Continuity of pullback and uniform attractors

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Abstract

We study the continuity of pullback and uniform attractors for non-autonomous dynamical systems with respect to perturbations of a parameter. Consider a family of dynamical systems parameterised by a complete metric space Λ such that for each $\lambda \in \Lambda$ there exists a unique pullback attractor $\mathscr{A}_\lambda(t)$. Using the theory of Baire category we show under natural conditions that there exists a residual set $\Lambda_* \subseteq \Lambda$ such that for every $t \in \mathbb{R}$ the function $\lambda \mapsto \mathscr{A}_\lambda(t)$ is continuous at each $\lambda \in \Lambda_*$ with respect to the Hausdorff metric. Similarly, given a family of uniform attractors $\mathbb{A}_\lambda$, there is a residual set at which the map $\lambda \mapsto \mathbb{A}_\lambda$ is continuous. We also introduce notions of equi-attraction suitable for pullback and uniform attractors and then show when Λ is compact that the continuity of pullback attractors and uniform attractors with respect to λ is equivalent to pullback equi-attraction and, respectively, uniform equi-attraction. These abstract results are then illustrated in the context of the Lorenz equations and the two-dimensional Navier–Stokes equations.

Keywords: Pullback attractor, uniform attractor.

1. Introduction

The theory of attractors plays an important role in understanding the long time behavior of dynamical systems, see Babin and Vishik [1], Billotti and LaSalle [3], Chueshov [6], Hale [10], Ladyzhenskaya [15], Robinson [19] and Temam [24]. For the autonomous theory, consider a family of dissipative dynamical systems parameterised by Λ such that for each $\lambda \in \Lambda$ the corresponding dynamical system possesses a unique compact global attractor $\mathscr{A}_\lambda \subseteq Y$, where Y is a complete metric space with metric d_Y. Under very mild assumptions, see for example [11] and the references therein, the map $\lambda \mapsto \mathscr{A}_\lambda$ is known to be upper semicontinuous. This means that

$$\rho_Y(\mathscr{A}_\lambda, \mathscr{A}_{\lambda_0}) \to 0 \qquad \text{as} \qquad \lambda \to \lambda_0$$

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where $\rho_Y(A,C)$ denotes the Hausdorff semi-distance

$$\rho_Y(A,C) = \sup_{a\in A} \inf_{c\in C} d_Y(a,c). \tag{1.1}$$

However, lower semicontinuity

$$\rho_Y(\mathscr{A}_{\lambda_0}, \mathscr{A}_\lambda) \to 0 \qquad \text{as} \qquad \lambda \to \lambda_0,$$

and hence full continuity with respect to the Hausdorff metric, is much harder to prove.

For autonomous systems, general results require strict conditions on the structure of the unperturbed global attractor, see Hale and Raugel [12] and Stuart and Humphries [22], which are rarely satisfied for complicated systems. However, in Babin and Pilyugin [2] and in our previous work [13], continuity can be shown to hold for λ_0 in a residual set $\Lambda_* \subseteq \Lambda$ using the theory of Baire category under natural conditions when Λ is a complete metric space. We recall this result for autonomous systems as Theorem 1.1 below.

Let Λ and X be complete metric spaces. Suppose $S_\lambda(\cdot)$ is a parametrised family of semigroups on X for $\lambda \in \Lambda$ and consider the following conditions:

(G1) $S_\lambda(\cdot)$ has a global attractor $\mathscr{A}_\lambda$ for every $\lambda \in \Lambda$;

(G2) there is a bounded subset D of X such that $\mathscr{A}_\lambda \subseteq D$ for every $\lambda \in \Lambda$; and

(G3) for $t > 0$, $S_\lambda(t)x$ is continuous in λ, uniformly for x in bounded subsets of X.

Note that condition (G2) can be strengthened and (G3) weakened by replacing *bounded* by *compact*. These modified conditions will be referred to as conditions (G2′) and (G3′).

Theorem 1.1. *Under assumptions* (G1–G3) *above—or under the assumptions* (G1), (G2′) *and* (G3′)*—$\mathscr{A}_\lambda$ is continuous in λ at all λ_0 in a residual subset of Λ. In particular the set of continuity points of $\mathscr{A}_\lambda$ is dense in Λ.*

The proof developed in [13] of the above theorem, which appears as Theorem 5.1 in that work, is more direct than previous proofs and can be modified to establish analogous results for the pullback attractors and uniform attractors of non-autonomous systems. This is the main purpose of the present paper. After briefly introducing some definitions and notations concerning attractors and Baire category theory in Section 2, we turn to Section 3 where Theorem 3.3, our main result concerning pullback attractors, is proved. Section 4 then contains Theorem 4.1, which provides similar results for uniform attractors. In addition, we investigate the continuity of pullback and uniform attractors on the entire parameter space Λ. It is proved by Li and Kloeden [17], see also our previous paper [13], that in case Λ is compact, the continuity of the global attractors on Λ is equivalent to equi-attraction of the semigroups. In Section 5, we extend this result and the notion of equi-attraction to non-autonomous and uniform attractors. Theorem 5.2 shows for pullback attractors that continuity is equivalent to pullback equi-attraction, while Theorem 5.3 shows for uniform attractors that continuity is equivalent to uniform equi-attraction.

We note that the continuity of pullback attractors is investigated by Carvalho et al. [4], who extend the autonomous results to non-autonomous systems, under strong conditions on the structure of the pullback attractors. Similarly, the notion of equi-attraction defined Section 5 is a difficult property to discern for any concrete family of dynamical system. In contrast, the continuity results from Sections 3 and 4 only require standard conditions that are met in many applications. We demonstrate this in Section 6 with the Lorenz system of ODEs and the two-dimensional Navier–Stokes equations.

Theorem 6.4 proves that the global attractors for two-dimensional incompressible Navier–Stokes equations with time-independent forces are continuous at a residual set of those forces. A similar result on the continuity of pullback and uniform attractors with respect to perturbations of the viscosity is established in Theorem 6.5 for time-dependent forces. The counterparts of these two theorems for the Lorenz system are Theorems 6.2 and 6.3. These results point out that, even though attractors are irregular and complicated objects, at least they possess a generic robustness with respect to perturbations of certain physical parameters. Thus, our results may be seen as a first step towards a general theory that covers the computational analysis of global attractors.

2. Preliminaries

We begin by setting some notation and defining the Hausdorff metric. Given a metric space (Y, d_Y), denote by $B_Y(y, r)$ the ball of radius r centred at y,

$$B_Y(y, r) = \{y \in Y : \ d_Y(x, y) < r\}.$$

Write Δ_Y for the symmetric Hausdorff distance

$$\Delta_Y(A, C) = \max(\rho_Y(A, C), \rho_Y(C, A)) \tag{2.1}$$

where ρ_Y is the semi-distance between two subsets A and C of Y defined in (1.1). Denote by $CB(Y)$ the collection of all non-empty closed, bounded subsets of a metric space Y, which is itself a metric space with metric given by the symmetric Hausdorff distance Δ_Y.

In the same way that a continuous semigroup may be used to describe an autonomous dynamical system, the concept of a non-autonomous process may be used to describe a non-autonomous dynamical system. In particular, we have

Definition 2.1. *Let (X, d_X) be a complete metric space. A* process $S(\cdot, \cdot)$ *on X is a two-parameter family of maps $S(t, s) \colon X \mapsto X$, $s \in \mathbb{R}$, $t \geq s$, such that*

(P1) $S(t, t) = \mathrm{id}$*;*

(P2) $S(t, \tau)S(\tau, s) = S(t, s)$ *for all $t \geq \tau \geq s$; and*

(P3) $S(t, s)x$ *is continuous in x, t, and s.*

Given a non-autonomous process, there are two common ways to characterise its asymptotic behaviour: Roughly speaking, the limit of $S(t, s)$ for a fixed t as $s \to -\infty$ leads to the definition of the pullback attractor, while the limit of $S(t + s, s)$ as $t \to \infty$ leads to the uniform attractor, given sufficient uniformity in s. While both characterisations describe the same object for autonomous dynamics, they may be different in the non-autonomous case. The pullback attractor is obtained by moving backward in time and given by

Definition 2.2. *A family of compact sets* $\mathscr{A}(\cdot) = \{\mathscr{A}(t) : t \in \mathbb{R}\}$ *in* X *is the* pullback attractor *for the process* $S(\cdot,\cdot)$ *if*

(A1) $\mathscr{A}(\cdot)$ *is invariant:* $S(t,s)\mathscr{A}(s) = \mathscr{A}(t)$ *for all* $t \geq s$*;*

(A2) $\mathscr{A}(\cdot)$ *is pullback attracting: for any bounded set* B *in* X *and* $t \in \mathbb{R}$

$$\rho_X\big(S(t,s)B, \mathscr{A}(t)\big) \to 0 \qquad \text{as} \qquad s \to -\infty;$$

and

(A3) $\mathscr{A}(\cdot)$ *is minimal, in the sense that if* $C(\cdot)$ *is any other family of compact sets that satisfies (A1) and (A2) then* $\mathscr{A}(t) \subseteq C(t)$ *for all* $t \in \mathbb{R}$*.*

The uniform attractor is obtained by taking limits forward in time and given by

Definition 2.3. *A set* $\mathbb{A} \subseteq X$ *is the* uniform attractor *if it is the minimal compact set such that*

$$\lim_{t\to\infty} \sup_{s\in\mathbb{R}} \rho_X\big(S(t+s,s)B, \mathbb{A}\big) = 0 \tag{2.2}$$

for any bounded $B \subseteq X$*.*

We finish the section on preliminaries by stating a few basic facts from the theory of Baire category including an abstract residual continuity result. Recall that a set is *nowhere dense* if its closure contains no non-empty open sets, and a set is *residual* if its complement is the countable union of nowhere dense sets. It is a well-known fact that any residual subset of a complete metric space is dense.

