ON THE CONTINUITY OF GLOBAL ATTRACTORS

LUAN T. HOANG, ERIC J. OLSON, AND JAMES C. ROBINSON

Abstract. Let Λ be a complete metric space, and let $\{S_\lambda(\cdot) : \lambda \in \Lambda\}$ be a parametrised family of semigroups with global attractors $\mathscr{A}_\lambda$. We assume that there exists a fixed bounded set D such that $\mathscr{A}_\lambda \subset D$ for every $\lambda \in \Lambda$. By viewing the attractors as the limit as $t \to \infty$ of the sets $S_\lambda(t)D$, we give simple proofs of the equivalence of 'equi-attraction' to continuity (when this convergence is uniform in λ) and show that the attractors $\mathscr{A}_\lambda$ are continuous in λ at a residual set of parameters in the sense of Baire Category (when the convergence is only pointwise).

1. Global attractors

The global attractor of a dynamical system is the unique compact invariant set that attracts the trajectories starting in any bounded set at a uniform rate. Introduced by Billotti & LaSalle [3], they have been the subject of much research since the mid-1980s, and form the central topic of a number of monographs, including Babin & Vishik [1], Hale [9], Ladyzhenskaya [13], Robinson [16], and Temam [18].

The standard theory incorporates existence results [3], upper semicontinuity [10], and bounds on the attractor dimension [7]. Global attractors exist for many infinite-dimensional models [18], with familiar low-dimensional ODE models such as the Lorenz equations providing a testing ground for the general theory [8].

While upper semicontinuity with respect to perturbations is easy to prove, lower semicontinuity (and hence full continuity) is more delicate, requiring structural assumptions on the attractor or the assumption of a uniform attraction rate. However, Babin & Pilyugin [2] proved that the global attractor of a parametrised set of semigroups is continuous at a residual set of parameters, by taking advantage of the known upper semicontinuity and then using the fact that upper semicontinuous functions are continuous on a residual set.

Here we reprove results on equi-attraction and residual continuity in a more direct way, which also serves to demonstrate more clearly why these results are true. Given equi-attraction the attractor is the uniform limit of a sequence of continuous functions, and hence continuous (the converse requires a generalised version of Dini's Theorem); more generally, it is the pointwise limit of a sequence of continuous functions, i.e. a 'Baire one' function, and therefore the set of continuity points forms a residual set.

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2. Semigroups and attractors

A semigroup $\{S(t)\}_{t\geq 0}$ on a complete metric space (X,d) is a collection of maps $S(t) : X \to X$ such that

- (S1) $S(t)0 = \mathrm{id}$;
- (S2) $S(t+s) = S(t)S(s) = S(s)S(t)$ for all $t, s \geq 0$; and
- (S3) $S(t)x$ is continuous in x and t.

A compact set $\mathscr{A} \subset X$ is the *global attractor* for $S(\cdot)$ if

- (A1) $S(t)\mathscr{A} = \mathscr{A}$ for all $t \in \mathbb{R}$; and
- (A2) for any bounded set B, $\rho_X(S(t)B, \mathscr{A}) \to 0$ as $t \to \infty$, where ρ_X is the semi-distance $\rho_X(A,B) = \sup_{a\in A} \inf_{b\in B} d(a,b)$.

When such a set exists it is unique, the maximal compact invariant set, and the minimal closed set that satisfies (A2).

3. Upper and lower semicontinuity

Let Λ be a complete metric space and $S_\lambda(\cdot)$ a parametrised family of semigroups on X. Suppose that

- (L1) $S_\lambda(\cdot)$ has a global attractor $\mathscr{A}_\lambda$ for every $\lambda \in \Lambda$;
- (L2) there is a bounded subset D of X such that $\mathscr{A}_\lambda \subset D$ for every $\lambda \in \Lambda$; and
- (L3) for $t > 0$, $S_\lambda(t)x$ is continuous in λ, uniformly for x in bounded subsets of X.

We can strengthen (L2) and weaken (L3) by replacing 'bounded' by 'compact' to yield conditions (L2$'$) and (L3$'$). A wide range of dissipative systems with parameters satisfy these assumptions, for example the 2D Navier–Stokes equations, the scalar Kuramoto–Sivashinsky equation, reaction-diffusion equations, and the Lorenz equations, all of which are covered in [18].

