Equi-homogeneity, Assouad Dimension and Non-autonomous Dynamics

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Abstract

A fractal, as originally described by Mandelbrot, is a set with an irregular and fragmented shape. Many fractals that have been extensively studied, such as self-similar sets, have the same degree of irregularity and fragmentation at all length scales. In contrast to this, an equi-homogeneous set is an irregular and fragmented shape that at each fixed scale is identical at every point.

We show that self-similar sets that arise from iterated function systems satisfying the Moran open set condition, canonical examples of 'fractal sets', are also equi-homogeneous. Such self-similar sets are also notable in that their Assouad and upper box-counting dimensions are equal. More generally, we show that the Assouad and upper box-counting dimensions are equal for equi-homogeneous sets, provided that the upper and lower box-counting dimensions are equal and are 'attained'. However, the concept of equi-homogeneity is distinct from any previously defined notion of dimensional equivalence.

Using this new theory, we analyse the geometry of a large class of pullback attractors of non-autonomous iterated function systems. These attractors are related to sets that satisfy the Moran structure conditions, include generalized Cantor sets, and possess different dimensional behavior at different length scales. We provide conditions under which the pullback attractor is equi-homogeneous, and use this to compute the Assouad dimension of a certain class of these highly non-trivial sets.

1 Introduction

In this paper we study in detail the notion of 'equi-homogeneity', which was introduced by Olson, Robinson, and Sharples in [22] to study the Assouad dimension of products of generalized Cantor sets. Generalized Cantor sets provide

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simple examples of equi-homogeneous sets $C \subset \mathbb{R}$ whose lower box-counting, upper box-counting, and Assouad dimensions can take arbitrary values satisfying

$$\dim_{\mathrm{LB}} C \le \dim_{\mathrm{B}} C \le \dim_{\mathrm{A}} C.$$

These sets also provide natural examples of sets that satisfy the Moran structure conditions as defined by Wen [26], which can be thought of as a non-autonomous version of the classical Moran open set condition. In this paper, we study the pullback attractors of non-autonomous iterated function systems. Pullback attractors include generalized Cantor sets and under natural assumptions satisfy the Moran structure conditions.

1.1 Pullback attractors

Pullback attractors (see Carvalho, Langa, and Robinson [4] or Kloeden & Rasmussen [13], for example) were introduced to characterize the possible states at time t of a non-autonomous dissipative continuous dynamical system once all previous initial conditions have been forgotten infinitely far in the past. Given an initial value problem of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(t,u), \qquad u(t_0) = u_0,$$

define a semi-process $S(t,t_0)$ for $t \ge t_0$ that maps initial values u_0 to their subsequent time evolution by

$$S(t,t_0)(u_0) = u(t) \qquad \text{for} \qquad t \ge t_0.$$

The pullback attractor is the unique collection of uniformly bounded compact sets A^t for $t \in \mathbb{R}$ such that $A^t = S(t,t_0)A^{t_0}$ for all $t \ge t_0$ and

$$\rho_H(S(t,t_0)(B), A^t) \to 0 \qquad \text{as} \qquad t_0 \to -\infty$$

for all bounded sets B. Here $\rho_H(X,Y)$ is the Hausdorff semi-distance

$$\rho_H(X,Y) = \sup_{x\in X} \inf_{y \in Y} |x-y|.$$

It is straightforward to adapt the idea of a pullback attractor to study iterated function systems whose maps change at each step in the iteration. Such systems arise in the construction of generalized Cantor sets and from the geometric similarities in the Moran structure conditions. The graph-directed Moran fractals studied by Olsen [20] may be likened to the situation in which a non-autonomous iterated function system has a skew product structure. A further generalization, in which the compositions of maps in a iterated function system are indexed by an infinite tree, arises, for example, from the use of the squeezing property to estimate the upper box-counting dimension of the global attractor of autonomous dissipative continuous dynamical system system (see Eden et al. [5]). In order to keep our notation simple and our presentation self-contained we do not consider skew products or iterated function systems indexed by trees here.

To set notation for the rest of this paper we now describe our non-autonomous iterated function systems. For each $i \in \mathbb{N}$ let $f_i : \mathbb{R}^d \to \mathbb{R}^d$ be a contraction with ratio $\sigma_i \in (0, 1)$. Thus

$$|f_i(x) - f_i(y)| \leq \sigma_i |x - y| \qquad \text{for all} \qquad i \in \mathbb{N}. \tag{1}$$

We say f_i is a similarity when the above inequality is, in fact, an equality. For each $k \in \mathbb{N}$ let $\mathcal{I}_k \subset \mathbb{N}$ be an index set with $\operatorname{card}(I_k) < \infty$.

Given $B \subseteq \mathbb{R}^d$ and $k < l$ define

$$\mathcal{S}^{k,l}(B) = \mathcal{S}^k \circ \ldots \circ \mathcal{S}^{l-1}(B) \qquad \text{and} \qquad \mathcal{S}^{k,k}(B) = B$$

where $\mathcal{S}^k(B) = \bigcup_{i \in \mathcal{I}_{k+1}} f_i(B)$. We note that $\mathcal{S}$ has the process structure

$$\mathcal{S}^{k,l} \circ \mathcal{S}^{l,m} = \mathcal{S}^{k,m} \qquad \text{for} \qquad k \leq l \leq m.$$

In particular, $\mathcal{S}^{k,l}$ is the discrete-time analogue of the continuous process operator $S(t, t_0)$, with the identification $t = -k$ and $t_0 = -l$.

Since pullback attractors are obtained in the limit $t_0 \to -\infty$, the switching of signs between l and t_0 allows us the convenience of working with positive indices throughout. This change of sign also results in a notation that is more consistent with the notations previously used when working with iterated function systems where the functions are the same at every step.

A pullback attractor of a non-autonomous iterated function system is a collection of sets F^k for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that

(i) each F^k is compact and uniformly bounded,

(ii) the $\{F^k\}$ are invariant, in the sense that $F^k = \mathcal{S}^k(F^{k+1})$ holds, and

(iii) $\rho_H(\mathcal{S}^{k,l}(B), F^k) \to 0$ as $l \to \infty$ for every bounded set $B \subset \mathbb{R}^d$.

It immediately follows from the definition that a pullback attractor, if it exists, is unique (Theorem 4.2).

1.2 Generalized Cantor sets

The generalized Cantor sets studied in [25] and [22] are illustrative examples of such pullback attractors. These sets are defined as follows. For $\lambda \in (0, 1/2)$ let the application of gen_λ to a disjoint set of compact intervals be the procedure in which the open middle $1 - 2\lambda$ proportion of each interval is removed. Given $c_n \in (0, 1/2)$ for all $n \in \mathbb{N}$, the generalized Cantor set C is given by

$$C = \bigcap_{n=1}^{\infty} C_n \quad \text{where} \quad C_{n+1} = \text{gen}_{c_n} C_n \quad \text{and} \quad C_1 = [0, 1].$$

To see C as a pullback attractor, consider the non-autonomous iterated function system given by

$$f_{2k-1}(x) = c_k x, \quad f_{2k}(x) = c_k x + 1 - c_k \tag{2}$$

with $\mathcal{I}_k = \{2k - 1, 2k\}$ for each $k \in \mathbb{N}$. It is not difficult to see that $C_n = \mathcal{S}^{0,n}([0, 1]) = \mathcal{S}^0 \circ \ldots \circ \mathcal{S}^{n-1}([0, 1])$, from which it is clear that applying the generator gen_{c_n} to the intervals C_n is equivalent to replacing the chain of maps

$\mathcal{S}^{0,n}$ by $\mathcal{S}^{0,n} \circ \mathcal{S}^n = \mathcal{S}^{0,n+1}$. By Theorem 4.3 this system possesses a pullback attractor F^k which, by Theorem 4.2, is unique.

We now construct this attractor explicitly. Let

$$C^k = \bigcap_{n=1}^{\infty} C_n^k \quad \text{where} \quad C_{n+1}^k = \text{gen}_{c_{n+k}} C_n^k \quad \text{and} \quad C_1^k = [0,1];$$

clearly $C^0 = C$. To finish the proof it is enough to show that C^k satisfies properties (i)–(iii). Clearly C^k is compact and uniformly bounded. Moreover, since $C_n^k = \mathcal{S}^{k,k+n-1}([0,1])$ for every $n \in \mathbb{N}$ then

$$\mathcal{S}^k C^{k+1} = \bigcap_{n=1}^{\infty} \mathcal{S}^k \mathcal{S}^{k+1,k+n}([0,1]) = \bigcap_{n=1}^{\infty} \mathcal{S}^{k,k+n}([0,1]) = \bigcap_{n=1}^{\infty} C_{n+1}^k = C^k$$

shows that C^k is invariant. Finally, let B be any bounded set. Then an argument from Section 4.2 shows that

$$\rho_H(S^{k,l}(B), S^{k,l}([0,1])) \leq 2^{k-l} \rho_H(B, [0,1])$$

(this is a consequence of (28) with $\sigma^* = 1/2$) which, coupled with the fact that $\rho_H(C_n^k, C^k) \to 0$ as $n \to \infty$ yields

$$\begin{aligned} \rho_H\big(S^{k,l}(B), C^k\big) &\leq \rho_H\big(S^{k,l}(B), S^{k,l}([0,1])\big) + \rho_H\big(S^{k,l}([0,1]), C^k\big) \\ &\leq 2^{k-l} \rho_H(B, [0,1]) + \rho_H(C_{l-k}^k, C^k) \to 0 \end{aligned}$$

as $l \to \infty$. Therefore, property (iii) is satisfied.

1.3 Equi-homogeneity

Pullback attractors can exhibit dimensionally different behavior at different length scales and are therefore not, in general, Ahlfors regular[1] (Heinonen [8] or Mackay & Tyson [18], for example). However, under two natural sets of hypotheses we are able to show that such sets are equi-homogeneous, which we now define.

Let (X, d_X) be a metric space and adopt the notation

$$B_\delta(x) = \{\, y \in X : d_X(x,y) \leq \delta \,\}$$

for the closed ball of radius δ with center $x \in X$. For brevity we refer to closed balls of radius δ as δ-balls. For each $\delta > 0$ we denote the minimum number of δ-balls with centers in F such that F is contained in their union by $\mathcal{N}(F, \delta)$. A set $F \subset X$ is *totally bounded* if for all $\delta > 0$ the quantity $\mathcal{N}(F, \delta) < \infty$, which is to say that F can be covered by finitely many balls of any radius.

A set $F \subseteq X$ is said to be *equi-homogeneous* if for all $\delta_0 > 0$ there exist constants $M \geq 1$ and $c_1, c_2 > 0$ such that

$$\sup_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \leq M \inf_{x \in F} \mathcal{N}(B_{c_1 \delta}(x) \cap F, c_2 \rho) \tag{3}$$

for all δ, ρ with $0 < \rho < \delta \leq \delta_0$.