The following result, an abstract version of Theorem 7.3 in Oxtoby [18], was proved as Theorem 5.1 in [13], and forms a key part of our proofs.

Theorem 2.4. *Let* $f_n : \Lambda \mapsto Y$ *be a continuous map for each* $n \in \mathbb{N}$*, where* Λ *is a complete metric space and* Y *is any metric space. If* f *is the pointwise limit of* f_n*, that is, if*

$$f(\lambda) = \lim_{n\to\infty} f_n(\lambda) \qquad \text{for each} \qquad \lambda \in \Lambda,$$

and the limit exists, then the points of continuity of f *form a residual subset of* Λ*.*

3. Residual continuity of pullback attractors

In this section we consider the continuity of pullback attractors. Let Λ be a complete metric space and $S_\lambda(\cdot,\cdot)$ a parametrised family of processes on X with $\lambda \in \Lambda$. Suppose that

(L1) $S_\lambda(\cdot,\cdot)$ has a pullback attractor $\mathscr{A}_\lambda(\cdot)$ for every $\lambda \in \Lambda$;

(L2) there is a bounded subset D of X such that $\mathscr{A}_\lambda(t) \subseteq D$ for every $\lambda \in \Lambda$ and every $t \in \mathbb{R}$; and

(L3) for every $s \in \mathbb{R}$ and $t \geq s$, $S_\lambda(t,s)x$ is continuous in λ, uniformly for x in bounded subsets of X.

We denote by (L2$'$) and (L3$'$) the assumptions (L2) and (L3), respectively, with *bounded* replaced by *compact*.

The following result is proved for the autonomous case as Lemma 3.1 in [13]; we omit the proof for the non-autonomous case, which is identical.

Lemma 3.1. *Assume either* (L2) *and* (L3)*, or* (L2$'$) *and* (L3$'$)*. Then for any* $s \in \mathbb{R}$ *and* $t \geq s$*, the map* $\lambda \mapsto \overline{S_\lambda(t,s)D}$ *is continuous from* Λ *into* $CB(X)$*.*

We also need the following related continuity result for $S_\lambda(t,s)B$. Note that the result only treats sets $B \in CB(K)$ for some compact K, which is crucial to the proof.

Lemma 3.2. *Assume that* (L3$'$) *holds, and let* K *be any compact subset of* X*. Then for any* $t \geq s$*, the mapping* $(\lambda, B) \mapsto S_\lambda(t,s)B$ *is (jointly) continuous in* $(\lambda, B) \in \Lambda \times CB(K)$*.*

Proof. Since every $B \in CB(K)$ is compact and $S_\lambda(t,s)x$ is continuous in x, then the image $S_\lambda(t,s)B$ is compact too. Now suppose $s \in \mathbb{R}$, $t \geq s$, $\lambda_0 \in \Lambda$, $B_0 \in CB(K)$ and $\epsilon > 0$. Condition (L3$'$) ensures that there exists a $\delta_1 \in (0,1)$ such that

$$d_\Lambda(\lambda_0, \lambda) < \delta_1 \quad \text{implies} \quad d_X\big(S_\lambda(t,s)x, S_{\lambda_0}(t,s)x\big) < \epsilon/2 \quad \text{for every} \quad x \in K.$$

Since K is compact, the map $x \mapsto S_{\lambda_0}(t,s)x$ is uniformly continuous on K; in particular, there is a $\delta_2 \in (0,1)$ such that

$$d_X(x,y) < \delta_2 \quad \text{with} \quad x, y \in K \quad \text{implies} \quad d_X\big(S_{\lambda_0}(t,s)x, S_{\lambda_0}(t,s)y\big) < \epsilon/2.$$

Set $\delta = \min\{\delta_1, \delta_2\}$.

Let $\lambda \in \Lambda$ with $d_\Lambda(\lambda, \lambda_0) < \delta$ and $B \in CB(K)$ with $\Delta_X(B, B_0) < \delta$. For any $b \in B$, there is $b_0 \in B_0$ such that $d_X(b, b_0) < \delta$. Therefore

$$\begin{aligned} d_X\big(S_\lambda(t,s)b, S_{\lambda_0}(t,s)b_0\big) &\leq d_X\big(S_\lambda(t,s)b, S_{\lambda_0}(t,s)b\big) \\ &\qquad + d_X\big(S_{\lambda_0}(t,s)b, S_{\lambda_0}(t,s)b_0\big) \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Hence,

$$\rho_X\big(S_\lambda(t,s)B, S_{\lambda_0}(t,s)B_0\big) \leq \epsilon. \tag{3.1}$$

The hypothesis $\Delta_X(B, B_0) < \delta$ also implies that for any $b_0 \in B_0$, there is $b \in B$ such that $d_X(b, b_0) < \delta$. Consequently

$$\rho_X\big(S_\lambda(t,s)B_0, S_{\lambda_0}(t,s)B\big) \leq \epsilon. \tag{3.2}$$

Combining (3.1) and (3.2) now yields $\Delta_X\big(S_\lambda(t,s)B, S_{\lambda_0}(t,s)B_0\big) \leq \epsilon$, which proves the joint continuity as claimed. □

As (L3) is a stronger hypothesis than (L3$'$) we note that Lemma 3.2 also holds under (L3). We now use Lemma 3.2 to prove the residual continuity of pullback attractors.

Theorem 3.3. *Let* $S_\lambda(\cdot,\cdot)$ *be a family of processes on* (X, d) *each satisfying* (P1–P3) *and suppose that* (L1) *and either*

(i) (L2′) *and* (L3′), *or*

(ii) (L2), (L3) *and that for any* $\lambda_0 \in \Lambda$ *and* $t \in \mathbb{R}$, *there exists* $\delta > 0$ *such that*

$$\overline{\bigcup_{B_\Lambda(\lambda_0,\delta)} \mathscr{A}_\lambda(t)} \qquad \textit{is compact.} \tag{3.3}$$

Then, there exists a residual set Λ_* *in* Λ *such that for every* $t \in \mathbb{R}$ *the function* $\lambda \mapsto \mathscr{A}_\lambda(t)$ *is continuous at each* $\lambda \in \Lambda_*$.

Proof. From Lemma 3.1 it follows for each $n \in \mathbb{Z}$ and $s < n$ that the function $\lambda \mapsto \overline{S_\lambda(n,s)D}$ is continuous. Moreover, since by either (L2) or (L2′) we have $D \supseteq \mathscr{A}_\lambda(s)$, then from the invariance of the attractor (A1) it follows that

$$\overline{S_\lambda(n,s)D} \supseteq S_\lambda(n,s)D \supseteq S_\lambda(n,s)\mathscr{A}_\lambda(s) = \mathscr{A}_\lambda(n) \qquad \text{for every} \qquad s \leq n. \tag{3.4}$$

Therefore, the pullback attraction property (A2) yields that

$$\mathscr{A}_\lambda(n) = \lim_{s\to-\infty} \overline{S_\lambda(n,s)D} \tag{3.5}$$

where the convergence is with respect to the Hausdorff metric. It follows from Theorem 2.4 that there is a residual set Λ_n of Λ at which the map $\lambda \in \Lambda \mapsto \mathscr{A}_\lambda(n)$ is continuous. Since the countable intersection of residual sets is still residual, then $\Lambda_* = \bigcap_{n\in\mathbb{Z}} \Lambda_n$ is a residual set at which $\lambda \mapsto \mathscr{A}_\lambda(n)$ is continuous for every $n \in \mathbb{Z}$.

