and $\rho > 0$ such that the inequalities

$$\sup\{ \|\Phi_k(x)\| : x \in B_\rho(a) \} \leq 3 \inf\{ \|\Phi_k(x)\| : x \in B_\rho(a) \} \tag{3.7}$$

and $\rho\|D\Phi_k\| \leq \|\Phi_k(a)\|$ are satisfied for all k. Defining Δ_k and δ_k as in (3.6) we may further express (3.7) as $\Delta_k \leq 3\delta_k$.

Given $\eta \in (0, 1)$, cover the unit sphere $\partial B_1(0)$ by a finite number M of balls $B_{\eta/2}(w_m)$ where $\|w_m\| = 1$ and $m = 1, 2, \ldots, M$. It follows that

$$\mathbb{R}^d = \bigcup_{m=1}^{M} \mathcal{C}(B_{\eta/2}(w_m)).$$

For each k choose m_k such that $\Phi_k(a)/\|\Phi_k(a)\| \in B_{\eta/2}(w_{m_k})$. Upon taking a subsequence, we obtain a single $w \in \partial B_1(0)$ such that

$$\Phi_k(a)/\|\Phi_k(a)\| \in B_{\eta/2}(w) \quad \text{for all} \quad k.$$

Define $f = S_\gamma = c_\gamma O_\gamma x + b_\gamma$ and note that $Df = c_\gamma O_\gamma$. Let $\rho' = \rho\eta/5$ and choose M so large that $m \geq M$ implies that $f^m(x) \in B_{\rho'}(a)$ for every $x \in F$. Let $\epsilon_2 = \eta/4$. By Lemma 3.9 there exists a finite set $\mathcal{J} \subseteq \mathcal{I}^*$ such that $U \in G$ implies there is $\alpha \in \mathcal{J}$ such that $\|u - O_\alpha\| < \epsilon_2$.

Given $\epsilon > 0$ we now define natural numbers k_j, m_j and maps g_j and h_j by induction on j. Let $g_0 = I$, $h_0 = I$, $d_0 = 1$ and $W_0 = I$. Note that $Dg_0 = d_0 W_0$. Choose k_j and $m_j \geq M$ so large that

$$d_{j-1} c_*^{-1} c_\gamma^{-m_j} \delta_{k_j} < \epsilon \leq d_{j-1} c_*^{-1} c_\gamma^{-m_j - 1} \delta_{k_j}$$

and let $\gamma_j \in \mathcal{J}$ be chosen so $\|O_{\gamma_j}^{-1} O_\gamma^{-m_j} W_{j-1} - I\| = \|W_{j-1} O_{\gamma_j}^{-1} O_\gamma^{-m_j} - I\| \leq \epsilon_2$. Define

$$g_j = g_{j-1} \circ S_{\gamma_j}^{-1} \circ f^{-m_j} \circ S_{\alpha_{k_j}}^{-1} \quad \text{and} \quad h_j = S_{\beta_{k_j}} \circ f^{m_j} \circ S_{\gamma_j} \circ h_{j-1}.$$

Let d_j be the ratio for g_j such that

$$\|g_j(x) - g_j(y)\| = d_j \|x - y\| \quad \text{for all} \quad x, y \in \mathbb{R}^d$$

and let W_j be the $d \times d$ orthogonal matrix such that $Dg_j = d_j W_j$.

We now show that the sequence of functions g_j and h_j defined above satisfy

$$c_* c_\gamma \epsilon \leq \|(g_j \circ h_j - g_{j-1} \circ h_{j-1})(x)\| \leq 3\epsilon \tag{3.8}$$

and

$$(g_j \circ h_j - g_{j-1} \circ h_{j-1})(x) \in \mathcal{C}(B_\eta(w)) \quad \text{for all} \quad x \in F \tag{3.9}$$

for every $j \in \mathbb{N}$. Since

$$f^{m_j} \circ S_{\gamma_j} \circ h_{j-1}(x) \in B_{\rho'}(a) \subseteq B_\rho(a) \quad \text{for all} \quad x \in F,$$