[1] $F \subseteq \mathbb{R}^d$ is called *Ahlfors regular* if there exist constants $a, b > 0$ such that for $s = \dim_{\mathrm{H}} F$, $ar^s \leq \mathcal{H}^s\big(B_r(x) \cap F\big) \leq br^s$, for all $x \in F$ and all $0 < r < 1$, where $\mathcal{H}^s$ refers to the s-dimensional Hausdorff measure. Ahlfors regularity is a sufficient condition for the coincidence of the Hausdorff and Assoud dimensions.

In [22] we demonstrate that the Assouad dimension of an equi-homogeneous set can be recovered from the more elementary upper and lower box-counting dimensions, provided that the box-counting dimensions are equal and are 'attained' (defined below). We recall this result in Theorem 3.5 below. We then give some simple examples showing that equi-homogeneity is a concept distinct from the equality of the lower-dimensional number of Larman to the Assouad dimension studied by Fraser [7] and other characterizations of dimensional regularity.

1.4 Equi-homogeneity of attractors for iterated function systems

Section 4.1 shows that the notion of equi-homogeneity is not overly restrictive in that self-similar sets satisfying the Moran open-set condition are equi-homogeneous.

We then treat a similar issue in a much more general setting, by considering the pullback attractors of non-autonomous iterated functions systems. After proving the existence and uniqueness of pullback attractors in this setting in some generality, we turn again to the case when the functions are contracting similarities.

Our first result concerning such non-autonomous iterated function systems requires the contraction ratios to coincide at each stage of the iteration; this generalises the construction of 'generalized Cantor sets' outlined above, and for this particular case the result of this theorem is contained in the analysis in [22], which includes examples where $\inf\{\, c_i : i \in \mathbb{N} \,\} = 0$. See the earlier discussion around (1) for the notation used in what follows.

Theorem 1.1. *Suppose that $\sigma_i = c_k$ for $i \in \mathcal{I}_k$ and that $\operatorname{card}(\mathcal{I}_k) \le N$ for $k \in \mathbb{N}$. Then under some additional assumptions, see Theorem 4.5, the pullback attractor is equi-homogeneous.*

We remark that the non-autonomous attractors covered by the above result include the generalized Cantor sets studied in [22] which include examples where We also note (and prove in Section 3.2) that while the resulting pullback attractors are equi-homogeneous, under certain choices for the sequence c_k, their Assouad and upper box-counting dimensions are not equal.

Finally we turn to the case where the contracting ratios σ_i may be different for indices within each $\mathcal{I}_k$. In this case we require some type of uniformity assumption: if we assume that for some $s > 0$

$$\textstyle\sum_{i\in\mathcal{I}_k} \sigma_i^s = 1 \qquad \text{for all} \qquad k \in \mathbb{N} \tag{4}$$

and that $\sigma_i \ge \sigma_* > 0$ for all i then the resulting pullback attractor is equi-homogeneous.

The hypothesis (4) can be weakened a little: let

$$\mathcal{J}_{k,l} = \mathcal{I}_k \times \cdots \times \mathcal{I}_l$$

and suppose there exist $s > 0$, n_0, and L such that

$$L^{-1} \le \textstyle\sum_{\alpha\in\mathcal{J}_{k,k+n}} \sigma_\alpha^s \le L \quad \text{for all} \quad k \in \mathbb{N} \quad \text{and} \quad n \ge n_0. \tag{5}$$

Under these assumptions we can still prove equi-homogeneity, and show that s is the Assouad dimension of the attractor.

Theorem 1.2. *Suppose* $\inf\{\sigma_i : i \in \mathbb{N}\} > 0$ *and that (5) holds. Then under some additional assumptions, see Theorem 4.10, the pullback attractor is equi-homogeneous and* $\dim_{\mathrm{A}}(F^k) = s$ *for every* $k \in \mathbb{N}_0$.

Note that assumption (4) is sufficient to overcome some problems with the proof in Li [15] and conclude that the Assouad and upper box-counting dimensions of the resulting pullback attractors are equal. A similar, but weaker, condition to (5) was used to compute the Assouad dimension by Li, Li, Miao, and Xi [16] in cases when the Assouad and upper box counting dimensions differ.

Intuitively (5) implies that (4) holds when averaged over long enough sequences of iterations. We end this introduction by loosely interpreting the results of Theorem 4.9 in terms of intermittency in turbulent fluid flows. There is increasing numerical evidence that suggests that for flows governed by the Navier–Stokes equations the rate of dissipation of energy in the small scales is intermittently much stronger than at other times, see, for example, Jiménez, Moisy, Tabeling, and Wilaime [11], Kerr [12], Yeung, Pope, Lamorgese, and Donzis [27] and references therein. At the same time, such quantities appear to be statistically stationary for fully developed turbulence and therefore a mean rate of dissipation of energy appears to be well defined.

We represent intermittency in a non-autonomous iterated function system by supposing at certain times $t_k = -k$ that the ratios σ_i for $i \in \mathcal{I}^k$ are much smaller than at other times. Stationary statistics and the existence of a mean rate of dissipation of energy are then represented by the hypothesis that there exists an s that satisfies (5). Our result that the resulting pullback attractor is equi-homogeneous but not self-similar, then provides a rigorous analog to various physical theories, see Roberto and Biferale [2], which suppose intermittency in homogeneous turbulence that is not self-similar.

2 Box-counting dimension

This section recalls the definition and some facts about the upper and lower box-counting dimensions in an arbitrary metric space (X, d_X). Recall that $\mathcal{N}(F, \delta)$ denotes the minimum number of δ-balls with centers in F such that F is covered by their union.

Definition 2.1. *For a totally bounded set* $F \subset X$ *the upper and lower box-counting dimensions are defined by*

$$\dim_{\mathrm{B}} F = \limsup_{\delta \to 0+} \frac{\log \mathcal{N}(F, \delta)}{-\log \delta} \tag{6}$$

$$\text{and} \qquad \dim_{\mathrm{LB}} F = \liminf_{\delta \to 0+} \frac{\log \mathcal{N}(F, \delta)}{-\log \delta}, \tag{7}$$

respectively.

The box-counting dimensions essentially capture the exponent $s \in \mathbb{R}^+$ for which $\mathcal{N}(F, \delta) \sim \delta^{-s}$. More precisely, it follows from Definition 2.1 that for all $\varepsilon > 0$ and all $\delta_0 > 0$ there exists a constant $C \geq 1$ such that

$$C^{-1} \delta^{-\dim_{\mathrm{LB}} F + \varepsilon} \leq \mathcal{N}(F, \delta) \leq C \delta^{-\dim_{\mathrm{B}} F - \varepsilon} \qquad \forall\ 0 < \delta \leq \delta_0. \tag{8}$$

For some bounded sets F the bounds (8) also hold for $\varepsilon = 0$ giving precise control of the growth of $\mathcal{N}(F,\delta)$. In [22] we distinguish this class of sets and say that they 'attain' their box-counting dimensions.

A useful quantity for proving lower bounds is $\mathcal{P}(F,\delta)$, the maximum number of disjoint δ-balls with centers in F. As we show in the following lemma, the quantity $\mathcal{P}(F,\delta)$ is, in fact, very closely related to $\mathcal{N}(F,\delta)$.

Lemma 2.2. *Let $F \subset X$ be totally bounded. For all $\delta > 0$*

$$\mathcal{N}(F,2\delta) \leq \mathcal{P}(F,2\delta) \leq \mathcal{N}(F,\delta). \tag{9}$$

Proof. Let $x_1,\ldots,x_{\mathcal{P}(F,\delta)} \in F$ be the centers of disjoint δ-balls. As each δ-ball $B_\delta(y)$ can cover at most one of the x_i, to cover F we need at least as many δ-balls are there are x_i, hence $\mathcal{N}(F,\delta) \geq \mathcal{P}(F,\delta)$, which is the first inequality of (9).

Next, with the same points $\{x_i\}$ observe that for each $x \in F$ the distance $d_X(x,x_i) \leq 2\delta$ for some $i = 1,\ldots,\mathcal{P}(F,\delta)$, otherwise the additional closed ball $B_\delta(x)$ would be disjoint from each of the $B_\delta(x_i)$. Consequently, the balls $B_{2\delta}(x_i)$ cover the set F, hence $\mathcal{N}(F,\delta) \leq \mathcal{P}(F,2\delta)$, which is the second inequality of (9). $\square$

In light of the inequalities (9), replacing $\mathcal{N}(F,\delta)$ with $\mathcal{P}(F,\delta)$ in the above gives an equivalent definition.

2.1 Homogeneity and the Assouad dimension

The Assouad dimension is a less familiar notion of dimension, in which we are concerned with 'local' coverings of a set F: for more details see Assouad [1], Bouligand [3], Luukkainen [17] Olson [21], or Robinson [24].

Definition 2.3. *A set $F \subset X$ is s-homogeneous if for all $\delta_0 > 0$ there exists a constant $C > 0$ such that*

$$\sup_{x\in F} \mathcal{N}(B_\delta(x) \cap F,\rho) \leq C(\delta/\rho)^s \qquad \forall\ \delta,\rho \quad \textit{with} \quad 0 < \rho < \delta \leq \delta_0. \tag{10}$$

Note that we do not require F to be bounded in order to be s-homogeneous, but minimally require each intersection $B_\delta(x) \cap F$ to be totally bounded. This trivially holds if X is a locally totally bounded space, which is to say that every ball $B_\delta(x) \subset X$ is totally bounded (for example, in Euclidean space $X = \mathbb{R}^n$).

The following technical lemma gives a relationship between the minimal size of covers of the set $B_\delta(x) \cap F$ for different length-scales, which will use in many of the subsequent proofs.

Lemma 2.4. *Let $F \subset X$. For all $\delta,\rho,r > 0$ and each $x \in F$*

$$\mathcal{N}(B_\delta(x) \cap F,\rho) \leq \mathcal{N}(B_\delta(x) \cap F,r) \sup_{x\in F} \mathcal{N}(B_r(x) \cap F,\rho) \tag{11}$$

Proof. The only non-trivial case occurs when $\rho < r < \delta$. Further, if $M := \mathcal{N}(B_\delta(x) \cap F, r) = \infty$ then there is nothing to prove. Assume that $M < \infty$ and let $x_1, \ldots, x_M \in F$ be the centers of the r-balls $B_r(x_j)$ that cover $B_\delta(x) \cap F$. Clearly

$$B_\delta(x) \cap F \subset \bigcup_{j=1}^{M} B_r(x_j) \cap F$$

so

$$\begin{aligned}\mathcal{N}(B_\delta(x) \cap F, \rho) &\leq \sum_{j=1}^{M} \mathcal{N}(B_r(x_j) \cap F, \rho) \\ &\leq M \sup_{x \in F} \mathcal{N}(B_r(x) \cap F, \rho)\end{aligned}$$

which is precisely (11). □

It will be useful to observe that in some cases s-homogeneity is equivalent to (10) holding only for *some* δ_0, which is easier to check.