We now use the invariance of $\mathscr{A}_\lambda(\cdot)$ to obtain continuity for every $t \in \mathbb{R}$. For $t \notin \mathbb{Z}$ there is $n \in \mathbb{Z}$ such that $t \in (n, n+1)$. Moreover,

$$\mathscr{A}_\lambda(t) = S_\lambda(t,n)\mathscr{A}_\lambda(n). \tag{3.6}$$

In case (i) set $K = D$; in case (ii) define

$$K = \overline{\bigcup_{B_\Lambda(\lambda_0,\delta)} \mathscr{A}_\lambda(t)}$$

where $\delta > 0$ has been chosen by (3.3) such that K is compact. Since $\mathscr{A}_\lambda(n)$ is continuous at $\lambda \in \Lambda_*$ and $\mathscr{A}_\lambda(n) \subseteq K$ for all $\lambda \in B_\Lambda(\lambda_0,\delta)$, then Lemma 3.2 guarantees that $S_\lambda(t,n)B$ is continuous in $(\lambda, B) \in \Lambda \times CB(K)$. Viewing (3.6) as a composition of continuous functions now yields that $\mathscr{A}_\lambda(t)$ is continuous at $\lambda \in \Lambda_*$. □

4. Residual continuity of uniform attractors

This section develops the theory of residual continuity of uniform attractors with respect to a parameter. A key component of the proof is the expression for the uniform attractor as a union of the uniform ω-limit sets given by

$$\mathbb{A}_\lambda = \overline{\bigcup_{n\in\mathbb{N}} \Omega_\lambda(B_X(0,n))}, \tag{4.1}$$

where, as in Chapter VII of [5], we define

$$\Omega_\lambda(B) = \bigcap_{\tau\in\mathbb{R}} \overline{\bigcup_{s\in\mathbb{R}} \bigcup_{t\geq\tau} S_\lambda(t+s,s)B} \qquad \text{for any set} \qquad B \subseteq X.$$

Our main result on uniform attractors can now be stated as

Theorem 4.1. *Suppose that there exists a compact set $K \subseteq X$ such that*

(a) *for every bounded $B \subseteq X$ and each $\lambda \in \Lambda$ there exists a $t_{B,\lambda}$ such that*

$$S_\lambda(t+s,s)B \subseteq K \qquad \textit{for all} \qquad t \geq t_{B,\lambda} \quad \textit{and} \quad s \in \mathbb{R}; \tag{4.2}$$

and

(b) *for any $t > 0$ the mapping $S_\lambda(t+s,s)x$ is continuous in $\lambda \in \Lambda$ uniformly in $s \in \mathbb{R}$ and $x \in K$. More specifically, given $\lambda_0 \in \Lambda$, $t > 0$ and $\epsilon > 0$, there is $\delta(\lambda_0, t, \epsilon) > 0$ such that for any $\lambda \in \Lambda$ with $d_\Lambda(\lambda, \lambda_0) < \delta$ one has*

$$d_X\big(S_\lambda(t+s,s)x, S_{\lambda_0}(t+s,s)x\big) < \epsilon \qquad \textit{for all} \qquad s \in \mathbb{R} \quad \textit{and} \quad x \in K.$$

Then, the uniform attractor $\mathbb{A}_\lambda$ is continuous in λ at a residual subset of Λ.

In the preceding theorem, assumption (a) is sufficient for the existence of a uniform attractor given by the uniform ω-limit (4.1) with λ fixed, and assumption (b) provides some uniform continuity of the processes S_λ in a way that depends only on the elapsed time.

Proof. Let $\lambda \in \Lambda$. Applying (2.2) to $S = S_\lambda$ gives

$$\lim_{t\to\infty} \sup_{s\in\mathbb{R}} \rho_X\big(S_\lambda(t+s,s)B, \mathbb{A}_\lambda\big) = 0$$

for any bounded $B \subseteq X$. Since

$$\rho_X\bigg(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)B}, \mathbb{A}_\lambda\bigg) = \rho_X\bigg(\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)B, \mathbb{A}_\lambda\bigg) \leq \sup_{s\in\mathbb{R}} \rho_X\big(S_\lambda(t+s,s)B, \mathbb{A}_\lambda\big),$$

we obtain

$$\lim_{t\to\infty} \rho_X\bigg(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)B}, \mathbb{A}_\lambda\bigg) = 0. \tag{4.3}$$

Let $T_n = \tau_{B,\lambda}$ for $n \in \mathbb{N}$ be given by (4.2) where $B = B_X(0,n)$. Then

$$\begin{aligned}
\Omega_\lambda(B_X(0,n)) &\subseteq \bigcap_{\tau\geq T_n} \overline{\bigcup_{s\in\mathbb{R}} \bigcup_{t\geq\tau} S_\lambda(t+s,s)B_X(0,n)} \\
&= \bigcap_{\tau\geq T_n} \overline{\bigcup_{s\in\mathbb{R}} \bigcup_{t\geq\tau} S_\lambda(t+s,T_n+s)S_\lambda(T_n+s,s)B_X(0,n)} \\
&\subseteq \bigcap_{\tau\geq T_n} \overline{\bigcup_{s\in\mathbb{R}} \bigcup_{t\geq\tau} S_\lambda(t+s,T_n+s)K} = \bigcap_{\tau\geq T_n} \overline{\bigcup_{\eta\in\mathbb{R}} \bigcup_{t\geq\tau} S_\lambda(t-T_n+\eta,\eta)K} \\
&= \bigcap_{\tau\geq T_n} \overline{\bigcup_{\eta\in\mathbb{R}} \bigcup_{s\geq\tau-T_n} S_\lambda(s+\eta,\eta)K} = \bigcap_{t\geq 0} \overline{\bigcup_{\eta\in\mathbb{R}} \bigcup_{s\geq t} S_\lambda(s+\eta,\eta)K}.
\end{aligned}$$

It follows from (4.1) that

$$\mathbb{A}_\lambda \subseteq \bigcap_{t\geq 0} \overline{\bigcup_{\eta\in\mathbb{R}} \bigcup_{s\geq t} S_\lambda(s+\eta,\eta)K}. \tag{4.4}$$

Let $T = \tau_{K,\lambda}$ in (4.2). By (4.4) we have for $t \geq 0$ that

$$\begin{aligned}
\mathbb{A}_\lambda &\subseteq \overline{\bigcup_{\eta\in\mathbb{R}} \bigcup_{s\geq t+T} S_\lambda(s+\eta,\eta)K} \\
&\subseteq \overline{\bigcup_{\eta\in\mathbb{R}} \bigcup_{s\geq t+T} S_\lambda\big(t+(s-t+\eta), s-t+\eta\big) S_\lambda(s-t+\eta,\eta)K} \\
&\subseteq \overline{\bigcup_{\eta\in\mathbb{R}} \bigcup_{s\geq t+T} S_\lambda\big(t+(s-t+\eta), s-t+\eta\big)K}.
\end{aligned}$$

Thus,

$$\mathbb{A}_\lambda \subseteq \overline{\bigcup_{z\in\mathbb{R}} S_\lambda(t+z,z)K}. \tag{4.5}$$

Taking $B = K$ in (4.3) yields

$$\Delta_X\left(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K}, \mathbb{A}_\lambda\right) \to 0 \qquad \text{as} \qquad t\to\infty. \tag{4.6}$$

Define $K_\lambda(t) = \bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K$, fix $\lambda_0 \in \Lambda$ and let $t > 0$. Given $\epsilon > 0$ choose $\delta > 0$ as in (b). Then for any $x \in K$, we have

$$d_X\big(S_\lambda(t+s,s)x, S_{\lambda_0}(t+s,s)x\big) < \epsilon \qquad \text{for any} \qquad s\in\mathbb{R}.$$

Hence

$$\rho_X\big(K_\lambda(t), K_{\lambda_0}(t)\big) \leq \epsilon \qquad \text{and} \qquad \rho_X\big(K_{\lambda_0}(t), K_\lambda(t)\big) \leq \epsilon$$

imply

$$\Delta_X\big(\overline{K_\lambda(t)}, \overline{K_{\lambda_0}(t)}\big) \leq \epsilon.$$

Consequently

$$\lambda \mapsto \overline{K_\lambda(t)} = \overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K}$$

is continuous from Λ into $BC(X)$. The result now follows from (4.6) and Theorem 2.4. $\square$

5. Continuity everywhere

In this section, we extend the notion of equi-attraction and the results on continuity of global attractors with respect to a parameter Λ in [17] to non-autonomous systems. We first show that that the continuity of pullback attractors with respect to a parameter $\lambda \in \Lambda$ is equivalent to pullback equi-attraction when Λ is compact. Next, we prove similar results for uniform equi-attraction and uniform attractors. Our methods are based on those used in Section 4 of [13].

Assume (L1) and (L2) throughout this section where D is the set specified in (L2). Now consider the following conditions:

(U1) Given $t \in \mathbb{R}$ there exists $s_0 \leq t$ and a bounded set B such that

$$S(t,s)D \subseteq B \qquad \text{for every} \qquad s \leq s_0 \quad \text{and} \quad \lambda \in \Lambda.$$

(U2) Pullback equi-attraction:

$$\lim_{s \to -\infty} \sup_{\lambda \in \Lambda} \rho_X\big(S_\lambda(t,s)D, \mathscr{A}_\lambda(t)\big) = 0 \qquad \text{for every} \qquad t \in \mathbb{R}.$$

(U3) There is a bounded set D_1 and a function $s_*(t)$ such that $s_*(t) \leq t$ and

$$S_\lambda(t,s)D_1 \subseteq D_1 \qquad \text{for every} \qquad s \leq s_*(t) \quad \text{and} \quad \lambda \in \Lambda. \tag{5.1}$$

We remark that (U3) is the uniform version of (U1) commonly obtained while proving the existence of pullback attractors. In the autonomous case (U3) is identical to condition (4.5) in [13] and was shown by Theorem 4.2 in that work to hold under even weaker conditions. The following version of Dini's theorem also appears in that work.

