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CONTINUOUS DATA ASSIMILATION WITH BLURRED-IN-TIME MEASUREMENTS OF THE SURFACE QUASI-GEOSTROPHIC EQUATION

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Abstract. An intrinsic property of almost any physical measuring device is that it makes observations which are slightly blurred in time. We consider a nudging-based approach for data assimilation that constructs an approximate solution based on a feedback control mechanism that is designed to account for observations that have been blurred by a moving time average. Analysis of this nudging model in the context of the subcritical surface quasi-geostrophic equation shows, provided the time-averaging window is sufficiently small and the resolution of the observations sufficiently fine, that the approximating solution converges exponentially fast to the observed solution over time. In particular, we demonstrate that observational data with a small blur in time possess no significant obstructions to data assimilation provided that the nudging properly takes the time averaging into account. Two key ingredients in our analysis are additional boundedness properties for the relevant interpolant observation operators and a non-local Gronwall inequality.

Dedicated to Professor Andrew Majda on the occasion of his 70th birthday

1. Introduction

The surface quasi-geostrophic (SQG) equation models the dynamics of the potential temperature on the two-dimensional horizontal boundaries of the three-dimensional quasi-geostrophic equations, which, in turn, are approximations to the shallow water equations in the limit of small Rossby number where the inertial forces are an order of magnitude smaller than the Coriolis and pressure forces. This is the regime of strong rotation, where the time scales associated with atmospheric flow over long distances are much larger than the time scales associated with the Earth's rotation (cf. [43]). The model of focus in our study of data assimilation is the subcritically dissipative SQG equation subject to periodic boundary conditions over the fundamental domain $\mathbb{T}^2 = [-\pi, \pi]^2$. In *non-dimensionalized* variables, it is given by

$$\begin{cases} \partial_t \theta + \kappa \Lambda^\gamma \theta + u\cdot \nabla \theta = f, \\ u = \mathcal{R}^\perp \theta, \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where $\Lambda^\gamma = (-\Delta)^{\gamma/2}$ corresponds to the Fourier muliplier operator $|k|^\gamma$, $\mathcal{R}^\perp = (-R_2, R_1)$ is the perpendicular Riesz transform, where each R_j corresponds to $(-ik_j/|\mathbf{k}|)_{\mathbf{k}\in\mathbb{Z}^2\setminus\{0\}}$, and the strength of dissipation satisfies $1 < \gamma \leq 2$. Note that $\gamma = 1$ gives the critical case while $0 < \gamma < 1$ gives the supercritical case. The scalar function θ represents the surface temperature or buoyancy of a fluid advected along the vector velocity field u. The parameter

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κ is a fixed positive quantity, which appears due to the phenomenon of Ekman pumping at the surface. Note, also, that if θ_0 has zero mean over $\mathbb{T}^2$, then the property $\frac{1}{4\pi^2}\int_{\mathbb{T}^2}\theta(t)dx = 0$ is propagated for all $t > 0$, so long as f has zero mean over $\mathbb{T}^2$ as well.

Since their introduction into the mathematical community by Constantin, Majda, and Tabak [19], the subcritical, critical and supercritical SQG equations have been thoroughly studied. Well-posedness and global regularity in various function spaces has been resolved in all but the supercritical case, (cf. [10, 17, 18, 20, 22, 24, 39, 40, 44]), and also for certain inviscid regularizations (cf. [38]). The long-time behavior in the subcritical and critical has been studied as well and in particular, a global attractor theory has been established for them (cf. [11, 13, 15, 20, 21, 34]). These equations have been used to simulate the production of fronts in geophysical flows and in spite of being a scalar model in two dimensions, possess solutions that behave in ways that are strikingly similar to fully three-dimensional flows. Therefore, equations (1.1) provide a physically-relevant dynamical context in which to analyze the performance of our model for data assimilation, that also supplies additional analytical difficulties that requires us to further develop the theoretical foundations of our approach.

Given a geophysical equation that describes some aspect of reality, the ability to predict the future using this equation requires an initial condition that accurately represents the current physical state. Although weather data has been collected nearly continuously in time since the 1960s, this data represents, at best, an incomplete picture of the current state of the atmosphere. Thus, rather than an exact initial condition, in practice one has a time series of low-resolution observations. Moreover, due to the nature of the measuring devices, the data itself may contain noise as well as systematic errors. Of particular interest to our present study is the fact that nearly all physical instrumentation produces measurements which are *manifestly blurred in time*. For example, the heat capacity of a thermometer naturally averages temperatures as they change over time while the rotational inertial of an anemometer similarly averages velocities. Time averages in satellite images result from finite shutter speeds and further averages result when satellite data is obtained by comparing two subsequent images. Blocher [7] shows both analytically and computationally that noisy, blurred-in-time observations of the X variable can be used to synchronize two copies of the three-dimensional Lorenz system of ordinary differential equations (ODEs) up to a factor of the variance of the noise, see also [8]. As the analysis of the SQG equation is more complicated, we do not consider noise or systematic errors in this work, as this was studied in [6] and [29], but instead focus solely on how to assimilate data that has been subject to a moving time average.