Under these mild assumptions it is easy to show that $\mathscr{A}_\lambda$ is upper semicontinuous,

$$\rho_X(\mathscr{A}_\lambda, \mathscr{A}_{\lambda_0}) \to 0 \qquad \text{as} \quad \lambda \to \lambda_0$$

see [1, 2, 6, 9, 10, 16, 18], for example. However, lower semicontinuity, that is

$$\rho_X(\mathscr{A}_{\lambda_0}, \mathscr{A}_\lambda) \to 0 \qquad \text{as} \quad \lambda \to \lambda_0,$$

requires more: either structural conditions on the attractor $\mathscr{A}_{\lambda_0}$ ($\mathscr{A}_{\lambda_0}$ is the closure of the unstable manifolds of a finite number of hyperbolic equilibria, see Hale & Raugel [11], Stuart & Humphries [17], or Robinson [16]) or the 'equi-attraction' hypothesis of Li & Kloeden [14] (see Section 4). As a result, the continuity of attractors,

$$\lim_{\lambda\to\lambda_0} d_{\mathrm{H}}(\mathscr{A}_\lambda, \mathscr{A}_{\lambda_0}) = 0,$$

where

$$d_{\mathrm{H}}(A,B) = \max(\rho_X(A,B), \rho_X(B,A)) \tag{3.1}$$

is the symmetric Hausdorff distance, is only known under restrictive conditions.

In this paper we view $\mathscr{A}_\lambda$ as a function from Λ into the space of closed bounded subsets of X, given as the limit of the continuous functions $\overline{S_\lambda(t_n)D}$ (see Lemma 3.1). Indeed, note that given any set $D \supseteq \mathscr{A}_\lambda$ it follows from the invariance of the attractor (A1) that

$$\overline{S_\lambda(t)D} \supseteq S_\lambda(t)D \supseteq S_\lambda(t)\mathscr{A}_\lambda = \mathscr{A}_\lambda \qquad \text{for every} \quad t > 0,$$

and so the the attraction property of the attractor in (A2) implies that

$$d_{\rm H}(\overline{S_\lambda(t)D},\mathscr{A}_\lambda)\to 0\qquad\text{as}\qquad t\to\infty. \tag{3.2}$$

Uniform convergence (with respect to λ) in (3.2) is essentially the 'equi-attraction' introduced in [14], and thus clearly related to continuity of the limiting function $\mathscr{A}_\lambda$ (Section 4). Given only pointwise (λ-by-λ) convergence in (3.2) we can still use the result from the theory of Baire Category that the pointwise limit of continuous functions (a 'Baire one function') is continuous at a residual set to guarantee that $\mathscr{A}_\lambda$ is continuous in λ on a residual subset of Λ (Section 5).

For both results the following simple lemma is fundamental. We let $CB(X)$ be the collection of all closed and bounded subsets of X, and use the symmetric Hausdorff distance $d_{\rm H}$ defined in (3.1) as the metric on this space.

Lemma 3.1. *Suppose that D is bounded and that* (L3) *holds. Then for any $t>0$ the map $\lambda\mapsto\overline{S_\lambda(t)D}$ is continuous from Λ into $CB(X)$. The same is true if D is compact and* (L3$'$) *holds.*

Proof. Given $t>0$, $\lambda_0\in\Lambda$, and $\epsilon>0$, (L3) ensures that there exists a $\delta>0$ such that $d_\Lambda(\lambda_0,\lambda)<\delta$ implies that $d_X(S_\lambda(t)x,S_{\lambda_0}(t)x)<\epsilon$ for every $x\in D$. It follows that

$$\rho_X(S_\lambda(t)D,S_{\lambda_0}(t)D)<\epsilon\qquad\text{and}\qquad\rho_X(S_{\lambda_0}(t)D,S_\lambda(t)D)<\epsilon,$$

and so

$$\rho_X(\overline{S_\lambda(t)D},\overline{S_{\lambda_0}(t)D})\le\epsilon\qquad\text{and}\qquad\rho_X(\overline{S_{\lambda_0}(t)D},\overline{S_\lambda(t)D})\le\epsilon,$$

from which $d_{\rm H}(\overline{S_\lambda(t)D},\overline{S_{\lambda_0}(t)D})\le\epsilon$ as required. □

4. Uniform convergence: continuity and equi-attraction

First we give a simple proof of the results in [14] on the equivalence between equi-attraction and continuity. In our framework these follow from two classical results: the continuity of the uniform limit of a sequence of continuous functions and Dini's Theorem in an abstract formulation.