Lemma 5.1 (Theorem 4.1 in [13]). *Let K be a compact metric space and Y be a metric space. For each $n \in \mathbb{N}$, let $f_n \colon K \mapsto Y$ be a continuous map. Assume f_n converges to a continuous function $f \colon K \mapsto Y$ as $n \to \infty$ in the following monotonic way*

$$d_Y\big(f_{n+1}(x), f(x)\big) \leq d_Y\big(f_n(x), f(x)\big) \qquad \text{for all} \qquad n \in \mathbb{N} \quad \text{and} \quad \forall x \in K.$$

Then f_n converges to f uniformly on K as $s \to \infty$.

First, we deal with the pullback attractors.

Theorem 5.2. *Given $t \in \mathbb{R}$, assume* (U1) *and* (U2)*, then $\mathscr{A}_\lambda(t)$ is continuous at every $\lambda \in \Lambda$. Conversely, assume Λ is a compact metric space and* (U3) *holds. If $\lambda \mapsto \mathscr{A}_\lambda(t)$ is continuous on Λ for $t \in \mathbb{R}$, then* (U2) *holds true with D being replaced with D_1.*

Proof. Let $t \in \mathbb{R}$ such that (U1) and (U2) hold true. Following the same arguments used to obtain (3.4) and (3.5) in the proof of Theorem 3.3, except with t replacing n, we have for each $\lambda \in \Lambda$ and $s \leq t$ that

$$\mathscr{A}_\lambda(t) \subseteq S_\lambda(t,s)D \tag{5.2}$$

and

$$\mathscr{A}_\lambda(t) = \lim_{s \to -\infty} \overline{S_\lambda(t,s)D}. \tag{5.3}$$

By (U2) the convergence in (5.3) is uniform in $\lambda \in \Lambda$ as $s \to -\infty$. Moreover, Lemma 3.1 implies for $s \leq t$ that the function $\lambda \mapsto \overline{S_\lambda(t,s)D}$ is continuous in λ on Λ. Thus, the limit function $\lambda \mapsto \mathscr{A}_\lambda(t)$ is continuous on Λ.

We now prove the converse. Assume (U3) and that $\lambda \mapsto \mathscr{A}_\lambda(t)$ is continuous on Λ for $t \in \mathbb{R}$. Let $s_0 = s_*(t) - 1$, $s_1 = s_*(s_0) - 1$ and $s_{n+1} = s_*(s_n) - 1$ for $n \geq 1$. Then the sequence $\{s_n\}_{n=0}^\infty$ is strictly decreasing and $s_n \to -\infty$ as $n \to \infty$. By (P2) and (U3) we have

$$S(t, s_{n+1})D_1 = S(t, s_n)S(s_n, s_{n+1})D_1 \subseteq S(t, s_n)D_1. \tag{5.4}$$

Replacing D by D_1 in (5.2) and (5.3) yields

$$\mathscr{A}_\lambda(t) \subseteq S_\lambda(t,s)D_1 \qquad \text{for all} \qquad s \leq t \tag{5.5}$$

and

$$\mathscr{A}_\lambda(t) = \lim_{s\to-\infty} \overline{S_\lambda(t,s)D_1}. \tag{5.6}$$

We infer from (5.4) and (5.5) that

$$\Delta_X\big(\overline{S(t,s_{n+1})D_1}, \mathscr{A}_\lambda(t)\big) \leq \Delta_X\big(\overline{S(t,s_n)D_1}, \mathscr{A}_\lambda(t)\big). \tag{5.7}$$

Therefore, the convergence given in (5.6) is monotonic along the sequence $s = s_n$, and consequently, Lemma 5.1 implies $\overline{S_\lambda(t,s_n)D_1}$ converges to $\mathscr{A}_\lambda(t)$ as $n \to \infty$ uniformly in $\lambda \in \Lambda$. Thus,

$$\lim_{n\to\infty} \sup_{\lambda\in\Lambda} \rho_X\big(\overline{S_\lambda(t,s_n)D_1}, \mathscr{A}_\lambda(t)\big) = 0. \tag{5.8}$$

To obtain (5.8) in the continuous limit as $s \to -\infty$ suppose $s \in (s_{n+2}, s_{n+1})$. Then by definition $s < s_*(s_n)$ so that by (U3) we obtain

$$\mathscr{A}_\lambda(t) \subseteq S(t,s)D_1 = S(t,s_n)S(s_n,s)D_1 \subseteq S(t,s_n)D_1.$$

Hence,

$$\rho_X\big(\overline{S_\lambda(t,s)D_1}, \mathscr{A}_\lambda(t)\big) \leq \rho_X\big(\overline{S_\lambda(t,s_n)D_1}, \mathscr{A}_\lambda(t)\big) \quad \text{for every} \quad \lambda \in \Lambda.$$

This and (5.8) prove

$$\lim_{s\to-\infty} \sup_{\lambda\in\Lambda} \rho_X\big(\overline{S_\lambda(t,s)D_1}, \mathscr{A}_\lambda(t)\big) = 0,$$

which is exactly (U2) with D replaced by D_1. □

For uniform attractors we work under the standing assumption that there exists a set K such that (a) of Theorem 4.1 holds. We say that $\mathbb{A}_\lambda$ is uniformly equi-attracting if

$$\lim_{t\to\infty} \sup_{\lambda\in\Lambda, s\in\mathbb{R}} \rho_X\big(S_\lambda(t+s,s)K, \mathbb{A}_\lambda\big) = 0. \tag{5.9}$$

In our analysis we further consider the case where any trajectory starting in K uniformly re-enters K within a certain time T_0. This is characterised by condition

(U4) Assume there exists $T_0 \geq 0$ such that

$$S_\lambda(t+s,s)K \subseteq K \qquad \text{for all} \qquad t \geq T_0, \quad \lambda \in \Lambda \quad \text{and} \quad s \in \mathbb{R}. \tag{5.10}$$

We are now ready to prove our main result on uniform equi-attraction.

Theorem 5.3. *If $\mathbb{A}_\lambda$ is uniformly equi-attracting, then $\mathbb{A}_\lambda$ is continuous on Λ. Conversely, if $\mathbb{A}_\lambda$ is continuous, Λ is compact and* (U4) *is satisfied, then $\mathbb{A}_\lambda$ is uniformly equi-attracting.*

Proof. Under our standing assumption about the existence of K, we have that (4.3), (4.5) and (4.6) hold. To prove that $\mathbb{A}_\lambda$ is continuous, recall from Theorem 4.1 that

$$\lambda \mapsto \overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K}$$

is continuous. By (4.5) and (5.9) the limit (4.6) is uniform in λ. In other words,

$$\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K} \to \mathbb{A}_\lambda \quad \text{as} \quad t\to\infty \quad \text{uniformly for} \quad \lambda\in\Lambda.$$

Therefore, $\mathbb{A}_\lambda$ is continuous in Λ.

Conversely, let T_0 be in (U4) and set $T_* = T_0 + 1 \geq 1$. Let $t_n = nT_*$ for all $n\in\mathbb{N}$. Then $t_n\to\infty$ as $n\to\infty$. For $s\in\mathbb{R}$,

$$\begin{aligned} S_\lambda(t_{n+1}+s,s)K = S_\lambda(t_n+T_*+s,s)K &= S_\lambda(t_n+T_*+s,T_*+s)S_\lambda(T_*+s,s)K \\ &\subseteq S_\lambda(t_n+T_*+s,T_*+s)K \subseteq \bigcup_{s\in\mathbb{R}} S_\lambda(t_n+s,s)K. \end{aligned}$$

Thus,

$$\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t_{n+1}+s,s)K} \subseteq \overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t_n+s,s)K}. \tag{5.11}$$

The inclusion (4.5) then yields

$$\Delta_X(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t_{n+1}+s,s)K}, \mathbb{A}_\lambda) \leq \Delta_X(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t_n+s,s)K}, \mathbb{A}_\lambda).$$

which is the monotonicity needed for Lemma 5.1. It follows that the convergence in (4.6) taken along the sequence $t=t_n$ is uniform in λ. In other words, that

$$\lim_{n\to\infty}\sup_{\lambda\in\Lambda} \Delta_X\left(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t_n+s,s)K}, \mathbb{A}_\lambda\right) = 0. \tag{5.12}$$