Lemma 2.5. *If $F \subset X$ is totally bounded or X is a locally totally bounded space then $F \subset X$ is s-homogeneous if and only if there exist constants $C, \delta_1 > 0$ such that*

$$\sup_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \leq C(\delta/\rho)^s \qquad \forall\, \delta, \rho \quad \textit{with} \quad 0 < \rho < \delta \leq \delta_1. \tag{12}$$

Proof. The 'if' direction is immediate from the definition of s-homogeneity. To prove the converse we let $\delta_0 > 0$ and $x \in F$ be arbitrary. If $\delta_0 \leq \delta_1$ then there is nothing to prove, so we assume that $\delta_0 > \delta_1$. Suppose that δ, ρ lie in the range $0 < \rho < \delta_1 < \delta \leq \delta_0$. From Lemma 2.4 with $r = \delta_1$ we obtain

$$\begin{aligned}\mathcal{N}(B_\delta(x) \cap F, \rho) &\leq \mathcal{N}(B_\delta(x) \cap F, \delta_1) \sup_{x \in F} \mathcal{N}(B_{\delta_1}(x) \cap F, \rho) \\ &\leq \mathcal{N}(B_\delta(x) \cap F, \delta_1) C(\delta_1/\rho)^s \\ &\leq \mathcal{N}(B_\delta(x) \cap F, \delta_1) C(\delta/\rho)^s \end{aligned} \tag{13}$$

which follows from (12) and the fact that $\delta > \delta_1$.

Now, if X is a locally totally bounded space then it follows from (13) that for $0 < \rho < \delta_1 < \delta \leq \delta_0$

$$\mathcal{N}(B_\delta(x) \cap F, \rho) \leq \mathcal{N}(B_{\delta_0}(0), \delta_1) C(\delta/\rho)^s,$$

and trivially for $\delta_1 \leq \rho < \delta \leq \delta_0$ that

$$\mathcal{N}(B_\delta(x) \cap F, \rho) \leq \mathcal{N}(B_\delta(x), \rho) \leq \mathcal{N}(B_{\delta_0}(x), \delta_1) \leq \mathcal{N}(B_{\delta_0}(0), \delta_1)(\delta/\rho)^s$$

as $\delta/\rho > 1$. Consequently, with $C_{\delta_0} = \mathcal{N}(B_{\delta_0}(0), \delta_1) \max(C, 1)$ we obtain

$$\sup_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \leq C_{\delta_0}(\delta/\rho)^s \qquad \forall\, \delta, \rho \quad \text{with} \quad 0 < \rho < \delta \leq \delta_0,$$

which, as $\delta_0 > 0$ was arbitrary, shows that F is s-homogeneous.

Next, if $F \subset X$ is totally bounded then it follows from (13) that for $0 < \rho < \delta_1 < \delta \leq \delta_0$

$$\mathcal{N}(B_\delta(x) \cap F, \rho) \leq \mathcal{N}(F, \delta_1) C(\delta/\rho)^s,$$

and again for $\delta_1 \le \rho < \delta \le \delta_0$ that

$$\mathcal{N}(B_\delta(x) \cap F, \rho) \le \mathcal{N}(F, \delta_1)(\delta/\rho)^s.$$

Consequently, the constant $C' = \mathcal{N}(F, \delta_1) \max(C, 1)$ is sufficient to extend (12) to all $0 < \rho < \delta \le \delta_0$, so we conclude that F is s-homogeneous. □

Corollary 2.6. *If $F \subset X$ is totally bounded then F is s-homogeneous if and only if there exists a constant C such that*

$$\sup_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \le C(\delta/\rho)^s \qquad \forall\, \delta, \rho \quad \textit{with} \quad 0 < \rho < \delta.$$

Proof. The 'if' direction is immediate from the definition. Conversely, we see in the above proof that the constant C' does not depend upon the upper bound δ_0, so the inequality is valid for all ρ, δ satisfying $0 < \rho < \delta$. □

Definition 2.7. *The Assouad dimension of a set $F \subset X$ is defined by*

$$\dim_{\mathrm{A}} F := \inf \left\{ s \in \mathbb{R}^+ : \ F \text{ is } s\text{-homogeneous} \right\}$$

It is known that for a bounded set $F \subset \mathbb{R}^n$ the three notions of dimension that we have now introduced satisfy

$$\dim_{\mathrm{LB}} F \le \dim_{\mathrm{B}} F \le \dim_{\mathrm{A}} F \tag{14}$$

(see, for example, Lemma 9.6 in Robinson [24]). The inequality (14) also holds for totally bounded subsets in general metric spaces.

Lemma 2.8. *If $F \subset X$ is totally bounded then $\dim_{\mathrm{B}} F \le \dim_{\mathrm{A}} F$.*

Proof. Let $s > \dim_{\mathrm{A}} F$ and let $x_1, \ldots, x_{\mathcal{N}(F,1)}$ be the centers of balls of radius 1 that form a cover of F. For all $\rho < 1$

$$\begin{aligned} \mathcal{N}(F, \rho) &\le \sum_{j=1}^{\mathcal{N}(F,1)} \mathcal{N}(B_1(x_j) \cap F, \rho) \le \mathcal{N}(F, 1) \sup_{x \in F} \mathcal{N}(B_1(x) \cap F, \rho) \\ &\le \mathcal{N}(F, 1) C(1/\rho)^s \end{aligned}$$

for some $C > 0$, hence $\dim_{\mathrm{B}} F \le s$. As $s > \dim_{\mathrm{A}} F$ was arbitrary we conclude that $\dim_{\mathrm{B}} F \le \dim_{\mathrm{A}} F$. □

An interesting example is given by countable compact subset of the real line $F_\alpha := \{n^{-\alpha}\}_{n \in \mathbb{N}} \cup \{0\}$ with $\alpha > 0$ for which

$$\begin{aligned} & \dim_{\mathrm{LB}} F_\alpha = \dim_{\mathrm{B}} F_\alpha = (1 + \alpha)^{-1} \\ \text{but} \qquad & \dim_{\mathrm{A}} F_\alpha = 1. \end{aligned}$$

(see Olson [21] and Example 13.4 in Robinson [23]).

3 Equi-homogeneity

From Definition 2.3 we see that homogeneity encodes the *maximum* size of a local optimal cover at a particular length-scale. However, the *minimal* size of a local optimal cover is not captured by homogeneity, and indeed this minimum size can scale very differently, as the example at the end of the previous section illustrates.

Example 3.1. *For each $\alpha > 0$ the set $F_\alpha := \{n^{-\alpha}\}_{n\in\mathbb{N}} \cup \{0\}$ has Assouad dimension equal to 1, so for all $\varepsilon > 0$*

$$\sup_{x\in F_\alpha} \mathcal{N}(B_\delta(x) \cap F_\alpha, \rho)(\delta/\rho)^{-(1-\varepsilon)}$$

is unbounded on δ, ρ with $0 < \rho < \delta$.

On the other hand $1 \in F_\alpha$ is an isolated point so

$$\inf_{x\in F_\alpha} \mathcal{N}(B_\delta(x) \cap F_\alpha, \rho) = 1$$

for all δ, ρ with $0 < \rho < \delta < 1 - 2^{-\alpha}$ as $B_\delta(1) \cap F_\alpha = \{1\}$ for such δ and this isolated point can be covered by a single ball of any radius.

We now define equi-homogeneous sets to be those sets for which the range of the number of sets required in the local covers is uniformly bounded at all length-scales.

Definition 3.2. *We say that a set $F \subset X$ is* equi-homogeneous *if for all $\delta_0 > 0$ there exist constants $M \geq 1$ and $c_1, c_2 > 0$ such that*

$$\sup_{x\in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \leq M \inf_{x\in F} \mathcal{N}(B_{c_1\delta}(x) \cap F, c_2\rho) \tag{15}$$

for all δ, ρ with $0 < \rho < \delta \leq \delta_0$.

Note that as $\mathcal{N}(B_\delta(x) \cap F, \rho)$ increases with δ and decreases with ρ, by replacing the c_i with 1 if necessary we can assume without loss of generality that $c_2 \leq 1 \leq c_1$ in (15).

3.1 Equivalent definitions

As with the definition of homogeneity, for a large class of sets it is sufficient that (15) holds only for *some* δ_0.

Lemma 3.3. *If $F \subset X$ is totally bounded or X is a locally totally bounded space then F is equi-homogeneous if and only if there exist constants $M \geq 1$ and $c_1, c_2, \delta_1 > 0$ such that*

$$\sup_{x\in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \leq M \inf_{x\in F} \mathcal{N}(B_{c_1\delta}(x) \cap F, c_2\rho)$$

for all ρ, δ satisfying $0 < \rho < \delta \leq \delta_1$.

Proof. The proof is substantially the same as that of Lemma 2.5. □

Again, if F is totally bounded then we can find $M \geq 1$ such that (15) holds for all ρ, δ with $0 < \rho < \delta$.

In normed spaces that are locally totally bounded (such as Euclidean space) there is an even more elementary formulation that does not require the constants c_1, c_2.

Lemma 3.4. *Let X be a normed space that is locally totally bounded. A set $F \subset X$ is equi-homogeneous if and only if there exists constants $M \geq 1$, $\delta_1 \geq 1$ such that*

$$\sup_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \leq M \inf_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \tag{16}$$

for all ρ, δ with $0 < \rho < \delta \leq \delta_1$.

Proof. The 'if' direction follows immediately from Lemma 3.3. To prove the converse fix $\delta_0 > 0$ and let $M \geq 1$ and $c_1, c_2 > 0$ with $c_2 \leq 1 \leq c_1$ be such that

$$\sup_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \leq M \inf_{x \in F} \mathcal{N}(B_{c_1\delta}(x) \cap F, c_2\rho)$$

for all $0 < \rho < \delta \leq \delta_0$.

First, observe that replacing δ by δ/c_1 we can assume that

$$\sup_{x \in F} \mathcal{N}(B_{\delta/c_1}(x) \cap F, \rho) \leq M \inf_{x \in F} \mathcal{N}(B_\delta(x) \cap F, c_2\rho) \tag{17}$$

for all δ, ρ with $0 < \rho < \delta/c_1$, $\delta \leq c_1\delta_0$. Note that if $\rho \geq \delta/c_1$ then the above inequality holds trivially, since the left-hand side is 1 and the right-hand side is at least $M \geq 1$; so in fact (17) holds for all $0 < \rho < \delta \leq \delta_1 := c_1\delta_0$.