Li & Kloeden require (L1), (L2'), a time-uniform version of (L3') (see Corollary 4.3), and in addition an 'equi-dissipative' assumption that there exists a bounded absorbing set K such that

$$S_\lambda(t)B\subset K\qquad\text{for every}\quad t\ge t_B, \tag{4.1}$$

where t_B does not depend on λ. They then show that when Λ is compact, continuity of $\mathscr{A}_\lambda$ in λ is equivalent to equi-attraction,

$$\lim_{t\to\infty}\sup_{\lambda\in\Lambda}\rho_X(S_\lambda(t)D,\mathscr{A}_\lambda)\to 0\qquad\text{as}\quad t\to\infty. \tag{4.2}$$

We now give our version of Dini's Theorem.

Theorem 4.1. *For each $n\in\mathbb{N}$ let $f_n:K\to Y$ be a continuous map, where K is a compact metric space and Y is any metric space. If f is continuous and is the monotonic pointwise limit of f_n, i.e. for every $x\in K$*

$$d_Y(f_n(x),f(x))\to 0\quad\textit{as}\quad n\to\infty\quad\textit{and}\quad d_Y(f_{n+1}(x),f(x))\le d_Y(f_n(x),f(x))$$

then f_n converges uniformly to f.

Proof. Given $\epsilon > 0$ define

$$E_n = \{x \in K : \ d_Y(f_n(x), f(x)) < \epsilon\}.$$

Since f_n and f are both continuous, E_n is open and non-decreasing. Since K is compact and $\cup_{n=1}^{\infty} E_n$ provides an open cover of K, there exists an $N(\epsilon)$ such that $K = \cup_{n=1}^{N} E_n$, and so $d_Y(f_n(x), f(x)) < \epsilon$ for all $x \in K$ for all $n \geq N(\epsilon)$. □

Our first result relates continuity to a slightly weakened form of equi-attraction through sequences. We remark that our proof allows us to dispense with the 'equi-dissipative' assumption (4.1) of [14].

Theorem 4.2. *Assume* (L1) *and* (L2–3) *or* (L2′–3′). *If there exist* $t_n \to \infty$ *such that*

$$\sup_{\lambda \in \Lambda} \rho_X(S_\lambda(t_n)D, \mathscr{A}_\lambda) \to 0 \qquad \text{as} \quad n \to \infty, \tag{4.3}$$

then $\mathscr{A}_\lambda$ *is continuous in* λ *for all* $\lambda \in \Lambda$. *Conversely, if* Λ *is compact then continuity of* $\mathscr{A}_\lambda$ *for all* $\lambda \in \Lambda$ *implies that there exist* $t_n \to \infty$ *such that (4.3) holds.*

Proof. Lemma 3.1 guarantees that $\lambda \mapsto \overline{S_\lambda(t_n)D}$ is continuous for each n, and we have already observed in (3.2) that

$$d_{\rm H}(\overline{S_\lambda(t_n)D}, \mathscr{A}_\lambda) \to 0 \qquad \text{as} \quad n \to \infty.$$

Therefore $\mathscr{A}_\lambda$ is the uniform limit of the continuous functions $\overline{S_\lambda(t_n)D}$ and so is continuous itself.

For the converse, let $D_1 = \{x \in X : \ \rho_X(x, D) < 1\}$. For each $\lambda_0 \in \Lambda$ it follows from (A2) and (L2) that there exists a time $t(\lambda_0) \in \mathbb{N}$ such that $S_{\lambda_0}(t)D_1 \subseteq D$ for all $t \geq t(\lambda_0)$. It follows from (L3) that there exists an $\epsilon(\lambda_0) > 0$ such that

$$S_\lambda(t(\lambda_0))D_1 \subset D_1 \tag{4.4}$$

for every λ with $d_\Lambda(\lambda, \lambda_0) < \varepsilon(\lambda_0)$.