The idea of finding the current physical state by combining a time-series of partial observations with knowledge about the dynamics dates back to a 1969 paper of Charney, Halem, and Jastrow [12]. Doing this optimally is the subject of data assimilation. Data assimilation has received considerable attention in both its theoretical development and practical use for the prediction of the weather (cf. Kalnay [35] and references therein). The approach of interest in this article computes an approximation using a "auxiliary system" obtained by taking the original model, which is assumed to coincide with the observations in the absence of measurement error, and applying feedback control based on the observations. This feedback control serves to nudge the solution towards the unknown but observed solution no matter what original initial condition was chosen for it. In theory, one could then integrate the approximate solution forward in time to obtain a good approximation of the current physical state. This approximation would then serve as an initial condition for subsequent forecasts.

The auxiliary system described above was first proposed as an approach to data assimilation for the model problem of the two-dimensional incompressible Navier–Stokes equations by Azouani, Olson, and Titi in [3]. In that work, exponential convergence of the approximating solution to the observed solution was shown under general conditions in which the observations were assumed to be taken continuously and instantaneously in time. By now this approach has been studied for several other physical systems such as the one-dimensional Chaffee-Infante equation, the two-dimensional Boussinesq, the three-dimensional Brinkman-Forchheimer extended Darcy equations, the three-dimensional Bénard convection in porous media, and the three-dimensional Navier–Stokes α-model (cf. [1, 2, 3, 25, 26, 27, 41]). Notably, Farhat, Lunasin, and Titi in [28], recently verified, in the case of the three-dimensional planetary geostrophic model, an earlier conjecture of Charney that posited that in simple atmospheric models, the temperature history determines all other state variables. The effects of noisy data were studied by Bloemker, Law, Stuart, and Zygalakis [9] and Bessaih, Olson and Titi [6]. A case related to the study undertaken by this paper, where observations are taken at discrete moments in time, rather than continuously, and with systematic deterministic errors, was studied in [29], while fully discretized versions were considered in [32]. Postprocessing methods were also applied to further ameliorate errors in this downscaling algorithm and in particular, obtain error bounds which are uniform-in-time (cf. [42]). See also [5] for a study into the continuous-time extended Kalman-Bucy filter in the setting of stochastic nonlinear ODEs. Observational measurements that have been blurred in time are studied here.

In continuation of the work in [33], we combine a feedback control based on time-averaged modal observables with the dynamics of the 2π-periodic subcritical SQG equation to obtain

$$\begin{cases}\partial_t\eta + \kappa\Lambda^\gamma\eta + v\cdot\nabla\eta = f - \mu\big(J_h^\delta(\eta) - J_h^\delta(\theta)\big),\\ v = \mathcal{R}^\perp\eta, \quad \eta(x,t)\big|_{t\in(-2\delta,0]} = g(x,t).\end{cases} \tag{1.2}$$

Here μ is a relaxation parameter, $J_h^\delta(\theta)$ represents an idealized interpolant based on modal measurements with observation resolution h along with a moving time average over intervals of width δ that represents the blur intrinsic to the measuring device used to obtain the data. It is natural to suppose that the observed solution, θ, represents the long-time evolution of the SQG equations, which is to say that θ belongs to the global attractor and therefore exists backward in time for all $t < 0$. For our analysis, however, it is sufficient to go back only as far as $t = -2\delta$. We therefore make the milder assumption that $\theta(\cdot, -2\delta)$ belongs to an absorbing ball for (1.1) with a sufficient regularity. Note also that in order to construct the data assimilation algorithm given by (1.2), we have assumed that the SQG equation is known in addition to the exact value of κ. What is not known, of course, is the initial condition for η represented by the function $g(x,t)$. Theoretically speaking one might as well take $g(x,t) = 0$; however, any 2π-periodic function with with mean zero that lies in the aforementioned absorbing set would be fine. Therefore, there may be better choices for g in practice. In particular, if we take $g(x,t) = \theta(x,t)$ for $t \in (-2\delta, 0]$, then $J_h^\delta(\eta) = J_h^\delta(\theta)$ in (1.2), so that $\eta(x,t) = \theta(x,t)$, for all $t > 0$; we refer the reader to Section 4.1 to help clarify this fact. Although there would be no need for data assimilation if $\theta(x,t)$ were already known, this cancellation is necessary to obtain the important mathematical property that, in the absence of noise or model error, η exactly synchronizes with θ over time.