To obtain uniformity in the continuous limit as $t\to\infty$ suppose $t\in(t_{n+1},t_{n+2})$. Then $t-t_n > T_0$ and

$$\begin{aligned} S_\lambda(t+s,s)K &= S_\lambda(t+s,t-t_n+s)S_\lambda(t-t_n+s,s)K \\ &\subseteq S_\lambda(t+s,t-t_n+s)K = S_\lambda(t_n+z,z)K \end{aligned}$$

with $z = t-t_n+s$. Thus,

$$S_\lambda(t+s,s)K \subseteq \bigcup_{z\in\mathbb{R}} S_\lambda(t_n+z,z)K.$$

Together with (4.5) we obtain that

$$\mathbb{A}_\lambda \subseteq \overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K} \subseteq \overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t_n+s,s)K}$$

and therefore

$$\Delta_X\left(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K}, \mathbb{A}_\lambda\right) \le \Delta_X\left(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t_n+s,s)K}, \mathbb{A}_\lambda\right). \tag{5.13}$$

Combining (5.13) with (5.12) yields

$$\lim_{t\to\infty}\sup_{\lambda\in\Lambda}\Delta_X\left(\overline{\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K}, \mathbb{A}_\lambda\right) = 0$$

and consequently

$$\lim_{t\to\infty}\sup_{\lambda\in\Lambda}\rho_X\left(\bigcup_{s\in\mathbb{R}} S_\lambda(t+s,s)K, \mathbb{A}_\lambda\right) = 0. \tag{5.14}$$

This implies that $\mathbb{A}_\lambda$ is uniformly equi-attracting. □

6. Applications

In this section we demonstrate the applicability of the abstract theory developed in the Sections 3 and 4 to some well-known systems of ordinary and partial differential equations. We also present some natural examples which are relevant for the autonomous theory developed in [13], see also [2]. We begin with the following simple observation about residual sets. The proof is elementary but included for the sake of clarity and completeness.

Lemma 6.1. *Let (X, d) be a metric space, and suppose that*

$$X = \bigcup_{j=1}^{\infty} X_j. \tag{6.1}$$

If $Y_j \subseteq X_j$ is residual in (X_j, d) for all $j \ge 1$, then the set

$$Y = \bigcup_{j=1}^{\infty} Y_j \tag{6.2}$$

is residual in X.

Proof. First, observe that if $X' \subseteq X$ and $Z \subseteq X'$ is nowhere dense in X' then Z is nowhere dense in X. Second, observe that if A is residual in X and $A \subseteq B$, then B is residual in X. By hypothesis $X_j \setminus Y_j = \bigcup_{i=1}^{\infty} A_{ji}$ where each A_{ji} is nowhere dense in X_j. It follows that

$$X \setminus Y = \bigcup_{j=1}^{\infty}(X_j \setminus Y) = \bigcup_{j=1}^{\infty}\bigcap_{k=1}^{\infty}(X_j \setminus Y_k) \subseteq \bigcup_{j=1}^{\infty}(X_j \setminus Y_j) = \bigcup_{j=1}^{\infty}\bigcup_{i=1}^{\infty} A_{ji}.$$

By the first observation each A_{ji} is nowhere dense in X. Since $X \setminus \bigcup_{j=1}^{\infty}\bigcup_{i=1}^{\infty} A_{ji} \subseteq Y$, the second observation implies that Y is residual in X. □

6.1. The Lorenz system

The first application of our theory concerns the system of three ordinary differential equations introduced by Lorenz in [16]. Namely, we consider

$$\begin{cases} x' = -\sigma x + \sigma y, \\ y' = rx - y - xz, \\ z' = -bz + xy, \end{cases} \tag{6.3}$$

where σ, b and r are positive constants. These equations have been widely studied as a model of deterministic nonperiodic flow. The standard bifurcation parameter of the Lorenz equations is r, see [21], but we will consider continuity of the global attractor of the autonomous system with respect to the full parameter set $\lambda = (\sigma, b, r)$. Since physical measurements and numerical computations in general employ only approximate values, then considering perturbations in all three parameters makes sense from a mathematical point of view. As an example, we point out that Tucker [25] considered an open neighborhood of the standard choice of parameters $\lambda = (10, 8/3, 28)$ in his work on the Lorenz equations.

As shown in Doering and Gibbon [9], see also Temam [24], for any $\lambda \in (0,\infty)^3$ the solutions to (6.3) generate a semigroup $S_\lambda(t)$ for which there exists a corresponding global attractor $\mathscr{A}_\lambda$. Therefore, the requirement (G1) of Theorem 1.1 is met. The estimates in that work also show for any compact subset Π of $(0,\infty)^3$ that there is a a bounded set D such that given any bounded set $B \in \mathbb{R}^3$ there is $T > 0$ such that

$$S_\lambda(t)B \subseteq D \qquad \text{for all} \qquad t \geq T \quad \text{and} \quad \lambda \in \Pi.$$

This guarantees that (G2) holds. Assumption (G3), the continuity of $S_\lambda(t)$ with respect to λ, can be verified by considering the equation for the difference of two solutions with different values of the parameters and using a standard Gronwall-type argument. Thus, the conditions of Theorem 1.1 are satisfied, and we may prove

Theorem 6.2. *There is a residual and dense subset Λ_* in $(0,\infty)^3$ such that the function $\lambda \mapsto \mathscr{A}_\lambda$ is continuous at every $\lambda \in \Lambda_*$.*

Proof. For each $n \in \mathbb{N}$ define $\Phi_n \colon [n^{-1}, n]^3 \mapsto CB(\mathbb{R}^3)$ by $\Phi_n(\lambda) = \mathscr{A}_\lambda$. Theorem 1.1 implies there is a residual set Λ_n in $[n^{-1}, n]^3$ such that Φ_n is continuous at each point in Λ_n with respect to the Hausdorff metric. Consequently, $\lambda \mapsto \mathscr{A}_\lambda$ viewed as a function with domain $(0,\infty)^3$ is continuous at each point in $\Lambda_{*,n} = \Lambda_n \cap (n^{-1}, n)^3$. Set

$$\Lambda_* = \bigcup_{n=2}^{\infty} \Lambda_{*,n}.$$

Then $\lambda \mapsto \mathscr{A}_\lambda$ is continuous on Λ_*. Since $\Lambda_{*,n}$ is residual and dense in $(n^{-1}, n)^3$, then Lemma 6.1 implies that Λ_* is a residual subset of $(0,\infty)^3$. Moreover, since each $\Lambda_{*,n}$ is dense in $(n^{-1}, n)^3$, then Λ_* is dense in $(0,\infty)^3$. $\square$

As a simple illustration of our non-autonomous theory, let $r(t)$ be a fixed C^1-function on $\mathbb{R}$ such that

$$|r(t)|, |r'(t)| \leq R_0 = const. \qquad \text{for all} \qquad t \in \mathbb{R}, \tag{6.4}$$

and consider the family of systems of ordinary differential equations given by

$$\begin{cases} x' = -\sigma x + \sigma y, \\ y' = r(t)x - y - xz, \\ z' = -bz + xy, \end{cases} \tag{6.5}$$

indexed by the parameter $\lambda = (\sigma, b)$.

Note that the model (6.5) and assumption (6.4) are relevant in some climate models, see e.g. [8]. In particular, the function $r(t)$ can be a finite sum of sinusoidal functions.

Making the standard change of variable $w = z - \sigma - r(t)$ we rewrite (6.5) as

$$\begin{cases} x' = -\sigma x + \sigma y, \\ y' = -y - \sigma x - xw, \\ w' = -bw + xy + F(t) \end{cases} \tag{6.6}$$

with

$$F(t) := -b(\sigma + r(t)) - r'(t)$$

satisfying, thanks to condition (6.4),

$$|F(t)| \leq F_0 := b(\sigma + R_0) + R_0 \quad \forall t \in \mathbb{R}. \tag{6.7}$$

Then similar estimates to those in [9] and [24], which are based on the formulation (6.6), show that for each $(\sigma, b) \in (0, \infty)^2$, the above system of equations (6.5) generates a process $S_{\sigma,b}(\cdot, \cdot)$, and there exist pullback attractors $\mathscr{A}_{\sigma,b}(t)$ for all $t \in \mathbb{R}$, and the uniform attractor $\mathbb{A}_{\sigma,b}$.

For the sake of completeness, we present explicit estimates here, which will also be needed in the next theorem.