Since Λ is compact

$$\Lambda = \bigcup_{\lambda \in \Lambda} B_{\epsilon(\lambda)}(\lambda) = \bigcup_{k=1}^{N} B_{\epsilon(\lambda_k)}(\lambda_k)$$

for some $N \in \mathbb{N}$ and $\lambda_k \in \Lambda$. If $T = \prod_{k=1}^{N} t(\lambda_k)$ then

$$S_\lambda(T)D_1 \subseteq D_1 \qquad \text{for every} \quad \lambda \in \Lambda, \tag{4.5}$$

since any $\lambda \in \Lambda$ is contained in $B_{\epsilon(\lambda_k)}(\lambda_k)$ for some k, and $T = mt(\lambda_k)$ for some $m \in \mathbb{N}$, from which (4.5) follows by applying $S_\lambda(t(\lambda_k))$ repeatedly ($m-1$ times) to both sides of (4.4).

It follows from (4.5) that for every $\lambda \in \Lambda$, $\overline{S_\lambda(nT)D_1}$ is a decreasing sequence of sets, and hence the convergence of $\overline{S_\lambda(nT)D_1}$ to $\mathscr{A}_\lambda$, ensured by (3.2), is in fact monotonic in the sense of our Theorem 4.1. Uniform convergence now follows, and finally the fact that $D \subseteq D_1$ yields

$$d_{\rm H}(S_\lambda(nT)D, \mathscr{A}_\lambda) \leq d_{\rm H}(S_\lambda(nT)D_1, \mathscr{A}_\lambda) \to 0$$

uniformly in λ as $n \to \infty$. □

With additional uniform continuity assumptions we can readily show that continuity implies equi-attraction in the sense of [14]. We give one version of this result.

Corollary 4.3. *Suppose that* (L1–3) *hold and that* Λ *is compact. Assume in addition that* $S_\lambda(t)x$ *is continuous in* x*, uniformly in* λ *and for* x *in bounded subsets of* X *and* $t \in [0,T]$ *for any* $T > 0$*. Then continuity of* $\mathscr{A}_\lambda$ *implies (4.2).*

Proof. Since $\mathscr{A}_\lambda \subset D$ and $\mathscr{A}_\lambda$ is invariant, given any $\epsilon > 0$, by assumption there exists a $\delta > 0$ such that

$$d_X(d, \mathscr{A}_\lambda) < \delta \qquad \Rightarrow \qquad d_X(S_\lambda(\tau)d, \mathscr{A}_\lambda) < \epsilon \tag{4.6}$$

any $d \in X$, for all $\lambda \in \Lambda$ and $\tau \in (0,T)$. Choose n_0 sufficiently large that

$$d_{\rm H}(S_\lambda(nT)D, \mathscr{A}_\lambda) < \delta \qquad \text{for all} \quad n \geq n_0;$$

now for any $t \in (nT, (n+1)T)$, $n \geq n_0$, we can write $t = nT + \tau$ for some $\tau \in (0,T)$, and it follows from (4.6) that

$$d_{\rm H}(S_\lambda(t)D, \mathscr{A}_\lambda) < \epsilon \qquad t \geq n_0 T,$$

with the convergence uniform in λ as required. □

5. Pointwise convergence and residual continuity

When the rate of attraction to $\mathscr{A}_\lambda$ is not uniform in λ we nevertheless have the convergence in (3.2) for each λ. In general, therefore, one can view the attractor as the 'pointwise' (λ-by-λ) limit of the sequence of continuous functions $\overline{S_\lambda(t)D}$. In the case of a sequence of continuous real functions, their pointwise limit is a 'Baire one function', and is continuous on a residual set. The same ideas in a more abstract setting yield continuity of $\mathscr{A}_\lambda$ on a residual subset of Λ.

We use the following abstract result, characterising the continuity of 'Baire one' functions, whose proof (which we include for completeness) is an easy variant of that given by Oxtoby [15]. A result in the same general setting as here can be found as Theorem 48.5 in Munkres [12]. Recall that a set is *nowhere dense* if its closure contains no open sets, and a set is *residual* if its complement is the countable union of nowhere dense sets. Any residual set is dense.

Theorem 5.1. *For each* $n \in \mathbb{N}$ *let* $f_n : \Lambda \to Y$ *be a continuous map, where* Λ *is a complete metric space and* Y *is any metric space. If* f *is the pointwise limit of* f_n*, i.e.* $f(\lambda) = \lim_{n\to\infty} f_n(\lambda)$ *for each* $\lambda \in \Lambda$ *(and the limit exists), then the points of continuity of* f *form a residual subset of* Λ.