We will assume that equation (1.2) governs the evolution of the approximating solution, η, used in our analysis of data assimilation for the SQG equation with observations that have been blurred in time and with 2π-periodic boundary conditions over $\mathbb{T}^2$. We will treat the subcritical case, when $\gamma \in (1,2)$. Our main results consist of the following two theorems:

(1) The data assimilation equations given by (1.2) are well posed (Theorem 1);
(2) For h sufficiently small, there exists a choice of μ and δ, for which the differences between η and θ vanish over time (Theorem 2).

Note that treating the critical case $\gamma = 1$ would, of course, also be very interesting for any type of observational data. However, this is beyond the scope of our present analysis.

We defer formal statements of our theorems to Section 3, after we have defined the mathematical setting of our problem in Section 2. Let us point out, however, that the presence of the moving time average introduces certain analytical difficulties. Firstly, it is difficult to control temporal oscillations in the approximating solution that arise due to deviations of the blurred-in-time observations from the exact values of the reference solution. For this, we must especially make use of more delicate boundedness properties of the interpolant operator, which we identify and prove in Section 2.2 and Appendix B, respectively. Second, a suitable non-local Gronwall inequality is required to control the difference between the approximating solution the observed solution. Theorem 2 shows that these obstacles can indeed be surmounted provided that δ is small enough. In this regime, (1.2) achieves exact asymptotic synchronization at an exponential rate and therefore performs similarly to the case studied in [33], where the observations are not blurred in time. Lastly, we emphasize that our approach to the analysis of this problem renders transparent which errors arise from the delay and which arise from the blurring, as well as the manner in which these errors transfer from one time-window to the next. Because of this, we are able to capture mathematically the role of the size of the averaging window.

2. Preliminaries

2.1. **Function spaces: L^p_{per}, V_σ, H^σ_{per}, $\dot{H}^\sigma_{per}$, C^∞_{per}.** Let $1 \leq p \leq \infty$, $\sigma \in \mathbb{R}$ and $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}) = [-\pi,\pi]^2$. Let $\mathcal{M}$ denote the set of real-valued Lebesgue measurable functions over $\mathbb{T}^2$. Since we will be working with periodic functions, define

$$\mathcal{M}_{per} := \{\phi \in \mathcal{M} : \phi(x,y) = \phi(x+2\pi, y) = \phi(x, y+2\pi) = \phi(x+2\pi, y+2\pi) \text{ a.e.}\}.$$

Let $C^\infty(\mathbb{R}^2)$ be the class of functions which are infinitely differentiable on $\mathbb{R}^2$. Define $C^\infty_{per}(\mathbb{T}^2)$ by

$$C^\infty_{per}(\mathbb{T}^2) := C^\infty(\mathbb{R}^2) \cap \mathcal{M}_{per}.$$

For $1 \leq p \leq \infty$, define the periodic Lebesgue spaces by

$$L^p_{per}(\mathbb{T}^2) := \{\phi \in \mathcal{M}_{per} : \|\phi\|_{L^p} < \infty\},$$

where

$$\|\phi\|_{L^p} := \left(\int_{\mathbb{T}^2} |\phi(x)|^p \, dx\right)^{1/p}, \quad 1 \leq p < \infty, \quad \text{and} \quad \|\phi\|_{L^\infty} := \operatorname*{ess\,sup}_{x \in \mathbb{T}^2} |\phi(x)|.$$

Let us also define

$$\mathcal{Z} := \{\phi \in L^1_{per} : \int_{\mathbb{T}^2} \phi(x) \, dx = 0\}. \tag{2.1}$$

For $\phi \in L^1_{per}(\mathbb{T}^2)$ let $\hat{\phi}(\mathbf{k})$ denote the Fourier coefficient of ϕ at wave-number $\mathbf{k} \in \mathbb{Z}^2$, i.e.,

$$\hat{\phi}(\mathbf{k}) := \frac{1}{4\pi^2} \int_{\mathbb{T}^2} e^{-i\mathbf{k}\cdot x} \phi(x)\, dx.$$

For any real number $\sigma \geq 0$, define the homogeneous Sobolev space, $\dot{H}^\sigma_{per}(\mathbb{T}^2)$, by

$$\dot{H}^\sigma_{per}(\mathbb{T}^2) := \{\phi \in L^2_{per}(\mathbb{T}^2) \cap \mathcal{Z} : \quad \|\phi\|_{\dot{H}^\sigma} < \infty\}, \tag{2.2}$$

where

$$\|\phi\|^2_{\dot{H}^\sigma} := 4\pi^2 \sum_{\mathbf{k}\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} |\mathbf{k}|^{2\sigma} |\hat{\phi}(\mathbf{k})|^2. \tag{2.3}$$