Let $v(t) = (x(t), y(t), z(t))$ be a solution of (6.3), and $u(t) = (x(t), y(t), w(t))$. Note that

$$|v| \leq |u| + \sigma + R_0 \quad \text{and} \quad |u| \leq |v| + \sigma + R_0.$$

We have from (6.6), (6.7) and by Cauchy's inequality that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(x^2 + y^2 + w^2) + \sigma x^2 + y^2 + bw^2 = Fw \leq \frac{b}{2}w^2 + \frac{F_0^2}{2b}.$$

Set $\sigma_0 = \min\{1, \sigma, b/2\}$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + 2\sigma_0|u|^2 \leq \frac{F_0^2}{b}.$$

This implies for all $t \geq 0$ that

$$|u(t)|^2 \leq |u(0)|^2 e^{-2\sigma_0 t} + \frac{F_0^2}{2\sigma_0 b},$$

hence

$$|u(t)| \leq |u(0)|e^{-\sigma_0 t} + \frac{F_0}{\sqrt{2\sigma_0 b}} \leq (|v(0)| + \sigma + R_0)e^{-\sigma_0 t} + \frac{F_0}{\sqrt{2\sigma_0 b}}. \tag{6.8}$$

Then

$$|v(t)| \le (|v(0)| + \sigma + R_0)e^{-\sigma_0 t} + \frac{F_0}{\sqrt{2\sigma_0 b}} + (\sigma + R_0). \tag{6.9}$$

By (6.8) and (6.9), we have for all $t \ge 0$ that

$$|u(t)|, |v(t)| \le R_1 := |v(0)| + 2(\sigma + R_0) + \frac{F_0}{\sqrt{2\sigma_0 b}}. \tag{6.10}$$

Next ,we consider the continuity in $\lambda = (\sigma, b)$. For $i = 1, 2$, let $\lambda_i = (\sigma_i, b_i) \in (0, \infty)^2$ be given, $v_i(t) = (x_i(t), y_i(t), z_i(t))$ be the corresponding solution of (6.3), and

$$u_i(t) = (x_i(t), y_i(t), w_i(t)) = (x_i(t), y_i(t), z_i(t) - \sigma_i - r(t)).$$

Let $\bar\lambda = \lambda_1 - \lambda_2 = (\bar\sigma, \bar b)$ and $\bar u = u_1 - u_2 = (\bar x, \bar y, \bar w)$. Then (6.6) induces that

$$\begin{cases} \bar x' = -\bar\sigma x_1 - \sigma_2 \bar x + \bar\sigma y_1 + \sigma_2 \bar y, \\ \bar y' = -\bar y - \bar x w_1 - x_2 \bar w - \bar\sigma x_1 - \sigma_2 \bar x, \\ \bar w' = -\bar b w_1 - b_2 \bar w + \bar x y_1 + x_2 \bar y - \bar b(\sigma_1 + r(t)) - b_2 \bar\sigma. \end{cases}$$

It follows this system that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|\bar u|^2 + \sigma_2 \bar x^2 + \bar y^2 + b_2 \bar w^2 &= -\bar\sigma(x_1 + y_1)\bar x - \bar x w_1 \bar y - \bar\sigma x_1 \bar y \\ &\quad - \bar b w_1 \bar w + \bar x y_1 \bar w - [\bar b(\sigma_1 + r(t)) + b_2 \bar\sigma]\bar w. \end{aligned} \tag{6.11}$$

By neglecting $\sigma_2 \bar x^2 + \bar y^2 + b_2 \bar w^2$ on the left-hand side of (6.11) and using estimate (6.10) to bound $|x_1|$, $|y_1|$, $|w_1|$ on the right-hand side, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|\bar u|^2 &\le 2R_1|\bar\lambda||\bar u| + R_1|\bar u|^2 + |\bar\lambda|R_1|\bar u| + |\bar\lambda|R_1|\bar u| + R_1|\bar u|^2 + (|\lambda_1| + R_0 + |\lambda_2|)|\bar\lambda||\bar u| \\ &= (4R_1 + R_0 + |\lambda_1| + |\lambda_2|)|\bar\lambda||\bar u| + 2R_1|\bar u|^2 \\ &\le 2R_2|\bar u|^2 + R_3^2|\bar\lambda|^2, \end{aligned}$$

where R_1 is defined in (6.10) with $v = v_1$,

$$R_2 = R_1 + \frac{1}{8} \quad \text{and} \quad R_3 = R_0 + 4R_1 + |\lambda_1| + |\lambda_2|. \tag{6.12}$$

By Gronwall's inequality, we obtain for $t \ge 0$ that

$$|\bar u(t)|^2 \le |\bar u(0)|^2 e^{2R_2 t} + R_3^2 t e^{2R_2 t}|\bar\lambda|^2. \tag{6.13}$$

Let $\bar v = v_1 - v_2 = (\bar x, \bar y, \bar z)$, which equals $\bar u + (0, 0, \sigma_1 - \sigma_2)$. Then it follows (6.13) for $t \ge 0$ that

$$\begin{aligned} |\bar v(t)| &\le |\bar u(t)| + |\bar\lambda| \le |\bar u(0)|e^{R_2 t} + R_3\sqrt{t}e^{R_2 t}|\bar\lambda| + |\bar\lambda| \\ &\le (|\bar v(0)| + |\bar\lambda|)e^{R_2 t} + R_3\sqrt{t}e^{R_2 t}|\bar\lambda| + |\bar\lambda|. \end{aligned}$$

Thus,

$$|\bar v(t)| \le e^{R_2 t}\Big\{|\bar v(0)| + (2 + R_3\sqrt{t})|\bar\lambda|\Big\} \quad \forall t \ge 0. \tag{6.14}$$

We are ready to obtain the continuity of $\mathscr{A}_{\sigma,b}(t)$ and $\mathbb{A}_{\sigma,b}$ in (σ, b).

Theorem 6.3. *There is a residual and dense subset R_* in $(0,\infty)^2$ such that the functions $(\sigma,b)\in(0,\infty)^2\mapsto\mathscr{A}_{\sigma,b}(t)$, for any $t\in\mathbb{R}$, and $(\sigma,b)\in(0,\infty)^2\mapsto\mathbb{A}_{\sigma,b}$ are continuous at every $(\sigma,b)\in R_*$.*

Proof. Denote $\lambda=(\sigma,b)$. Let P be any compact subset of $(0,\infty)^2$. Set $\Lambda=P$.

Suppose $0<\delta\le\sigma,b\le\mu$ and $|v(0)|\le M$. Then

$$F_0\le F_*:=\mu(\mu+R_0)+R_0,\quad \sigma_0\ge\sigma_*:=\min\{1,\delta/2\},\quad b\ge 2\sigma_*.$$

These yield

$$R_1\le R_{1,*}:=M+2(\mu+R_0)+\frac{F_*}{2\sigma_*}.$$

Similarly, suppose $0<\delta\le\sigma_i,b_i\le\mu$ and $|v_i(0)|\le M$ for $i=1,2$. Then from (6.12),

$$R_2\le R_{2,*}:=R_{1,*}+\frac{1}{8}\quad\text{and}\quad R_3\le R_{3,*}:=R_0+4R_{1,*}+2\sqrt{2}\mu.$$

With these bounds, ones can use (6.9), resp., (6.14), to verify requirement (L2), resp., (L3) of section 3, as well as condition (a), and resp., (b) of Theorem 4.1 for $\lambda\in P$. Now, arguing as in the proof of Theorem 6.2 with the use of approximation sets $[n^{-1},n]^2$ for $(0,\infty)^2$ instead of $[n^{-1},n]^3$ for $(0,\infty)^3$, we obtain the the desired statement. $\square$

6.2. The two-dimensional Navier-Stokes equations

We now turn to the two-dimensional Navier–Stokes equations. Let Ω be a bounded, open and connected set in $\mathbb{R}^2$ with C^2 boundary, such that $\partial\Omega$ can be represented locally as the graph of a C^2 function. Consider the two-dimensional incompressible Navier–Stokes equations in Ω with no-slip Dirichlet boundary conditions

$$\begin{cases} u_t-\nu\Delta u+(u\cdot\nabla)u=-\nabla p+f & \text{on }\Omega\\ \nabla\cdot u=0 & \text{on }\Omega\\ u=0 & \text{on }\partial\Omega,\end{cases}\tag{6.15}$$

where $u=u(x,t)$ is the Eulerian velocity field, $p=p(x,t)$ is the pressure, $\nu>0$ is the kinematic viscosity and $f=f(x,t)$ is the body force.

Define

$$\mathscr{V}=\{\,v\in[C_c^\infty(\Omega)]^2:\nabla\cdot v=0\,\}$$

and let H and V be the closures of $\mathscr{V}$ in the norms of $[L^2(\Omega)]^2$ and $[H^1(\Omega)]^2$, respectively. Note that H is a Hilbert space with inner product $(\cdot,\cdot)$ and corresponding norm $\|\cdot\|$ inherited from $[L^2(\Omega)]^2$. Similarly V is a Hilbert space, however, in this case we shall use the norm $v\mapsto\|\nabla v\|$, which is equivalent to the $[H^1(\Omega)]^2$ norm on V due to the Poincaré inequality. The Rellich–Kondrachov Theorem implies V is compactly embedded into H. Denote the corresponding dual of V by V^* with the pairing $\langle u,v\rangle$ for $u\in V^*$ and $v\in V$.