then

$$\begin{aligned}(g_j \circ h_j - g_{j-1} \circ h_{j-1})(x) &= g_{j-1} \circ S_{\gamma_j}^{-1} \circ f^{-m_j}(S_{\alpha_{k_j}}^{-1} \circ S_{\beta_{k_j}} \circ f^{m_j} \circ S_{\gamma_j} \circ h_{j-1}(x)) \\ &\qquad - g_{j-1} \circ S_{\gamma_j}^{-1} \circ f^{-m_j}(f^{m_j} \circ S_{\gamma_j} \circ h_{j-1}(x)) \\ &= d_{j-1} c_{\gamma_j}^{-1} c^{-m_j} W_{j-1} O_{\gamma_j}^{-1} O_\gamma^{-m_j} \Phi_{k_j}(f^{m_j} \circ S_{\gamma_j} \circ h_{j-1}(x))\end{aligned}$$

which implies

$$c_* c_\gamma \epsilon \leq d_{j-1} c_{\gamma_j}^{-1} c^{-m_j} \delta_{k_j} \leq \|(g_j \circ h_j - g_{j-1} \circ h_{j-1})(x)\| \leq d_{j-1} c_{\gamma_j}^{-1} c^{-m_j} \Delta_{k_j} \leq 3\epsilon$$

for all $x \in F$.

Write $y = f^{m_j} \circ S_{\gamma_j} \circ h_{j-1}(x)$, $z = \Phi_{k_j}(y)$ and $\tilde{z} = W_j O_{\gamma_j}^{-1} O_\gamma^{-m_j} z$. Then

$$\|z - \Phi_{k_j}(a)\| = \|\Phi_{k_j}(y) - \Phi_{k_j}(a)\| \leq \|D\Phi_{k_j}\| \|y - a\| \leq \rho' \|D\Phi_{k_j}\|$$

which implies that

$$\|\tilde{z} - z\| \leq \epsilon_2 \|z\| \leq \|z - \Phi_{k_j}(a)\| + \|\Phi_{k_j}(a)\| \leq \epsilon_2 \rho' \|D\Phi_{k_j}\| + \epsilon_2 \|\Phi_{k_j}(a)\|.$$

Since $\eta < 1$, $\rho' = \rho\eta/5$, $\epsilon_2 = \eta/4$ and $\rho \|D\Phi_{k_j}\| \leq \|\Phi_{k_j}(a)\|$ it follows that

$$\begin{aligned}\|\tilde{z} - \Phi_{k_j}(a)\| &\leq (1 + \eta/4)(\eta/5)\rho \|D\Phi_{k_j}\| + \eta/4 \|\Phi_{k_j}(a)\| \\ &\leq \big((5/4)(\eta/5) + \eta/4\big) \|\Phi_{k_j}(a)\| = (\eta/2) \|\Phi_{k_j}(a)\|.\end{aligned}$$

Consequently

$$\frac{W_j O_{\gamma_j}^{-1} O_\gamma^{-m_j} \Phi_{k_j}\big(f^{m_j} \circ S_{\gamma_j} \circ h_{j-1}(x)\big)}{\|\Phi_{k_j}(a)\|} \in B_{\eta/2}\bigg(\frac{\Phi_{k_j}(a)}{\|\Phi_{k_j}(a)\|}\bigg) \subseteq B_\eta(w). \tag{3.10}$$

It follows that

$$(g_j \circ h_j - g_{j-1} \circ h_{j-1})(x) \in \mathcal{C}(B_\eta(w)) \quad \text{for all} \quad x \in F.$$

Figure 1 shows representative locations for $(g_1 \circ h_1 - I)(a)$ and $(g_2 \circ h_2 - g_1 \circ h_1)(a)$ in the cone $\mathcal{C}(B_\eta(w))$. Note that the norm of the projection of the points $g_j \circ h_j(a)$ along the w direction always increases as j increases. In particular, we have

$$c_* c_\gamma \xi \epsilon \leq w \cdot (g_j \circ h_j - g_{j-1} \circ h_{j-1})(a) \leq 3\epsilon$$

where $\xi = \sqrt{1 - \eta^2}$.

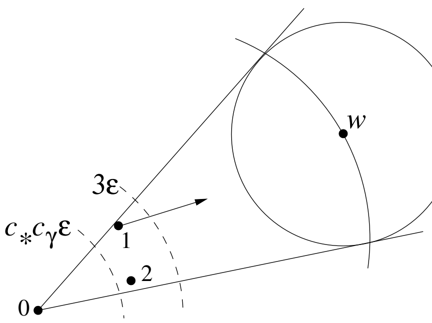

Figure 1: Locations of $(g_1 \circ h_1 - I)(a)$ and $(g_2 \circ h_2 - g_1 \circ h_1)(a)$ denoted by points 1 and 2. The arrow points to the location of $(g_2 \circ h_2 - I)(a)$.