Before the proof we make the following observation: if U and V are open subsets of Λ with $V \subset \overline{U}$, then $U \cap V \neq \emptyset$. Otherwise V^c, the complement of V in Λ, is a closed set containing U, and since $\overline{U}$ is the intersection of all closed sets that contain U, $\overline{U} \subset V^c$, a contradiction.

Proof. We show that for any $\delta > 0$ the set

$$F_\delta = \{\lambda_0 \in \Lambda : \lim_{\epsilon \to 0} \sup_{d_\Lambda(\lambda,\lambda_0) \leq \epsilon} d_Y(f(\lambda), f(\lambda_0)) \geq 3\delta\}$$

is nowhere dense. From this it follows that

$$\cup_{n\in\mathbb{N}} F_{1/n} = \{\text{discontinuity points of } f\}$$

is nowhere dense, and so the set of continuity points is residual.

To show that F_δ is nowhere dense, i.e. that its closure contains no open set, let

$$E_n(\delta) = \{\lambda \in \Lambda : \sup_{i,j \geq n} d_Y(f_i(\lambda), f_j(\lambda)) \leq \delta\}.$$

Note that E_n is closed, $E_{n+1} \supset E_n$, and $\Lambda = \cup_{n=0}^{\infty} E_n$. Choose any open set $U \subset \Lambda$, and consider $\overline{U} = \cup_{n=0}^{\infty} \overline{U} \cap E_n$. Since $\overline{U}$ is a complete metric space, it follows from the Baire Category Theorem that there exists an n such that $\overline{U} \cap E_n$ contains an open set V'. From the remark before the proof, $V := V' \cap U$ is an open subset of $\overline{U} \cap E_n$ that is in addition a subset of U.

Since $V \subset E_n$, it follows that $d_Y(f_i(\lambda), f_j(\lambda)) \leq \eta$ for all $\lambda \in V$ and $i, j \geq n$. Fixing $i = n$ and letting $j \to \infty$ it follows that

$$d_Y(f_n(\lambda), f(\lambda)) \leq \eta \qquad \text{for all} \quad \lambda \in V.$$

Now, since $f_n(\lambda)$ is continuous in λ, for any $\lambda_0 \in V$ there is a neighbourhood $N(\lambda_0) \subset V$ such that

$$d_Y(f_n(\lambda), f_n(\lambda_0)) \leq \eta \qquad \text{for all} \quad \lambda \in N(\lambda_0).$$

Thus by the triangle inequality

$$d_Y(f(\lambda_0), f(\lambda)) \leq 3\eta \qquad \text{for all} \quad \lambda \in N(\lambda_0).$$

It follows that no element of $N(\lambda_0)$ belongs to F_δ, which implies, since $N(\lambda_0) \subset V \subset U$ that U contains an open set that is not contained in F_δ. This shows that F_δ is nowhere dense, which concludes the proof. □

Theorem 5.2. *Under assumptions* (L1–3) *above – or* (L1), (L2′), *and* (L3′) *–* $\mathscr{A}_\lambda$ *is continuous in* λ *for all* λ_0 *in a residual subset of* Λ. *In particular the set of continuity points of* $\mathscr{A}_\lambda$ *is dense in* Λ.

Proof. We showed in Lemma 3.1 that for every $t > 0$ the map $\lambda \mapsto \overline{S_\lambda(n)D}$ is continuous from Λ into $BC(X)$, and observed in (3.2) the pointwise convergence

$$d_{\mathrm{H}}(\overline{S_\lambda(t)D}, \mathscr{A}_\lambda) \to 0 \qquad \text{as} \quad t \to \infty.$$

The result follows immediately from Theorem 5.1, setting $f_n(\lambda) = \overline{S_\lambda(n)D}$ and $f(\lambda) = A_\lambda$ for every $\lambda \in \Lambda$. □

Residual continuity results also hold for the pullback attractors [4] and uniform attractors [5] that occur in non-autonomous systems. We will discuss these results in the context of the two-dimensional Navier–Stokes equations in a future paper.

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Department of Mathematics and Statistics, Texas Tech University, Box 41042, Lubbock, TX 79409-1042. USA.
E-mail address: luan.hoang@ttu.edu

Department of Mathematics/084, University of Nevada, Reno, NV 89557. USA.
E-mail address: ejolson@unr.edu

Mathematics Institute, University of Warwick, Coventry CV4 7AL. UK.
E-mail address: j.c.robinson@warwick.ac.uk