Now, it follows from (11) with $r = \delta/c_1$ that

$$\mathcal{N}(B_\delta(x) \cap F, \rho) \leq \mathcal{N}(B_\delta(x), \delta/c_1) \sup_{x \in F} \mathcal{N}(B_{\delta/c_1}(x) \cap F, \rho)$$

for all $x \in F$, so setting $N_1 := \mathcal{N}(B_\delta(x), \delta/c_1) = \mathcal{N}(B_1(0), 1/c_1)$, which follows as X is a normed space, we obtain

$$\sup_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \leq N_1 \sup_{x \in F} \mathcal{N}(B_{\delta/c_1}(x) \cap F, \rho). \tag{18}$$

It also follows from (11) that for any $r > 0$

$$\begin{aligned} \mathcal{N}(B_\delta(x) \cap F, c_2\rho) &\leq \mathcal{N}(B_\delta(x) \cap F, r) \sup_{x \in F} \mathcal{N}(B_r(x) \cap F, c_2\rho) \\ &\leq \mathcal{N}(B_\delta(x) \cap F, r) \sup_{x \in F} \mathcal{N}(B_r(x), c_2\rho) \\ &= \mathcal{N}(B_\delta(x) \cap F, r)\mathcal{N}(B_r(0), c_2\rho) \end{aligned}$$

so taking $r = \rho$, setting $N_2 = \mathcal{N}(B_\rho(0), c_2\rho) = \mathcal{N}(B_1(0), c_2)$, which again follows as X is a normed space, and taking the infimum over $x \in F$ we obtain

$$\inf_{x \in F} \mathcal{N}(B_\delta(x) \cap F, c_2\rho) \leq \inf_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho) N_2. \tag{19}$$

It follows from (17), (18) and (19) that for all ρ, δ with $0 < \rho < \delta \leq \delta_1$

$$\sup_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \leq M \frac{N_1}{N_2} \inf_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho)$$

so we conclude from Lemma 3.4 that F is equi-homogeneous. □

We note that it was shown in [22] that for reasonable choices of product metric, the product of two equi-homogeneous sets is also equi-homogeneous.

3.2 Equi-homogeneity and the Assouad dimension

From Definition 3.2 we see that a set is equi-homogeneous if the difference between the minimal and maximal local covers is uniformly bounded at all length-scales. Roughly, this means that at each fixed length-scale the set exhibits identical dimensional detail near every point. As a consequence we can find the Assouad dimension of equi-homogeneous sets without needing to account for the variance in local detail. More precisely, it is clear that if $F \subset X$ is an equi-homogeneous set and $x \in F$ is an arbitrary point then F is s-homogeneous if and only if there exists a constant $C > 0$ such that

$$\mathcal{N}\left(B_\delta\left(x\right) \cap F, \rho\right) \leq C\left(\delta/\rho\right)^s \qquad \forall\ \delta, \rho \quad \text{with} \quad 0 < \rho < \delta \leq \delta_0.$$

In fact, for equi-homogeneous sets we can find the Assouad dimension in terms of the more elementary box-counting dimensions, which do not account for variance in local detail, provided that the box-counting dimensions are suitably 'well behaved', as shown in the following theorem (from [22]).

Theorem 3.5. *If a totally bounded set $F \subset X$ is equi-homogeneous, F attains both its upper and lower box-counting dimensions, and* $\dim_{\mathrm{LB}} F = \dim_{\mathrm{B}} F$*, then* $\dim_{\mathrm{A}} F = \dim_{\mathrm{B}} F = \dim_{\mathrm{LB}} F$.

Note that our notion of equi-homogeneity is related to, but distinctly different from, the coincidence of the Assouad dimension with Larman's minimal dimension number $\dim_{\mathrm{LA}} F$. The quantity $\dim_{\mathrm{LA}} F$ was defined in [14] as the supremum over all s for which there exists constants c and $\delta_0 > 0$ such that

$$\inf_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \geq c(\delta/\rho)^s \qquad \text{for all} \qquad 0 < \rho < \delta \leq \delta_0.$$

Corollary 2.11 of Fraser [7] shows that self-similar sets F that satisfy the Moran open-set condition also satisfy $\dim_{\mathrm{A}} F = \dim_{\mathrm{LA}} F$ and we will show in Proposition 4.1 that sets in this large class are also equi-homogeneous. However, we will now show that equi-homogeneity is not equivalent to the condition $\dim_{\mathrm{A}} F = \dim_{\mathrm{LA}} F$.

The minimal dimension number and the Assouad dimension encode the scaling of the two distinct quantities $\inf_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho)$ and $\sup_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho)$, which can scale very differently as we have seen in Example 3.1. However, if $F \subset X$ is equi-homogeneous then the difference between the minimal and maximal local cover is negligible, in which case the minimal dimension number and the Assouad dimension encode the scaling of the single quantity $\mathcal{N}\left(B_\delta\left(x\right) \cap F, \rho\right)$, in a similar way to the box-counting dimension scaling law (8).

Even though the minimal and maximal local covers scale identically for an equi-homogeneous set, the minimal dimension number and the Assouad dimension can be distinct. Rather than being the consequence of the different scaling of the minimal and maximal local cover (as in Example 3.1), any difference in these dimensions is due to an oscillation in the growth of the covers $\mathcal{N}\left(B_\delta\left(x\right) \cap F, \rho\right)$. If this oscillation is sufficiently large then the upper and lower bounds on the growth of the covers (provided respectively by the Assouad and minimal dimension number) are distinct.

We now give an example of the oscillating growth of local covers, and the resulting difference in the Assouad dimension and the minimal dimension number.

As shown in the introduction, the generalized Cantor sets that were constructed in Section 3 of [22] may also be obtained as the pullback attractors of the non-autonomous iterated function systems given by (2). These systems satisfy the hypothesis of Theorem 4.5 (with Moran open sets $U^k = (0,1)$ for all k) and are therefore equi-homogeneous (we give a direct proof that generalized Cantor sets are equi-homogeneous in [22]). At the same time, these sets can have minimal dimension numbers that differ from their Assouad dimensions. In fact, even the Assouad and the upper box-counting dimension can differ for these sets. We give a simple example here to demonstrate this.

Using the notation from Section 1.2 in the Introduction, define

$$c_k = \begin{cases} 1/3 & \text{for} \quad k \in [2^{2(n-1)}, 2^{2n-1}) \\ 1/9 & \text{for} \quad k \in [2^{2n-1}, 2^{2n}). \end{cases}$$

Let $\pi_k = c_1 \cdots c_k$. By results from [22], see also Hua, Rao, Wen, and Wu [9], the box-counting dimension of F^0 is given by

$$\dim_{\mathrm{B}} F^0 = \limsup_{k\to\infty} s_k \qquad \text{where} \qquad s_k = \frac{k \log 2}{\log(1/\pi_k)}.$$

Since s_k is a non-increasing function on $[2^{2(n-1)}, 2^{2n-1})$ and an increasing function on $[2^{2n-1}, 2^{2n})$, it follows that we may take the limit supremum along the subsequence $k = 4^n$. The calculation

$$\begin{aligned} \log(1/\pi_{4^n}) &= \sum_{j=1}^{n} \Big(2^{2(j-1)} \log 3 + 2^{2j-1} \log 9 \Big) \\ &= \frac{5}{4} \sum_{j=1}^{n} 4^j \log 3 = \frac{5}{3}(4^n - 1) \log 3 \end{aligned}$$

therefore implies

$$\dim_{\mathrm{B}} F^0 = \lim_{n\to\infty} \frac{4^n \log 2}{(5/3)(4^n - 1)\log 3} = \frac{3}{5} \frac{\log 2}{\log 3}.$$

On the other hand, $\dim_{\mathrm{A}} F^0 \geq \log 2/\log 3$. Suppose, for contradiction, that $\dim_{\mathrm{A}} F^0 < s < \log 2/\log 3$. Then, there would exist a C such that

$$\mathcal{N}(B_\delta(x) \cap F^0, \rho/2) \leq C(\delta/\rho)^s \quad \text{for all} \quad 0 < \rho < \delta.$$

Choose $\delta_n = \pi_{2^{2(n-1)}}$ and $\rho_n = \pi_{2^{2n-1}}$. Then

$$\delta_n/\rho_n = 3^{2^{2(n-1)}} \qquad \text{and} \qquad \mathcal{N}(B_{\delta_n}(x) \cap F^0, \rho_n/2) \geq 2^{2^{2(n-1)}}$$

would imply that

$$1 \leq C 2^{-2^{2(n-1)}} (3^{2^{2(n-1)}})^s = C(3^{2^{2(n-1)}})^{s - \log 2/\log 3} \to 0 \quad \text{as} \quad n \to \infty$$

which is a contradiction.

We note that the above set F^0 satisfies the Moran structure conditions of Wen [26] with $c_* = \inf\{ c_k : k \in \mathbb{N} \} = 1/9 > 0$. Therefore an additional assumption, such as (4), is needed in order to complete the proof of Li [15] and conclude that the Assouad and upper box-counting dimensions are equal.

We now give a simple example of a set that satisfies $\dim_{\rm A} F = \dim_{\rm LA} F$ but that is not equi-homogeneous. Taken together these two examples demonstrate that the notion of equi-homogeneity is entirely distinct from the coincidence of these two dimensions.

Proposition 3.6. *Let* $F = \{0,1\} \cup \{\, 2^{-n} : n \in \mathbb{N}\,\}$. *Then*

$$\dim_{\rm A} F = \dim_{\rm LA} F = 0$$

but F *is not equi-homogeneous.*

Proof. Let $\delta = 1/4$. Then

$$B_\delta(1) \cap F = \{1\} \quad \text{implies that} \quad \inf_{x\in F} \mathcal{N}(B_\delta(x) \cap F, \rho) = 1$$

for every $\rho > 0$. On the other hand, for $0 < \rho < 1/4$, let K be chosen so that

$$2^{-K-1} \le \rho < 2^{-K}.$$

Then $K \ge 2$ and

$$B_\delta(0) \cap F \supseteq \{\, 2^{-n} : n = 2, \dots, K\,\}.$$

Moreover $2^{-n+1} - 2^{-n} = 2^{-n} \ge 2^{-K} > \rho$ for $n \le K$ implies that at least one set of diameter ρ is required to cover each of the $K-1$ points above. Therefore

$$\sup_{x\in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \ge K - 1 \ge \frac{\log(1/\rho)}{\log 2} - 2.$$

This shows there is no value for M independent of ρ that could appear in Definition 3.2 for this set, and so F is not equi-homogeneous.

Clearly $\dim_{\rm LA}(F) = 0$. The equality $\dim_{\rm A}(F) = 0$ is stated as Fact 4.3 in Olson [21] without proof. We include the proof here and remark that the logarithmic terms that occur in the course of the argument can also be used to show that F does not 'attain' its box-counting dimension (i.e. that (8) does not hold with $\varepsilon = 0$.).

Let $x \in [0,1]$ and $0 < \rho < \delta < 1/4$. Define

$$G = \{\, 2^{-n} : \max(0, x-\delta) < 2^{-n} \le \rho\,\}$$

and

$$H = \{\, 2^{-n} : \max(\rho, x-\delta) < 2^{-n} < \min(x+\delta, 1)\,\}.$$

Then $B_\delta(x) \cap F \subseteq \{0,1\} \cup G \cup H$. Now depending on ρ, x, and δ it may happen that either or both of the sets H and G are empty. As covering an empty set is trivial, we need only consider the cases when these sets are non-empty.