Following, for example [23], we write (6.15) in functional form as the equation

$$u_t+\nu Au+B(u,u)=P_Lf\tag{6.16}$$

in V^* where $v(t) \in V$. Here P_L is the (Helmholtz–Leray) orthogonal projection from $[L^2(\Omega)]^2$ onto H and A and B are the continuous extensions of the operators given by

$$Au = P_L(-\Delta u) \qquad \text{and} \qquad B(u,v) = P_L((u \cdot \nabla)v) \qquad \text{for} \qquad u, v \in \mathscr{V}$$

such that $A\colon V \mapsto V^*$ and $B\colon V \times V \mapsto V^*$.

For convenience we assume $f(t) \in H$ for all time so that $P_L f = f$ in (6.16). Let $\lambda_1 > 0$ be the first eigenvalue of the Stokes operator. It follows that Poincaré's inequality may be written as $\|\nabla v\|^2 \geq \lambda_1 \|v\|^2$ for all $v \in V$.

Time-independent body force.

Let $\nu > 0$ be fixed. Assume $f(t) = f \in H$ for all $t \geq 0$, i.e., it is time-independent. The equation (6.15) generates a semigroup $S_f(t)v = u(t)$ in H, which is the solution of (6.16) with initial data $u(0) = v \in H$. This semigroup possesses a global attractor $\mathscr{A}_f \subseteq V \subseteq H$.

The relevant calculations that show the existence of absorbing sets in H and V can be found in [24] pages 109–111 (similar calculations can be found in [7, 14, 19, 20, 23], among others); we reproduce some of these here since the resulting bounds will be required to check (G1–3).

Let $u_0 \in H$ and $u(t) = S_f(t)u_0$. Taking the inner product of (6.16) with u and using the orthogonality property

$$\langle B(u,v), v\rangle = 0 \tag{6.17}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu\|\nabla u\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1}. \tag{6.18}$$

Integrating this in time from 0 to t gives a bound on the first derivative,

$$\nu \int_0^t \|\nabla u(\tau)\|^2 \,\mathrm{d}\tau \leq \|u_0\|^2 + \frac{\|f\|^2}{\nu\lambda_1} t. \tag{6.19}$$

By Poincaré's inequality we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu\lambda_1\|u\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1},$$

which, using Gronwall's inequality, yields

$$\|u(t)\|^2 \leq \|u_0\|^2 e^{-\nu\lambda_1 t} + \rho_0^2, \quad \text{where} \quad \rho_0 = \frac{\|f\|}{\nu\lambda_1}. \tag{6.20}$$

It follows that given a fixed $f \in H$, for each bounded subset B of H there exists a time $t_0(B)$ such that

$$\|u(t)\|^2 \leq (\rho_0 + 1)^2 \qquad \text{for all} \qquad t \geq t_0(B). \tag{6.21}$$

The argument ones use to show the existence of an absorbing set in V, which is required to prove the existence of a global attractor, can be found in the above references; we will not need any of the estimates in what follows (other than the fact that they are finite). We therefore have (G1), i.e. the existence of $\mathscr{A}_f$, and (G2) will follow from (6.20).

To prove that (G3) holds we consider the difference of two solutions u_1 and u_2 with forcing functions f_1 and f_2. Then $w = u_1 - u_2$ satisfies

$$\frac{\mathrm{d}w}{\mathrm{d}t} + \nu Aw + B(u_1, w) + B(w, u_2) = f, \qquad \text{where} \quad f = f_1 - f_2. \tag{6.22}$$

Recall the inequality

$$|\langle B(u,v), w\rangle| \leq c_B \|u\|^{1/2} \|\nabla u\|^{1/2} \cdot \|\nabla v\| \cdot \|w\|^{1/2} \|\nabla w\|^{1/2}, \tag{6.23}$$

where $c_B > 0$ (this follows from the 2D Ladyzhenskaya inequality $\|u\|_{L^4} \leq c\|u\|^{1/2}\|\nabla u\|^{1/2}$).

Taking the inner product of (6.22) with w and using (6.17) and (6.23) gives

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \nu\|\nabla w\|^2 &\leq |\langle B(w,u_2), w\rangle| + \|f\|\|w\| \leq c_B\|w\|\|\nabla w\|\|\nabla u_2\| + \lambda_1^{-1/2}\|f\|\|\nabla w\| \\ &\leq \frac{\nu}{4}\|\nabla w\|^2 + \frac{c_B^2\|w\|^2\|\nabla u_2\|^2}{\nu} + \frac{\nu}{4}\|\nabla w\|^2 + \frac{\|f\|^2}{\nu\lambda_1}.\end{aligned}$$

Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 \leq \frac{2c_B^2\|\nabla u_2\|^2}{\nu}\|w\|^2 + \frac{2\|f\|^2}{\nu\lambda_1}.$$

Applying Gronwall's inequality we obtain

$$\|w(t)\|^2 \leq \frac{2\|f\|^2}{\nu\lambda_1}\int_0^t \exp\left(\frac{2c_B^2}{\nu}\int_\tau^t \|\nabla u_2(s)\|^2\,\mathrm{d}s\right)\mathrm{d}\tau. \tag{6.24}$$

Theorem 1.1 now yields the following continuity of $\mathscr{A}_f$ with respect to f.

Theorem 6.4. *There is a residual (and dense) set H_* in H such that the map $f \mapsto \mathscr{A}_f$ from H into $CB(H)$ is continuous at every $f \in H_*$.*

Proof. Let $\rho > 0$; we apply Theorem 1.1 to $\Lambda = \bar{B}_H(0,\rho)$, $X = H$, $\lambda = f \in \Lambda$.

Conditions (G1) and (G2) are discussed above. To finish the proof of (G3), let $f_1, f_2 \in \bar{B}_H(0,\rho)$, and use the same notation as in (6.18) above. For $\|u_0\| \leq M$, we have from (6.24), using estimate (6.19) for u_2, that

$$\begin{aligned}\|S_{f_1}(t)u_0 - S_{f_2}(t)u_0\|^2 &\leq \frac{2\|f\|^2}{\nu\lambda_1}t\exp\left\{\frac{2c_B^2}{\nu^2}\left(\|u_0\|^2 + \frac{\|f_2\|^2}{\nu\lambda_1}t\right)\right\} \\ &\leq \frac{2t}{\nu\lambda_1}\exp\left\{\frac{2c_B^2}{\nu^2}\left(M^2 + \frac{\rho^2}{\nu\lambda_1}t\right)\right\}\|f_1 - f_2\|^2.\end{aligned}$$

Therefore (G3) is met.

By Theorem 1.1 there is a residual set H_ρ in $\bar{B}_H(0,\rho)$ such that the map $f \in \bar{B}_H(0,\rho) \mapsto \mathscr{A}_f$ is continuous on H_ρ. Set $H_* = \bigcup_{n=1}^\infty (H_n \setminus \partial B_H(0,n))$. Then the map $f \in H \mapsto \mathscr{A}_f$ is continuous on H_*. Since $H_n \setminus \partial B_H(0,n)$ is residual in $B_H(0,n)$ for all $n \in \mathbb{N}$, then, by Lemma 6.1, H_* is residual in H. $\square$

Time-dependent body force.

As an illustration of the results of this paper, we once more consider the Navier–Stokes equations (6.15), but now with a fixed time-dependent forcing $f = f(x,t) \in L^\infty(\mathbb{R}, H)$, taking ν as the parameter.

For each $\nu > 0$, the system (6.15) generates a process $S_\nu(t,s)$ defined by $S_\nu(t,s)v = u(t)$, for $v \in H$ and $t \geq s$, which is the solution of (6.15) on $[s,\infty)$ with $u(s) = v$. Moreover, there exists a pullback attractor $\{\mathscr{A}_\nu(t) : t \in \mathbb{R}\}$ and a uniform attractor $\mathbb{A}_\nu$ for $S_\nu(t,s)$, see [4]; the bounds on individual solutions are obtained in a similar way to those in the case that f is time independent, and are the same with $\|f\|$ replaced by $\|f\|_\infty := \|f\|_{L^\infty(\mathbb{R};H)}$.