We finish by constructing a weak tangent to F that has Assouad dimension 1. The inclusion

$$\{a\} \cup \{\, g_j \circ h_j(a) : j = 1, \ldots, n \,\} \subseteq \{\, g_n \circ S_\beta(a) : \beta \in \mathcal{I}^* \,\}$$

holds just as (3.4) held in the proof for the real line. However, unlike our proof for the real line, we will construct a weak tangent rather than a weak pseudo-tangent. This is done in order to make use of the fact that the space of all compact subsets of a compact set in $\mathbb{R}^d$ is a compact metric space with respect to the Hausdorff metric.

Let $g_{j,n}$ be the functions g_j defined above when $\epsilon = (c_* c_\gamma \xi n)^{-1}$ and define $T_n = g_{n,n}$. Note that the cone $\mathcal{C}(B_\eta(w))$ in the construction does not depend on ϵ. As before, it follows that

$$T_n F \supseteq \{a\} \cup \{\, g_{j,n} \circ h_{j,n}(a) : j = 1, \ldots n\}.$$

Since

$$n^{-1} \leq w \cdot (g_{j,n} \circ h_{j,n} - g_{j-1,n} \circ h_{j-1,n})(a) \leq 3(c_* c_\gamma \xi n)^{-1}$$

we obtain

$$\mathrm{p}_{\mathcal{H}}([a_w, a_w + 1], w \cdot T_n(F)) \leq 3(c_* c_\gamma \xi n)^{-1} \to 0, \tag{3.11}$$

where $a_w = w \cdot a$. However $w \cdot T_n$ is not a similarity, so this is not a weak pseudo-tangent and we shall make use of one more compactness argument to finish the proof.

Let

$$X = \overline{(\{a\} + \mathcal{C}(B_\eta(w))) \cap B_{1/\xi}(a)}$$

and consider the sets $B_n = T_n(F) \cap X$. Since the space of all compact subsets of X is a compact metric space with respect to the Hausdorff metric, there exists a subsequence n_k and a compact set $\hat{F} \in X$ such that

$$\mathrm{d}_{\mathcal{H}}(\hat{F}, T_{n_k}(F) \cap X) \to 0 \quad \text{as} \quad k \to \infty.$$

Thus, $\hat{F}$ is a weak tangent to F. Since (3.11) still holds along any subsequence, we conclude that $\hat{F}$ is not empty and in particular $w \cdot \hat{F} = [a_w, a_w + 1]$. Since

$$\mathcal{N}_{\hat{F}}(r, \rho) \geq \mathcal{N}_{B_r(a) \cap \hat{F}}(\rho) \geq \mathcal{N}_{[a_w, a_w + \xi r)}(\rho) \geq (2^{-1}\xi)(r/\rho)$$

for all $0 < \rho < r < 1$, it follows immediately that $\dim_A(\hat{F}) \geq 1$. □

4 Examples

If the WSP is satisfied, then our result is sharp, and if the WSP is not satisfied, then our result is sharp if we restrict our attention to $\mathbb{R}$. In this section we present some examples which show, under the hypothesis that the WSP does not hold, that our result in $\mathbb{R}^d$ is also sharp, in some sense. We conclude by mentioning our hope for future work that leads to a refinement of our understanding of the Assouad dimension in $\mathbb{R}^d$ when the weak separation property does not hold.

4.1 Intermediate Assouad dimension in the plane

In light of Theorems 2.1 and 3.1, one might imagine that the Assouad dimension of a self-similar set is either equal to the Hausdorff dimension, or maximal (equal to the ambient spatial dimension). Trivially, this is false by noting that a self-similar set in $\mathbb{R}$ with Assouad dimension equal to 1 and similarity dimension strictly less than 1, is also a self-similar subset of $\mathbb{R}^d$ via the natural embedding and this does not alter the dimensions. However, this is not very enlightening so we give a more convincing example, where the self-similar set is not contained in any hyperplane. The key is to make the weak separation property fail 'only in one direction'. We achieve this by modifying an example of Bandt and Graf [3], which shows that the weak separation property can fail in the line, even if all the contraction ratios are equal. Consider the following four similarity maps on $[0, 1]^2$:

$$S_1(x) = x/5, \qquad S_2(x) = x/5 + (t/5, 0),$$

$$S_3(x) = x/5 + (4/5, 0), \qquad S_4(x) = x/5 + (0, 4/5)$$

where $t \in [0, 4]$. Let F denote the attractor of this system and note that it is not contained in any line and has similarity dimension equal to $\log 4/ \log 5$, which is an upper bound for the Hausdorff dimension. Let F_1 denote the projection of F onto the first coordinate and F_2 denote the projection of F onto the second coordinate. F_1 is the self-similar attractor of the iterated function system consisting of the three maps

$$S'_1(x) = x/5, \qquad S'_2(x) = x/5 + t/5, \qquad S'_3(x) = x/5 + 4/5$$

defined on $[0, 1]$ and F_2 is the self-similar attractor of the iterated function system consisting of the two maps

$$S''_1(x) = x/5, \qquad S''_2(x) = x/5 + 4/5$$

defined on $[0, 1]$, which satisfies the open set condition and thus has Hausdorff dimension and Assouad dimension equal to $\log 2/ \log 5$. Following Bandt and Graf [3, Section 2 (5)], if t is rational, then the iterated function system for F_1 satisfies the weak separation property. However, it is possible to construct irrational t for which the weak separation property is not satisfied. Assume for a moment that we have done this for some $t \in [0, 4]$. It follows from Theorem 3.1 that $\dim_A F_1 = 1$. Note that F is contained in $F_1 \times F_2$ and contains an isometrically embedded copy of F_1. Since the

Assouad dimension is subadditive on product sets and monotone, the Assouad dimension of F is bounded above by the sum of the Assouad dimensions of F_1 and F_2 and below by the Assouad dimension of F_1. This gives

$$\dim_{\mathrm{H}} F \leq \log 4/\log 5 < 1 \leq \dim_{\mathrm{A}} F \leq 1 + \log 2/\log 5 < 2$$

as required. To see how to choose t such that the weak separation property fails for the defining iterated function system for F_1, again following Bandt and Graf [3, Section 2 (5)], let

$$t = 4\sum_{k=0}^{\infty} 5^{-2^k} = 0.9664102400\ldots$$

and observe that elements $(S'_{\boldsymbol{i}})^{-1} \circ S'_{\boldsymbol{j}} \in \mathcal{E}$ where $|\boldsymbol{i}| = |\boldsymbol{j}| = n$ are precisely maps of the form

$$x \mapsto x + \sum_{k=0}^{n-1} 5^k a_k$$

for any sequence a_k over $\{0, \pm 4, \pm t, \pm(4-c)\}$. Let $n = 2^m + 1$ for some large $m \in \mathbb{N}$ and choose the coefficients $\{a_k\}_{k=0}^{n-1}$ as follows:

$$a_k = \begin{cases} -4 & \text{if } k = 2^m - 2^l \text{ for some } l = 0, \ldots, m \\ t & \text{if } k = n-1 \\ 0 & \text{otherwise} \end{cases}$$

This renders the map above equal to

$$x \mapsto x + 5^{2^m} t - 4\sum_{k=0}^{m} 5^{2^m - 2^k} = x + 4\sum_{k=m+1}^{\infty} 5^{2^m - 2^k}$$

which approximates the identity map arbitrarily closely by making m (and thus n) large.

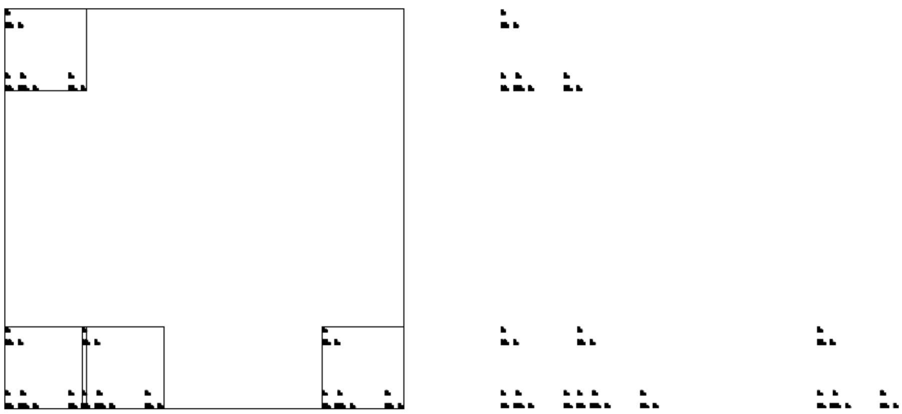

Figure 2: A self-similar set in the plane with intermediate Assouad dimension where $t = 0.9664102400\ldots$ is chosen as above to guarantee the failure of the weak separation property for F_1. The version on the left includes squares to indicate the images of the four maps in the iterated function system.