If $G \ne \emptyset$ then $x - \delta < \rho$, and it follows that

$$\mathcal{N}(G, \rho) \le \frac{\rho - \max(0, x-\delta)}{\rho} + 1 \le 2. \tag{20}$$

Similarly if $H \ne \emptyset$ then

$$\mathcal{N}(H, \rho) \le \frac{1}{\log 2} \log\left\{ \frac{\min(x+\delta, 1)}{\max(\rho, x-\delta)} \right\} + 1.$$

If $x + \delta \geq 1$ then $x - \delta \geq 1 - 2\delta \geq 1/2$. Thus $\mathcal{N}(H, \rho) \leq 2$. If $x - \delta \leq \rho$ then $x + \delta \leq \rho + 2\delta < 3\delta < 1$. Thus $\mathcal{N}(H, \rho) \leq (\log 2)^{-1} \log(3\delta/\rho) + 1$. Otherwise, $\rho + \delta < x < 1 - \delta$. On this interval $x \mapsto \log\left\{(x + \delta)/(x - \delta)\right\}$ is a decreasing function. Therefore, in general,

$$\mathcal{N}(H, \rho) \leq 2\log(\delta/\rho) + 3. \tag{21}$$

Combining (20) with (21) we obtain

$$\mathcal{N}(B_\delta(x) \cap F, \rho) \leq 2\log(\delta/\rho) + 7$$

Since for every $s > 0$ there exists $C > 0$ such that

$$2\log(\delta/\rho) + 7 \leq C(\delta/\rho)^s \qquad \text{for every} \qquad 0 < \rho < \delta < 1/4,$$

taking $\delta_0 = 1/4$ in Lemma 2.5 shows that $\dim_{\mathrm{A}}(F) = 0$. □

4 Equi-homogeneity and dynamical systems

We now demonstrate that the notion of equi-homogeneity is not overly restrictive: it is enjoyed by all self-similar sets that satisfy the Moran open set condition, generalized Cantor sets, and the pullback attractors for a certain class of non-autonomous iterated function systems that satisfy a suitably generalized version of the Moran open set condition.

4.1 Autonomous systems

We will begin with self-similar sets, which are a much studied and canonical class of fractal sets. A (contracting) similarity is a map $f_i \colon \mathbb{R}^d \to \mathbb{R}^d$ of the form

$$f_i(x) = \sigma_i O_i x + \beta_i, \tag{22}$$

where $\sigma_i \in (0, 1)$, $\beta_i \in \mathbb{R}^d$, and $O_i \in O(d)$, the set of all $d \times d$ orthogonal matrices.

Using the notation given in the introduction we consider a collection of index sets $\mathcal{I}_k \subset \mathbb{N}$ with $\operatorname{card}(\mathcal{I}_k) < \infty$ for all $k \in \mathbb{N}$. We also introduce some additional notation: define

$$\mathcal{I}^0 = \bigcup_{n=1}^{\infty} \mathcal{I}_1 \times \cdots \times \mathcal{I}_n.$$

For $\alpha = (i_1, \ldots, i_n) \in \mathcal{I}^0$ let

$$f_\alpha = f_{i_1} \circ \cdots \circ f_{i_n} \qquad \text{and} \qquad \sigma_\alpha = \sigma_{i_1} \cdots \sigma_{i_n},$$

and, if $n \geq 2$, we denote the truncation $(i_1, \ldots, i_{n-1})$ by α'.

If we assume that $\mathcal{I}_k = \mathcal{I}$ for each $k \in \mathbb{N}$ we obtain classical iterated function systems. Define the set function $\mathcal{T}$ and its iterates as

$$\mathcal{T}(B) = \bigcup_{i \in \mathcal{I}} f_i(B) \qquad \text{and} \qquad \mathcal{T}^{n+1}(B) = \mathcal{T} \circ \mathcal{T}^n(B)$$

where $\mathcal{T}^0(B) = B$. It is well known, see Falconer [6], for example, that if the f_i are contractions then there exists a unique non-empty compact set F such that

$$F = \mathcal{T}(F), \tag{23}$$

and further, that this set satisfies $\rho_H(\mathcal{T}^l(B), F) \to 0$ as $l \to \infty$ for any bounded set $B \subset \mathbb{R}^d$. In this case, defining $F^k = F$ for all $k \in \mathbb{N}_0$ and noting that $\mathcal{T}^l = \mathcal{S}^{0,l}$ yields a collection of compact sets that satisfy properties 1–3 for a pullback attractor. Conversely, as pullback attractors are unique (which we prove in Theorem 4.2) it follows that for classical iterated function systems the pullback attractor can be identified with the invariant set.

We shall say that F is self-similar if it is the attractor of a classical system of contracting similarities. Reasonable self-similar sets result when we impose some separation properties on the iterated function system, see Falconer [6] or Hutchinson [10], for example. The simplest such property is the Moran open-set condition [19]: there exists an open set U such that $F \subset \overline{U}$, $f_i(U) \subseteq U$, and

$$f_i(U) \cap f_j(U) = \emptyset \qquad \text{when} \qquad i \neq j.$$

We now show that these self-similar sets are equi-homogeneous.

Proposition 4.1. *Self-similar sets that satisfy the Moran open-set condition are equi-homogeneous.*

Proof. Let $\sigma_* = \min\{\, \sigma_i : i \in \mathcal{I}\,\}$ and $\eta = \operatorname{diam}(\overline{U})$. For $\delta \le \sigma_*\eta$ define

$$\mathcal{J}_\delta = \{\, \alpha \in \mathcal{I}^0 : \sigma_\alpha \eta < \delta \le \sigma_{\alpha'}\eta \,\}.$$

Note that $n \ge 2$ for any $\alpha \in \mathcal{J}_\delta$. Moreover, for $\alpha, \beta \in \mathcal{J}_\delta$ we have

$$f_\alpha(U) \cap f_\beta(U) = \emptyset \qquad \text{when} \qquad \alpha \neq \beta. \tag{24}$$

This follows from the open set condition as we now show.

Write

$$\alpha = (i_1, \ldots, i_n) \qquad \text{and} \qquad \beta = (j_1, \ldots, j_m)$$

and assume without loss of generality that $m \le n$. Let k be the smallest integer such that $i_k \neq j_k$. Such a k exists because if not, then $\alpha \neq \beta$ would imply $m < n$ and consequently that $\sigma_{\alpha'} \le \sigma_\beta$. This would imply that $\delta \le \sigma_{\alpha'}\eta \le \sigma_\beta\eta < \delta$, which is a contradiction. If $k = 1$ then $i_1 \neq j_1$ and the open set condition implies that

$$f_\alpha(U) \cap f_\beta(U) \subseteq f_{i_1}(U) \cap f_{j_1}(U) = \emptyset.$$

If $k > 1$ define $\gamma = (i_1, \ldots, i_{k-1})$ and again

$$f_\alpha(U) \cap f_\beta(U) \subseteq f_\gamma \circ f_{i_k}(U) \cap f_\gamma \circ f_{j_k}(U) = f_\gamma(\emptyset) = \emptyset.$$

Thus we have shown that (24) holds.

We next claim that

$$F = \bigcup_{\alpha \in \mathcal{J}_\delta} f_\alpha(F). \tag{25}$$

This follows from (23) and induction. Given $x \in F$ choose $i_1 \in \mathcal{I}$ such that $x \in f_{i_1}(F)$. Assume that $x \in f_{i_1} \circ \cdots \circ f_{i_k}(F)$; then $f_{i_k}^{-1} \circ \cdots \circ f_{i_1}^{-1}(x) \in F$

implies that we can choose $i_{k+1} \in \mathcal{I}$ such that $f_{i_k}^{-1} \circ \cdots \circ f_{i_1}^{-1}(x) \in f_{i_{k+1}}(F)$. It follows that $x \in f_{i_1} \circ \cdots \circ f_{i_{k+1}}(F)$. Given the sequence i_k chosen above, there is exactly one choice of n such that $\alpha = (i_1, \cdots, i_n)$ satisfies $\sigma_\alpha \eta < \delta \le \sigma_{\alpha'}\eta$. We conclude that $x \in f_\alpha(F)$ for some $\alpha \in \mathcal{J}_\delta$, which proves the claim.

We now use (24) and (25) to show that F is equi-homogeneous. Let $x \in F$ be arbitrary. Then $x \in f_\alpha(F)$ for some $\alpha \in \mathcal{J}_\delta$ and consequently

$$\operatorname{diam}(f_\alpha(F)) = \sigma_\alpha \operatorname{diam}(F) \le \sigma_\alpha \eta < \delta,$$

which implies that $f_\alpha(F) \subseteq B_\delta(x)$. It follows that

$$B_\delta(x) \cap F = B_\delta(x) \cap \bigcup_{\beta \in \mathcal{J}_\delta} f_\beta(F) \supseteq B_\delta(x) \cap f_\alpha(F) = f_\alpha(F).$$

Therefore

$$\mathcal{N}(B_\delta(x) \cap F, \rho) \ge \mathcal{N}(f_\alpha(F), \rho) = \mathcal{N}(F, \rho/\sigma_\alpha) \ge \mathcal{N}(F, c_1\rho/\delta)$$

where $c_1 = \eta/\sigma_*$ implies that

$$\inf_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \ge \mathcal{N}(F, c_1\rho/\delta). \tag{26}$$

Now let $A_\delta = \{\, \alpha \in \mathcal{J}_\delta : B_\delta(x) \cap f_\alpha(\overline{U}) \ne \emptyset \,\}$. Then $\alpha \in A_\delta$ implies that

$$f_\alpha(\overline{U}) \subseteq B_{\delta + \operatorname{diam} f_\alpha(\overline{U})}(x) \subseteq B_{2\delta}(x).$$

Therefore by (24) we obtain

$$\begin{aligned}\lambda(B_{2\delta}(x)) &\ge \lambda\Big(\bigcup_{\alpha \in A_\delta} f_\alpha(U)\Big) = \sum_{\alpha \in A_\delta} \lambda\big(f_\alpha(U)\big)\\ &= \lambda(U) \sum_{\alpha \in A_\delta} (\sigma_\alpha)^d \ge \lambda(U)(\delta/c_1)^d \operatorname{card}(A_\delta)\end{aligned}$$

where λ is the d-dimensional Lebesgue measure. Consequently

$$\begin{aligned}\mathcal{N}(B_\delta(x) \cap F, \rho) &\le \sum_{\alpha \in A_\delta} N\big(f_\alpha(F), \rho\big) = \sum_{\alpha \in A_\delta} \mathcal{N}(F, \rho/\sigma_\alpha)\\ &\le \operatorname{card}(A_\delta)\mathcal{N}(F, \eta\rho/\delta) \le M\mathcal{N}(F, \eta\rho/\delta)\end{aligned}$$

where $M = (2c_1\eta)^d \lambda(B_1(x))/\lambda(U)$. It follows from (26) that

$$\sup_{x \in F} \mathcal{N}(B_\delta(x) \cap F, \rho) \le M\mathcal{N}(F, \eta\rho/\delta) \le M \inf_{x \in F} \mathcal{N}(B_{c_1\delta}(x) \cap F, c_2\rho)$$

where $c_2 = \eta$. In light of Lemma 3.3 the proof is complete. □

4.2 Non-autonomous systems

We now consider pullback attractors of 'non-autonomous' iterated function systems, recalling first the notation from the Introduction. Given $B \subseteq \mathbb{R}^d$ and $k < l$ we define

$$\mathcal{S}^{k,l}(B) = \mathcal{S}^k \circ \ldots \circ \mathcal{S}^{l-1}(B) \qquad \text{and} \qquad \mathcal{S}^{k,k}(B) = B$$

where $\mathcal{S}^k(B) = \bigcup_{i \in \mathcal{I}_{k+1}} f_i(B)$. The *pullback attractor* of a non-autonomous iterated function system is a collection of sets $\{F^k\}_{k \in \mathbb{N} \cup \{0\}}$ such that

(i) each F^k is compact and uniformly bounded[2],

(ii) the $\{F^k\}$ are invariant, in the sense that $F^k = \mathcal{S}^k(F^{k+1})$ holds, and

(iii) $\rho_H(\mathcal{S}^{k,l}(B), F^k) \to 0$ as $l \to \infty$ for every bounded set $B \subset \mathbb{R}^d$.