We recall here the needed estimates for $\|\nabla u(t)\|$. (Again, calculations can be found, for example, in [24] pages 109–111.) Let $\nu \geq \varepsilon > 0$. Define

$$\rho_{0,\varepsilon} = \frac{\|f\|_\infty}{\varepsilon\lambda_1}, \quad \rho'_{0,\varepsilon} = \rho_{0,\varepsilon} + 1,$$

$$a_{3,\varepsilon} = \lambda_1 \rho_{0,\varepsilon}^2 + \frac{{\rho'}^2_{0,\varepsilon}}{\varepsilon}, \quad a_{2,\varepsilon} = \frac{2\|f\|_{L^\infty}^2}{\varepsilon}, \quad a_{1,\varepsilon} = \frac{2c'_1}{\varepsilon^3}{\rho'}^2_{0,\varepsilon} a_{3,\varepsilon}.$$

For $R > \rho'_{0,\varepsilon}$, denote

$$t_{1,\varepsilon}(R) = 1 + \frac{1}{\varepsilon\lambda_1}\log\frac{R^2}{2\rho_{0,\varepsilon}+1}.$$

Then for $\|v\| \leq R$ and all $t \geq t_{1,\varepsilon}(R)$,

$$\|\nabla u(t+s)\|^2 \leq \rho_{1,\varepsilon}^2 := (a_{3,\varepsilon} + a_{2,\varepsilon})e^{a_{1,\varepsilon}}. \tag{6.25}$$

Next, we consider the continuity in ν. Given $\nu_1, \nu_2 > 0$, let $u_1(t), u_2(t)$ be the corresponding solutions on $[s,\infty)$ with same initial data $u_1(s) = u_2(s) = u_0$. Let $w = u_1 - u_2$ and $\nu = \nu_1 - \nu_2$; then we have

$$\frac{\mathrm{d}w}{\mathrm{d}t} + \nu_1 Aw + \nu Au_2 + B(u_1, w) + B(w, u_2) = 0. \tag{6.26}$$

Then taking the inner product of (6.26) with w and using (6.17) and (6.23) we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \nu_1\|\nabla w\|^2 &= -\langle Au_2, w\rangle - \langle B(w,u_2), w\rangle \\ &\leq \nu\|\nabla w\|\|\nabla u_2\| + c_B\|w\|\|\nabla w\| \cdot \|\nabla u_2\| \\ &\leq \frac{\nu_1\|\nabla w\|^2}{4} + \frac{\nu^2\|\nabla u_2\|^2}{\nu_1} + \frac{\nu_1\|\nabla w\|^2}{4} + \frac{c_B^2\|w\|^2\|\nabla u_2\|^2}{\nu_1},\end{aligned}$$

using Cauchy's inequality in the final line. This implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \frac{\nu_1}{2}\|\nabla w\|^2 \leq \frac{\nu^2\|\nabla u_2\|^2}{\nu_1} + \frac{c_B^2\|w\|^2\|\nabla u_2\|^2}{\nu_1},$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 \leq \frac{2c_B^2\|\nabla u_2\|^2}{\nu_1}\|w\|^2 + \frac{2\nu^2\|\nabla u_2\|^2}{\nu_1}. \tag{6.27}$$

Theorem 6.5. *There is a residual and dense set Λ_* in $(0,\infty)$ such that all the maps $\nu \mapsto \mathscr{A}_\nu(t)$, with $t \in \mathbb{R}$, and $\nu \mapsto \mathbb{A}_\nu$, considered as maps from $(0,\infty)$ into $CB(H)$, are continuous at every $\nu \in \Lambda_*$.*

Proof. The proof is based on Theorems 3.3 and 4.1, and similar arguments as in Theorem 6.2.

Step 1. Given any $\varepsilon > 0$, we apply Theorem 3.3 to $\Lambda = [\varepsilon, \infty)$ with (L2′) and (L3′). We need to check these two conditions.

Condition (L2′). Let $\nu \geq \varepsilon$. We use Theorems 11.3 and 2.12 of [4]. In this case

$$\mathscr{A}_\nu(t) = \overline{\bigcup \{\omega(B,t) :\ B \text{ is a bounded set in } H\}},$$

where, according to Definition 2.2 in [4],

$$\omega(B,t) = \bigcap_{\sigma \leq t} \overline{\bigcup_{s \leq \sigma} S_\lambda(t,s)B}.$$

Define $K = \bar{B}_V(0, \rho_{1,\varepsilon})$ which is a compact set in H.

Fix $t \in \mathbb{R}$ and and a bounded set B in H. Let $R > \rho'_{0,\varepsilon}$ such that $B \subseteq B_H(0,R)$.

Take $\sigma_0 = t - t_{1,\varepsilon}(R)$. For $s \leq \sigma_0$, we have $t - s \geq t_{1,\varepsilon}(R)$, and by (6.25),

$$S_\nu(t,s)B \subseteq K.$$

Hence

$$\omega(B,t) \subseteq \overline{\bigcup_{s \leq \sigma_0} S_\nu(t,s)B} \subseteq K,$$

and consequently, $\mathscr{A}_\nu(t) \subseteq K$. Therefore, (L2′) is satisfied.

Condition (L3′). Let $\nu_1, \nu_2 \geq \varepsilon$. Let $s \in \mathbb{R}$, $v \in H$ with $\|v\| \leq R$. Let $u_i(t) = S_{\nu_i}(t,s)v$.

Let $w = u_1 - u_2$. Then for $t \geq 0$, we have from (6.27) and Gronwall's inequality that

$$\begin{aligned}
\|w(t+s)\|^2 &\leq \frac{2(\nu_1 - \nu_2)^2}{\nu_1} \int_s^{t+s} \exp\Big(\frac{2c_B^2}{\nu_1} \int_\tau^{t+s} \|\nabla u_2(z)\|^2 \,\mathrm{d}z\Big) \|\nabla u_2(\tau)\|^2 \,\mathrm{d}\tau \\
&\leq \frac{2(\nu_1 - \nu_2)^2}{\nu_1} \exp\Big(\frac{2c_B^2}{\nu_1} \int_s^{t+s} \|\nabla u_2(\tau)\|^2 \,\mathrm{d}\tau\Big) \int_s^{t+s} \|\nabla u_2(\tau)\|^2 \,\mathrm{d}\tau.
\end{aligned}$$

Similar to (6.19),

$$\int_s^{t+s} \|\nabla u_2(\tau)\|^2 \,\mathrm{d}\tau \leq \frac{\|v\|^2}{\nu_2} + \frac{\|f\|_\infty^2}{\nu_2^2 \lambda_1} t \leq E_{\varepsilon,R}(t) := \frac{R^2}{\varepsilon} + \frac{\|f\|_\infty^2}{\varepsilon^2 \lambda_1} t.$$

Therefore, for all $s \in \mathbb{R}$ and $t \geq 0$

$$\|S_{\nu_1}(t+s,s)v - S_{\nu_2}(t+s,s)v\|^2 = \|w(t+s)\|^2 \leq \frac{2E_{\varepsilon,R}(t)}{\varepsilon} \exp\Big(\frac{2c_B^2 E_{\varepsilon,R}(t)}{\varepsilon}\Big)(\nu_1 - \nu_2)^2.$$

This proves that for any $t \geq 0$, the mapping $S_\lambda(t+s,s)v$ is continuous in $\nu \in [\varepsilon, \infty)$, uniformly in $s \in \mathbb{R}$ and $v \in \bar{B}_H(0,R)$ for any $R > 0$. Condition (L3′) is an obvious consequence.

Then, by Theorem 3.3, there is a residual set Λ_ε^p in $[\varepsilon,\infty)$ such that the function $\nu \in [\varepsilon,\infty) \mapsto \mathscr{A}_\nu(t)$ is continuous on Λ_ε^p for all $t \in \mathbb{R}$.

Step 2. For the uniform attractors, given any $\varepsilon > 0$, we apply Theorems 4.1 to $\Lambda = [\varepsilon,\infty)$. We check requirements (a) and (b).

Condition (a). Let $K = \bar{B}_V(0,\rho_{1,\varepsilon})$, then K is compact in H. For any $R > \rho'_{0,\varepsilon}$, $\lambda \in \Lambda_\varepsilon$, $s \in \mathbb{R}$, $t \ge t_{1,\varepsilon}(R)$, we have from (6.25) that $S_\lambda(t+s,s)(\bar{B}_H(0,R)) \subseteq K$. Therefore condition (a) is satisfied.

Condition (b). This is obtained in the proof of (L3$'$) in step 1.

Then, by Theorem 4.1, there is a residual set Λ_ε^u in $[\varepsilon,\infty)$ such that the function $\nu \in [\varepsilon,\infty) \mapsto \mathbb{A}_\nu$ is continuous on Λ_ε^u.

Step 3. For $\varepsilon > 0$, let $\Lambda_\varepsilon = (\Lambda_\varepsilon^p \cap \Lambda_\varepsilon^u) \setminus \{\varepsilon\}$. Then Λ_ε is residual and dense in (ε,∞), and all mappings $\nu \in (0,\infty) \mapsto \mathscr{A}_\nu(t)$, with $t \in \mathbb{R}$, and $\nu \in (0,\infty) \mapsto \mathbb{A}_\nu$ are continuous on Λ_ε.

Set
$$\Lambda_* := \bigcup_{n=1}^\infty \Lambda_{1/n}.$$
Then the function $\nu \in (0,\infty) \mapsto \mathscr{A}_\nu(t)$, for any $t \in \mathbb{R}$, and $\nu \in (0,\infty) \mapsto \mathbb{A}_\nu$ are continuous on Λ_*. By Lemma 6.1, Λ_* is residual in $(0,\infty)$. Since each $\Lambda_{1/n}$ is dense in $(1/n,\infty)$, the set Λ_* is dense in $(0,\infty)$. The proof is complete. □

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