4.2 Full Assouad dimension in $\mathbb{R}^d$

Here we show how to generalise the example given in Fraser [9, Section 3.1] to arbitrary dimensions. In particular, we show how to construct a self-similar set in $\mathbb{R}^d$ with arbitrarily small Hausdorff dimension, but full Assouad dimension.

Let $d \in \mathbb{N}$ and $\Lambda = \big\{z = (z_1, \dots, z_d) : z_i \in \{0,1\} \text{ for } i = 1, \dots, d\big\}$ be the set of 2^d corners of the unit hypercube $[0,1]^d$. Let $\alpha, \beta, \gamma \in (0,1)$ be such that $(\log \beta)/(\log \alpha) \notin \mathbb{Q}$ and define similarity maps S_α, S_β and S_γ^z $(z \in \Lambda \setminus \{(0,\dots,0)\})$ mapping $[0,1]^d$ into itself by

$$S_\alpha(x) = \alpha x, \qquad S_\beta(x) = \beta x \qquad \text{and} \qquad S_\gamma^z(x) = \gamma x + (1-\gamma)z.$$

Let $F \subseteq [0,1]^d$ be the self-similar attractor of the iterated function system consisting of these $1 + 2^d$ maps, which we will now prove has full Assouad dimension, $\dim_{\mathrm{A}} F = d$, by showing that $[0,1]^d$ is a weak tangent to F. Let $X = [0,1]^d$ and assume without loss of generality that $\alpha < \beta$. For each $k \in \mathbb{N}$ let T_k on $\mathbb{R}^d$ be defined by

$$T_k(x) = \beta^{-k} x$$

and we will now show that $T_k(F) \cap [0,1]^d \to_{d_{\mathcal{H}}} [0,1]^d$. Let

$$E_k^d := \bigcup_{z \in \Lambda} \Big(\big\{\alpha^m \beta^n z : m \in \mathbb{N},\, n \in \{-k, \dots, \infty\}\big\} \cap [0,1]^d \Big)$$

and observe that $E_k^d \subseteq T_k(F) \cap [0,1]^d$ for each $k \in \mathbb{N}$. However,

$$E_k^d = \prod_{l=1}^{d} \Big(\big\{\alpha^m \beta^n : m \in \mathbb{N},\, n \in \{-k, \dots, \infty\}\big\} \cap [0,1] \Big)$$

where the set $\big\{\alpha^m \beta^n : m \in \mathbb{N},\, n \in \{-k, \dots, \infty\}\big\} \cap [0,1]$ is precisely the pre-tangent set E_k which appears in [9, Section 3.1]. There it was shown that $E_k \to_{d_{\mathcal{H}}} [0,1]$ from which it follows immediately that $E_k^d \to_{d_{\mathcal{H}}} [0,1]^d$ and thus $T_k(F) \cap [0,1]^d \to_{d_{\mathcal{H}}} [0,1]^d$, proving the desired result. Proving that $E_k \to_{d_{\mathcal{H}}} [0,1]$ follows by applying Dirchlet's Theorem on Diophantine approximation and using the fact that α and β are not log-commensurable. We omit the details but refer the reader to [9, Section 3.1].

The similarity dimension of F, which is an upper bound for the Hausdorff dimension, is the unique solution, s, of

$$\alpha^s + \beta^s + (2^d - 1)\gamma^s = 1$$

which, by making α, β and γ small, can be made arbitrarily small (but positive) whilst retaining the non-log-commensurable condition on α and β. Finally, observe that despite the fact we were able to utilise the product *sub*structure of F and thus the product structure of the tangent, F itself is not a product, nor could we generalise the example in [9, Section 3.1] by simply taking the d-fold product of the one dimensional version because the result would not be a self-similar set, but a strictly self-affine set.

4.3 Future work

If the weak separation property does not hold, then the lower bound in Theorem 3.2 is sharp in the sense that the Assouad dimension need not be maximal. We suspect, however, that there is a natural additional condition one could add to 'failure of the weak separation property' that would guarantee, for example, that $\dim_A F \geq 2$. Exactly what this additional condition might be is an open question and a topic of further study.

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