Before we give conditions ensuring the existence of a pullback attractor, we first show that the definition is sufficient to ensure uniqueness.

Theorem 4.2. *The pullback attractor of a non-autonomous iterated function system, if it exists, is unique.*

Proof. Suppose the collection of sets G^k for $k \in \mathbb{N}_0$ also satisfy properties 1–3 above. Given $k \in \mathbb{N}_0$ fixed, consider any point $x_k \in G^k$. By property 2 there is $x_{k+1} \in G^{k+1}$ and $i_{k+1} \in \mathcal{I}_{k+1}$ such that $x_k = f_{i_{k+1}}(x_{k+1})$. By induction define $x_j \in F^j$ and $i_j \in \mathcal{I}^j$ such that $x_j = f_{i_{j+1}}(x_{j+1})$ for all $j > k$. Since the $x_j \in G^j$ and the $\{G^j\}$ are uniformly bounded then $B = \{\, x_j : j > k \,\}$ is a bounded set. Moreover, $x_k \in \mathcal{S}^{k,l}(B)$ for every $l > k$. Consequently

$$\rho_H(\{x_k\}, F^k) \le \rho_H(\mathcal{S}^{k,l}(B), F^k) \to 0 \quad \text{as} \quad l \to \infty$$

by property 3. Since F^k is closed, it follows that $x_k \in F^k$. Therefore $G^k \subseteq F^k$. Switching the roles of F^k and G^k yields that $F^k = G^k$. □

We now show under natural conditions that the pullback attractor of a non-autonomous iterated function system exists. For each $i \in \mathbb{N}$ let b_i be the unique fixed point of f_i such that $f_i(b_i) = b_i$.

Theorem 4.3. *If*

$$M = \sup\{\, |b_i| : i \in \mathbb{N} \,\} < \infty \qquad \textit{and} \qquad \sigma^* = \sup\{\, \sigma_i : i \in \mathbb{N}, \} < 1,$$

then the pullback attractor exists.

Proof. We first find a compact set K such that for any bounded set $B \subset \mathbb{R}^d$ and any $k \in N$ there exists a corresponding value of $l(k, B) \ge k$ such that

$$\mathcal{S}^{k,l}(B) \subseteq K \qquad \text{for all} \qquad l \ge l(k, B). \tag{27}$$

Let $R = 2(1 + \sigma^*)M/(1 - \sigma^*)$ and K be the closed ball of radius R centered at the origin. Consider a bounded set $B \subset \mathbb{R}^d$ with $|x| \le L$ for all $x \in B$. If $|x| \ge R$, then

$$\begin{aligned} |f_i(x)| &\le |f_i(x) - f_i(b_i)| + |b_i| \le \sigma^*|x - b_i| + |b_i| \\ &\le \sigma^*|x| + (1 + \sigma^*)|b_i| \le ((1 + \sigma^*)/2)|x|. \end{aligned}$$

It follows that (27) holds for $l(k, B) = k + \lceil \log(L/R)/\log(2/(1 + \sigma^*)) \rceil$.

[2] We note that the requirement that the sets $\{F_k\}$ are uniformly bounded is not normally part of the definition of the continuous-time pullback attractor (see [4]), but in our setting this restriction is both natural and convenient. Without such an assumption the pullback attractor need not attract itself, and then an additional property (such as minimality) is required to ensure uniqueness. Even within the continuous-time setting attractors that are uniformly bounded 'in the past' (for all $t \le t_0$ for each $t_0 \in \mathbb{R}$) are convenient to avoid various possible pathologies, and this corresponds to uniform boundedness for our iterated function systems which are only defined for indices that correspond to $t \le 0$.

Next we show that $\mathcal{S}^k$ is a contraction with respect to the Hausdorff metric. Let $A, B \subset \mathbb{R}^d$ be compact. For each $a \in A$ choose $\pi(a) \in B$ such that

$$|a - \pi(a)| = \min\{\, |a - b| : b \in B \,\}.$$

Any $x \in \mathcal{S}^k(A)$ may be written as $x = f_i(a)$ for some $a \in A$ and $i \in \mathcal{I}_k$. Consequently

$$\rho_H(\{x\}, \mathcal{S}^k(B)) \le |f_i(a) - f_i(\pi(a))| \le \sigma_i |a - \pi(a)| \le \sigma^* \rho_H(A, B)$$

implies

$$\rho_H(\mathcal{S}^k(A), \mathcal{S}^k(B)) \le \sigma^* \rho_H(A, B) \le \sigma^* \rho_H(A, B). \tag{28}$$

Interchanging the roles of A and B yields

$$\rho_H(\mathcal{S}^k(A), \mathcal{S}^k(B)) \le \sigma^* \rho_H(A, B).$$

Given k fixed, define $K_m = \mathcal{S}^{k,k+m}(K)$ for $m \in \mathbb{N}$ to obtain a sequence of compact subsets of $\mathbb{R}^d$. If $m \le n$ then

$$\begin{aligned}\rho_H(K_m, K_n) &= \rho_H(\mathcal{S}^{k,k+m}(K), \mathcal{S}^{k,k+m} \circ \mathcal{S}^{k+m,k+n}(K)) \\ &\le (\sigma^*)^m \rho_H(K, \mathcal{S}^{k+m,k+n}(K)) \le (\sigma^*)^m 2R,\end{aligned}$$

which shows that K_m is a Cauchy sequence. The completeness of the Hausdorff metric on the space of all compact subsets of $\mathbb{R}^d$ then yields a compact limit set, which we call F^k.

It remains to show that F^k satisfies the properties required of the pullback attractor. Clearly F^k is compact. Moreover, since $K_m \subseteq K$ for all m then $F^k \subseteq K$ and so F^k is uniformly bounded. To show the invariance property (ii) note that

$$\begin{aligned}&\rho_H(F^k, \mathcal{S}^k(F^{k+1})) \\ &\qquad \le \rho_H(F^k, \mathcal{S}^{k,k+m}(K)) + \rho_H(\mathcal{S}^k \circ \mathcal{S}^{k+1,k+m}(K), \mathcal{S}^k(F^{k+1})) \\ &\qquad \le \rho_H(F^k, \mathcal{S}^{k,k+m}(K)) + \sigma^* \rho_H(\mathcal{S}^{k+1,k+m}(K), F^{k+1}) \\ &\qquad \to 0 \quad \text{as} \quad m \to \infty.\end{aligned}$$

Given $\varepsilon > 0$ choose m so large that $(\sigma^*)^m 2R < \varepsilon$. Now choose $l \ge k + m$ so large that $\mathcal{S}^{k+m,l}(B) \subseteq K$. It follows that

$$\rho_H(\mathcal{S}^{k,l}(B), F^k) \le \rho_H(\mathcal{S}^{k,k+m}(K), F^k) = \rho_H(K_m, F^k) \le (\sigma^*)^m 2R < \varepsilon,$$

and therefore property (iii) holds. □

We are now ready to formulate conditions on non-autonomous iterated function systems that guarantee that the resulting non-autonomous attractor is equi-homogeneous.

We say that a non-autonomous iterated function system satisfies the (generalized) Moran open set condition if there exists a uniformly bounded sequence of non-empty open sets $U^k \subset \mathbb{R}^d$ for $k \in \mathbb{N}_0$ such that

(i) $S^k(U^{k+1}) \subseteq U^k$;

(ii) $f_i(U^k) \cap f_j(U^k) = \emptyset$ for $i, j \in \mathcal{I}_k$ such that $i \ne j$; and

(iii) $\lambda(U^k) \geq \epsilon_0 > 0$ for all $k \in \mathbb{N}_0$, where λ is the d-dimensional Lebesgue measure.

If a classical ('autonomous') iterated function system satisfies the Moran open set condition, then it satisfies the above condition with $U^k = U$, $\mathcal{I}_k = \mathcal{I}$ and $F^k = F$ for every $k \in \mathbb{N}_0$. Thus, this definition of the Moran open set reduces to the classical one when the functions are the same at every step of the iteration, and hence there is no ambiguity in referring to the generalized definition above simply as the 'Moran open set condition'.

Lemma 4.4. *If a non-autonomous iterated function system satisfies the Moran open set condition and the hypotheses of Theorem 4.3, then* $F^k \subseteq \overline{U^k}$.

Proof. Let L be a uniform bound on U^k and F^k for all $k \in \mathbb{N}$. Since $\mathcal{S}^{k,l}(U^l) \subseteq U^l$ and $\mathcal{S}^{k,l}(F^l) = F^k$ for $k \leq l$ then

$$\rho_H(F^k, U^k) \leq \rho_H(S^{k,l}(F^l), S^{k,l}(U^l)) \leq (\sigma^*)^{l-k}\rho_H(F^l, U^l) \leq (\sigma^*)^{l-k}2L.$$

Taking $l \to \infty$ it follows that $\rho_H(F^k, U^k) = 0$ and consequently $F^k \subseteq \overline{U^k}$. □

We now consider the case where the contractions f_i are similarities of the form given by (22). We begin by treating the simplest situation, in which $\sigma_i = c_k$ for all $\sigma_i \in \mathcal{I}_k$. Thus, all the contraction ratios are the same at each level k of the iteration. Note that this class of non-autonomous iterated function systems includes pullback attractors equal to the generalized Cantor sets which appear in [22]. Intuitively, each step of the iteration corresponds to a different scale, and since all the maps contract in the same way at that scale, it is natural that the pullback attractor is equi-homogeneous. The following theorem is the full statement of Theorem 1.1 which appears in the introduction.

Theorem 4.5. *Suppose that we have a non-autonomous iterated function system of similarities for which there exists an* $N \in \mathbb{N}$ *such that* $\mathrm{card}(\mathcal{I}_k) \leq N$ *for every* k *and* $\sigma_i = c_k$ *for all* $i \in \mathcal{I}_k$. *Then if the Moran open set condition is satisfied the pullback attractor* F^k *is equi-homogeneous for every* $k \in \mathbb{N}_0$.

Proof. Since all the hypotheses are uniform in k it is sufficient to show that F^0 is equi-homogeneous. Let

$$\pi_n = \textstyle\prod_{j=1}^n c_j \qquad \text{and} \qquad \eta = \sup\{\,\mathrm{diam}(\overline{U^k}) : k \in \mathbb{N}\,\}.$$

For $\delta \leq c_1\eta$ there exists $n \geq 2$ such that $\pi_n\eta < \delta \leq \pi_{n-1}\eta$. Define

$$\mathcal{J}_n = \mathcal{I}_1 \times \cdots \times \mathcal{I}_n.$$

From the open set condition we have

$$f_\alpha(U^n) \cap f_\beta(U^n) = \emptyset \quad \text{for} \quad \alpha, \beta \in \mathcal{J}_n \quad \text{with} \quad \alpha \neq \beta \tag{29}$$

and by property 2 for pullback attractors, we obtain

$$\textstyle\bigcup_{\alpha \in \mathcal{J}_\delta} f_\alpha(F^n) = \mathcal{S}^{0,n}(F^n) = F^0. \tag{30}$$

We now use (29) and (30) to show that F^0 is equi-homogeneous. Let $x \in F^0$ be arbitrary. Then $x \in f_\alpha(F^n)$ for some $\alpha \in \mathcal{J}_n$ and consequently

$$\mathrm{diam}(f_\alpha(F^n)) = \pi_n \mathrm{diam}(F^n) \leq \pi_n\eta < \delta,$$

which implies that $f_\alpha(F^n) \subseteq B_\delta(x)$. It follows that

$$B_\delta(x) \cap F^0 = B_\delta(x) \cap \bigcup_{\beta \in \mathcal{J}_n} f_\beta(F^n) \supseteq B_\delta(x) \cap f_\alpha(F^n) = f_\alpha(F^n).$$

Therefore

$$\mathcal{N}(B_\delta(x) \cap F^0, \rho) \geq \mathcal{N}(f_\alpha(F^n), \rho) = \mathcal{N}(F^n, \rho/\pi_n)$$

implies that

$$\inf_{x \in F^0} \mathcal{N}(B_\delta(x) \cap F^0, \rho) \geq \mathcal{N}(F^n, \rho/\pi_n). \tag{31}$$

Let $A_n = \{\, \alpha \in \mathcal{J}_n : B_\delta(x) \cap f_\alpha(\overline{U^n}) \neq \emptyset \,\}$. Then $\beta \in A_{n-1}$ implies that

$$f_\beta(\overline{U^{n-1}}) \subseteq B_{\delta + \operatorname{diam} f_\beta(\overline{U^{n-1}})}(x) \subseteq B_{2\eta\pi_{n-1}}(x).$$

Therefore by (29) we obtain

$$\begin{aligned}\lambda(B_{2\eta\pi_{n-1}}(x)) &\geq \lambda\Big(\bigcup_{\beta \in A_{n-1}} f_\beta(U^{n-1}) \Big) = \sum_{\beta \in A_{n-1}} \lambda\big(f_\beta(U^{n-1})\big) \\ &= \operatorname{card}(A_{n-1})(\pi_{n-1})^d \lambda(U^{n-1}) \geq \operatorname{card}(A_{n-1})(\pi_{n-1})^d \epsilon_0.\end{aligned}$$

Consequently

$$\operatorname{card}(A_n) \leq \operatorname{card}(\mathcal{I}_n)\operatorname{card}(A_{n-1}) \leq M$$

where $M = (2\eta)^d N \lambda(B_1(0))/\epsilon_0$ is independent of n. Therefore

$$\begin{aligned}\mathcal{N}(B_\delta(x) \cap F^0, \rho) &\leq \sum_{\alpha \in A_n} N\big(f_\alpha(F^n), \rho\big) = \sum_{\alpha \in A_n} \mathcal{N}(F^n, \rho/\pi_n) \\ &= \operatorname{card}(A_n)\mathcal{N}(F^n, \rho/\pi_n) \leq M\mathcal{N}(F^n, \rho/\pi_n).\end{aligned}$$

Taking the supremum of F^0 and combining this with (31) we obtain

$$\sup_{x \in F^0} \mathcal{N}(B_\delta(x) \cap F^0, \rho) \leq M\mathcal{N}(F^n, \rho/\pi_n) \leq \inf_{x \in F^0} \mathcal{N}(B_\delta(x) \cap F^0, \rho)$$

which completes the proof of the theorem. □

Corollary 4.6. *Suppose that we have a non-autonomous iterated function system of similarities for which $\sigma_i = c_k \geq c_* > 0$ for all $i \in \mathcal{I}_k$. Then if the Moran open set condition is satisfied the pullback attractor F^k is equi-homogeneous for every $k \in \mathbb{N}_0$.*

Proof. The proof is the same as the proof of Theorem 4.5 with the estimate of $\operatorname{card}(A_n)$ in terms of $\operatorname{card}(A_{n-1})$ replaced by the more direct estimate used in the proof of Proposition 4.1.

Alternatively, let B be a bounded set such that $U^k \subseteq B$ for every $k \in \mathbb{N}_0$. The open-set condition then implies

$$\begin{aligned}\lambda(B) &\geq \lambda(U^{k-1}) \geq \lambda(\textstyle\bigcup_{i \in \mathcal{I}_k} f_i(U^k)) \\ &\geq \textstyle\sum_{i \in \mathcal{I}_k} \lambda(f_i(U^k)) \geq c_* \sum_{i \in \mathcal{I}_k} \lambda(U^k) \geq \operatorname{card}(I_k) c_* \epsilon_0.\end{aligned}$$

Therefore $\operatorname{card}(\mathcal{I}_k) \leq N$ where $N = \lambda(B)/(c_* \epsilon_0)$ and the result now follows from the application of Theorem 4.5. □

We now examine the case when the σ_i need not coincide for all the functions in each step of the iteration. This situation requires more refined analysis which we make easier by introducing the following notation. Denote

$$\mathcal{I}^k = \bigcup_{m\in\mathbb{N}} \mathcal{J}^{k,k+m} \qquad \text{where} \qquad \mathcal{J}^{k,n} = \mathcal{I}^{k+1} \times \cdots \times \mathcal{I}^n$$

Given $\alpha \in \mathcal{J}^{k,n}$ denote $k_\alpha = k$ and $n_\alpha = n$. Note for k_α and n_α to be well defined, we assume as we may that $\mathcal{I}_k \cap \mathcal{I}_n = \emptyset$ for $k \neq n$. Our analysis will further make use of the set

$$\mathcal{J}_\delta^k = \{\, \alpha \in \mathcal{I}^k : \sigma_\alpha \eta < \delta \leq \sigma_{\alpha'} \eta \,\}. \tag{32}$$

As in the proof of Theorem 4.1 we have the following facts: if $\alpha, \beta \in \mathcal{J}_\delta^k$ then

$$f_\alpha(U^{n_\alpha}) \cap f_\beta(U^{n_\beta}) = \emptyset \qquad \text{when} \qquad \alpha \neq \beta$$

and

$$F^k = \bigcup_{\alpha\in\mathcal{J}_\delta^k} f_\alpha(F^{n_\alpha}).$$

We shall need, as before, an estimate on the cardinality of

$$A_\delta^k = \{\, \alpha \in \mathcal{J}_\delta^k : f_\alpha(\overline{U^{n_\alpha}}) \cap B_\delta(x) \neq \emptyset \,\},$$

which is provided by the following lemma.

Lemma 4.7. *Let f_i be a non-autonomous iterated function system of similarities that satisfies the Moran open set condition with open sets U^k and has a pullback attractor F^k. Then*

$$\operatorname{card}(A_\delta^k) \leq \kappa_0,$$

where κ_0 is independent of k and δ.

Proof. For $\alpha \in A_\delta^k$ we have $f_\alpha(\overline{U^{n_\alpha}}) \subseteq B_{2\delta}(x)$. Therefore

$$\begin{aligned}\lambda(B_{2\delta}(x)) &\geq \lambda\Big(\bigcup_{\alpha\in A_\delta^k} f_\alpha(U^{n_\alpha})\Big)\\ &= \sum_{\alpha\in A_\delta^k} \lambda\big(f_\alpha(U^{n_\alpha})\big) \geq \operatorname{card}(A_\delta^k)(\delta\sigma_*/\eta)^d \epsilon_0\end{aligned}$$

implies

$$\operatorname{card}(A_\delta^k) \leq \lambda(B_1(0))(2\eta)/\sigma_*)^d/\epsilon_0 = \kappa_0$$

where $\kappa_0 = \lambda(B_1(0))(2\eta/\sigma_*)^d/\epsilon_0$. □

We also need a bound on the cardinality of $\mathcal{J}_\delta^k$ (as defined in (32)). In order to obtain this bound we make the uniformity assumption that for some $s > 0$

$$\sum_{i\in\mathcal{I}_k} \sigma_i^s = 1 \qquad \text{for all} \qquad k \in \mathbb{N} \tag{33}$$

(this was (4) in the Introduction) and consider a sequence of probability measures μ^k with support on F^k defined by the relationships

$$\mu^k(f_i(B)) = \mu^{k+1}(B) \frac{\operatorname{diam}(f_i(B))^s}{\sum_{j\in\mathcal{I}_{k+1}} \operatorname{diam}(f_j(B))^s} = \mu^{k+1}(B)\sigma_i^s$$

where $k \in \mathbb{N}_0$, $i \in \mathcal{I}_{k+1}$ and B is any Borel set. Note for $\alpha \in \mathcal{I}^{k,n}$ that

$$\mu^k(f_\alpha(F^n)) = \mu_n(F^n)(\sigma_\alpha)^s = (\sigma_\alpha)^s.$$

We are now ready to estimate $\operatorname{card}(\mathcal{J}_\delta^k)$.

Lemma 4.8. *Let f_i be a non-autonomous iterated function system of similarities that has a pullback attractor F^k and satisfies (33) and the Moran open set condition with open sets U^k. Then*

$$\kappa_1\delta^{-s} \le \operatorname{card}(\mathcal{J}_\delta^k) \le \kappa_2\delta^{-s},$$

where κ_1 and κ_2 are independent of k and δ.

Proof. For the lower bound we estimate

$$\begin{aligned} 1 = \mu^k(F^k) = \mu^k\Big(\textstyle\bigcup_{\alpha\in\mathcal{J}_\delta^k} f_\alpha(F^{n_\alpha})\Big) &\le \textstyle\sum_{\alpha\in\mathcal{J}_\delta^k}\mu^k\big(f_\alpha(F^{n_\alpha})\big)\\ &= \textstyle\sum_{\alpha\in\mathcal{J}_\delta^k}(\sigma_\alpha)^s < \sum_{\alpha\in\mathcal{J}_\delta^k}(\delta/\eta)^s = \operatorname{card}(J_\delta^k)(\delta/\eta)^s. \end{aligned}$$

Therefore

$$\operatorname{card}(\mathcal{J}_\delta^k) > \kappa_1\delta^{-s}$$

where $\kappa_1 = \eta^s$. For the lower bound we will use Lemma 4.7 to count the non-empty intersections where

$$f_\alpha(F^{n_\alpha}) \cap f_\beta(F^{n_\beta}) \ne \emptyset \qquad \text{and} \qquad \beta \ne \alpha.$$

We will do this inductively. Let $J_1 = \mathcal{J}_\delta^k$ and pick $\alpha_1 \in J_1$. Define

$$J_{i+1} = \{\, \beta \in J_i : f_{\alpha_i}(F^{n_{\alpha_i}}) \cap f_\beta(F^{n_\beta}) = \emptyset \,\}.$$

Since $f_{\alpha_k}(F^{n_{\alpha_k}}) \subseteq B_\delta(x)$ for some x it follows from Lemma 4.7 that

$$\operatorname{card}(J_{i+1}) \ge \operatorname{card}(J_i) - \operatorname{card}(A_\delta^k) \ge \operatorname{card}(J_i) - \kappa_0 \ge \operatorname{card}(J_1) - i\kappa_0.$$

We can continue choosing $a_{i+1} \in J_{i+1}$ until $i = i_0$ where

$$(i_0 - 1)\kappa_0 < \operatorname{card}(J_1) \le i_0\kappa_0.$$

By construction, it follows that

$$f_{\alpha_i}(F^{n_{\alpha_i}}) \cap f_{\alpha_j}(F^{n_{\alpha_j}}) = \emptyset \qquad \text{for} \qquad i \ne j.$$

Therefore

$$\begin{aligned} 1 = \mu^k(F^k) \ge \mu^k\Big(\textstyle\bigcup_{i=1}^{i_0} f_{\alpha_i}(F^{n_{\alpha_i}})\Big) &= \textstyle\sum_{i=1}^{i_0}\mu^k\big(f_{\alpha_i}(F^{n_{\alpha_i}})\big)\\ &= \textstyle\sum_{i=1}^{i_0}(\sigma_{\alpha_i})^s \ge \sum_{i=1}^{i_0}(\sigma_*\delta/\eta)^s \ge \operatorname{card}(\mathcal{J}_\delta^k)(\sigma_*\delta/\eta)^s/\kappa_0. \end{aligned}$$

It follows that

$$\operatorname{card}(\mathcal{J}_\delta^k) \le \kappa_0(\eta/\sigma_*)^s\delta^{-s}.$$

Taking $\kappa_2 = \kappa_0(\eta/\sigma_*)^s$ finishes the proof. □

We are now ready to prove sufficient conditions for the equi-homogeneity of pullback attractors in the case when the contraction ratios need not coincide within each stage of the iteration.

Theorem 4.9. *Given a non-autonomous iterated function system that satisfies the Moran open set condition, suppose that*

$$\inf\{\,\sigma_i : i\in\mathbb{N}\,\} = \sigma_* > 0$$

and that there exists an $s>0$ *such that*

$$\sum_{i\in\mathcal{I}_k}\sigma_i^s = 1 \quad \textit{for all} \quad i\in\mathcal{I}_k. \tag{34}$$

Then F^k *is equi-homogeneous and* $\dim_{\mathrm{A}}(F^k)=s$ *for every* $k\in\mathbb{N}_0$.

Proof. We first estimate

$$\inf_{x\in F^0}\mathcal{N}(F^0\cap B_\delta(x),\rho).$$

Let $x\in F_0$. Then there is $\alpha\in\mathcal{J}_\delta^0$ such that $x\in f_\alpha(F^{n_\alpha})$ and consequently $F^0\cap B_\delta(x)\subseteq f_\alpha(F^{n_\alpha})$. Define

$$\tilde{A}_\delta^k = \{\,\alpha\in\mathcal{J}_\delta^k : f_\alpha(\overline{U^{\alpha_k}})\cap B_{2\delta}(x)\neq\emptyset\,\}.$$

Following the same proof as in Lemma 4.7 there exists $\tilde\kappa_0$ which is independent of δ and k such that $\operatorname{card}(\tilde{A}_\delta^k)\le\tilde\kappa_0$. Following the same proof as in Lemma 4.8 we can find a sequence $\gamma_i\in\mathcal{J}^{n_\alpha}_{\rho/\sigma_\alpha}$ up to $i=\tilde\imath_0$ where

$$(\tilde\imath_0-1)\tilde\kappa_0 < \operatorname{card}(\mathcal{J}^{n_\alpha}_{\rho/\sigma_\alpha})\le\tilde\imath_0\tilde\kappa_0$$

such that $f_{\gamma_i}(F^{n_{\gamma_i}})\subseteq B_\delta(x_i)$ for $x_i\in f_{\gamma_i}(F^{n_{\gamma_i}})$ and

$$B_{2\delta(x_i)}\cap F_{\gamma_j}(F^{n_{\gamma_j}})=\emptyset \qquad \text{for} \qquad i\neq j.$$

In particular, we have found $x_i\in F^{n_\alpha}$ such that

$$B_\delta(x_i)\cap B_\delta(x_j)=\emptyset \qquad \text{for} \qquad i\neq j.$$

It follows that

$$\begin{aligned}\mathcal{P}(F^0\cap B_\delta(x),\rho) &\ge \mathcal{P}(f_\alpha(F^{n_\alpha}),\rho)=\mathcal{P}(F^{n_\alpha},\rho/\sigma_\alpha)\ge\tilde\imath_0\\ &\ge \operatorname{card}(\mathcal{J}^{n_\alpha}_{\rho/\sigma_\alpha})/\tilde k_0 \ge (\kappa_1/\tilde\kappa_0)(\sigma_\alpha/\rho)^s \ge (\kappa_1/\tilde\kappa_0)\sigma_*^s(\delta/\rho)^s\end{aligned}$$

Therefore

$$\inf_{x\in F^0}\mathcal{N}(F^0\cap B_\delta(x),\rho)\ge\kappa_3(\delta/\rho)^s$$

where $\kappa_3=\kappa_1(2\sigma_*)^s/\tilde\kappa_0$.

We now estimate

$$\sup_{x\in F^0}\mathcal{N}(F^0\cap B_\delta(x),\rho).$$

Let $x\in F^0$. Applying Lemma 4.8 and Lemma 4.7 we obtain

$$\begin{aligned}\mathcal{N}(F^0\cap B_\delta(x),\rho) &\le \textstyle\sum_{\beta\in A_\delta^0}\mathcal{N}(f_\beta(F^{n_\beta}),\rho)=\sum_{\beta\in A_\delta^0}\mathcal{N}(F^{n_\beta},\rho/\sigma_\beta)\\ &= \textstyle\sum_{\beta\in A_\delta^0}\mathcal{N}\Big(\bigcup_{\gamma\in\mathcal{J}^{n_\beta}_{\rho/\gamma_\beta}} f_\gamma(F^{n_\gamma}),\rho/\sigma_\beta\Big)\\ &\le \textstyle\sum_{\beta\in A_\delta^0}\sum_{\gamma\in\mathcal{J}^{n_\beta}_{\rho/\gamma_\beta}}\mathcal{N}\big(F^{n_\gamma},\rho/(\sigma_\beta\sigma_\gamma)\big)\\ &= \textstyle\sum_{\beta\in A_\delta^0}\sum_{\gamma\in\mathcal{J}^{n_\beta}_{\rho/\gamma_\beta}}\mathcal{N}\big(F^{n_\gamma},\eta\big)\\ &\le \textstyle\sum_{\beta\in A_\delta^0}\operatorname{card}(\mathcal{J}^{n_\beta}_{\rho/\gamma_\beta})\le\sum_{\beta\in A_\delta^0}\kappa_2(\sigma_\beta/\rho)^s\\ &\le \kappa_0\kappa_2\eta^{-s}(\delta/\rho)^s.\end{aligned}$$

Taking $\kappa_4 = \kappa_0\kappa_2\eta^{-s}$ and $M = \kappa_4/\kappa_3$ yields

$$\sup_{x\in F^0} \mathcal{N}(F^0 \cap B_\delta(x), \rho) \leq \kappa_4(\delta/\rho)^s \leq M \inf_{x\in F^0} \mathcal{N}(F^0 \cap B_\delta(x), \rho).$$

We finish by noting that the above inequality also shows $\dim_{\mathrm{A}}(F^k) = s$. □

It is worth remarking that the proof of Theorem 4.9, in addition to proving that F^k is equi-homogeneous, also shows that F^k attains its upper and lower box-counting dimensions. Thus, Theorem 3.5 implies that $\dim_{\mathrm{B}}(F^k) = \dim_{\mathrm{A}}(F^k)$ for all k.

For the final result in this section we note that the hypothesis (34) on s in Theorem 4.9 can be weakened without changing the details of the proof.

Theorem 4.10. *Given a non-autonomous iterated function system that satisfies the Moran open set condition, suppose that* $\inf\{\, \sigma_i : i \in \mathbb{N} \,\} = \sigma_* > 0$ *and that there exist* $s > 0$, $n_0 \in \mathbb{N}$, *and* $L > 0$ *such that*

$$L^1 \leq \textstyle\sum_{\alpha\in\mathcal{J}_{k,n}} \sigma_i^s \leq L \quad \textit{for all} \quad i \in \mathcal{I}_k \quad \textit{and} \quad n \geq n_0. \tag{35}$$

Then F^k *is equi-homogeneous and* $\dim_{\mathrm{A}}(F^k) = s$ *for every* $k \in \mathbb{N}_0$.

Note that if an s exists that satisfies (35), then it is unique. Let $\sigma^* = \sup\{\, \sigma_i : i \in \mathbb{N} \,\} < 1$. Suppose s_0 satisfies (35) and $\delta \neq 0$. If $\delta > 0$ then

$$\textstyle\sum_{\alpha\in\mathcal{J}_{k,k+n}} \sigma_\alpha^{s_0+\delta} \leq (\sigma^*)^{\delta n} \sum_{\alpha\in\mathcal{J}_{k,k+n}} \sigma_\alpha^{s_0} \leq (\sigma^*)^{\delta n} L \to 0 \quad \text{as} \quad n \to \infty$$

shows that the lower bound in (35) could not hold for $s = s_0 + \delta$. On the other hand, if $\delta < 0$ then

$$\textstyle\sum_{\alpha\in\mathcal{J}_{k,k+n}} \sigma_\alpha^{s_0+\delta} \geq (\sigma^*)^{\delta n} \sum_{\alpha\in\mathcal{J}_{k,k+n}} \sigma_\alpha^{s_0} \geq (\sigma^*)^{\delta n} L \to \infty \quad \text{as} \quad n \to \infty$$

shown that the upper bound could not hold. We conclude that there is at most one value for s such that (35) holds.

Note that (35) is essentially the statement that (34) holds uniformly when averaged over long enough sequences of iterations. A similar, but weaker, condition appears in Li, Li, Miao, and Xi [16] which is sufficient to compute the Assouad dimension of a set that satisfies the Moran structure conditions. Our theorem shows equi-homogeneity and requires a stronger assumption.

5 Conclusion

We have demonstrated that the equi-homogeneous sets include a large class of attractors of iterated functions systems, both autonomous and non-autonomous, in addition to the generalized Cantor sets considered in [22]. Further, as equi-homogeneous sets have identical dimensional detail at all points at each fixed length scale, we have shown that the calculation of their Assouad dimensions can be much simplified. Finally, we have demonstrated that equi-homogeneity is independent of any previously defined notion of dimensional equivalence, establishing equi-homogeneity as a novel and useful tool in the analysis of fractal sets.

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