Similarly, for $\sigma \geq 0$, we define the inhomogeneous Sobolev space, $H^\sigma_{per}(\mathbb{T}^2)$, by

$$H^\sigma_{per}(\mathbb{T}^2) := \{\phi \in L^2_{per}(\mathbb{T}^2) : \|\phi\|_{H^\sigma} < \infty\}, \tag{2.4}$$

where

$$\|\phi\|^2_{H^\sigma} := 4\pi^2 \sum_{\mathbf{k}\in\mathbb{Z}^2} (1 + |\mathbf{k}|^2)^\sigma |\hat{\phi}(\mathbf{k})|^2. \tag{2.5}$$

Let $\mathcal{V}_0 \subset \mathcal{Z}$ denote the set of trigonometric polynomials with mean zero over $\mathbb{T}^2$ and set

$$V_\sigma := \overline{\mathcal{V}_0}^{H^\sigma}, \tag{2.6}$$

where the closure is taken with respect to the norm given by (2.5). Observe that the mean-zero condition can be equivalently stated as $\hat{\phi}(\mathbf{0}) = 0$. Thus, $\|\cdot\|_{\dot{H}^\sigma}$ and $\|\cdot\|_{H^\sigma}$ are equivalent as norms over V_σ. Moreover, by Plancherel's theorem we have

$$\|\phi\|_{\dot{H}^\sigma} = \|\Lambda^\sigma \phi\|_{L^2}.$$

Finally, for $\sigma \geq 0$, we identify $V_{-\sigma}$ as the dual space, $(V_\sigma)'$, of V_σ, which can be characterized as the space of all bounded linear functionals, ψ, on V_σ represented by the Fourier coefficients $\hat{\psi}(\mathbf{k})$ with duality paring

$$\langle \psi, \phi \rangle = 4\pi^2 \sum_{\mathbf{k}\in\mathbb{Z}^2\setminus\mathbf{0}} \overline{\hat{\psi}(\mathbf{k})} \cdot \hat{\phi}(\mathbf{k}) \quad \text{such that} \quad \|\psi\|_{\dot{H}^{-\sigma}} = 4\pi^2 \sum_{\mathbf{k}\in\mathbb{Z}^2\setminus\mathbf{0}} |\mathbf{k}|^{-2\sigma} \big|\hat{\phi}(\mathbf{k})\big| < \infty.$$

Given our use of non-dimensional variables and the 2π spatial periodicity of our functions, the Poincaré inequality may be written with a non-dimensional constant equal to one as

$$\|\phi\|_{\dot{H}^{\sigma'}} \leq \|\phi\|_{\dot{H}^\sigma} \qquad \text{for} \qquad \sigma' \leq \sigma. \tag{2.7}$$

Moreover, we have the following continuous embeddings

$$V_\sigma \hookrightarrow V_{\sigma'} \hookrightarrow V_0 \hookrightarrow V_{-\sigma'} \hookrightarrow V_{-\sigma} \qquad \text{when} \qquad 0 \leq \sigma' \leq \sigma.$$

Remark 2.1. *Since we will be working over V_σ and $\|\cdot\|_{\dot{H}^\sigma}$, $\|\cdot\|_{H^\sigma}$ determine equivalent norms over V_σ, we will often denote $\|\cdot\|_{\dot{H}^\sigma}$ simply by $\|\cdot\|_{H^\sigma}$ for convenience. Similarly, we will often abuse notation and denote $L^p_{per}(\mathbb{T}^2)$ simply by L^p.*

2.2. **General Interpolant Observables.** We will consider general interpolant observables, J_h, which are defined as those which satisfy certain boundedness and approximation-of-identity properties. The canonical examples of such observables include projection onto local spatial averages or projection onto finitely many Fourier modes. It was shown in [33] that such projections do in fact satisfy the properties we impose on J_h.

Let $0 < h < \pi/3$ and $1 \leq q \leq p \leq \infty$. Let $J_h : L^p(\mathbb{T}^2) \to L^p(\mathbb{T}^2)$ be a linear operator satisfying

$$\sup_{h>0} \|J_h\phi\|_{L^p} \leq C\|\phi\|_{L^p}, \tag{2.8}$$

$$\|J_h\phi\|_{L^p} \leq Ch^{2(1/p-1/q)}\|\phi\|_{L^q}, \tag{2.9}$$

where $C > 0$ represents a constant independent of ϕ, h. Note that $1/p - 1/q < 0$ when $q < p$ in which case the bound in (2.9) gets worse as h becomes smaller. In addition to (2.8) and (2.9), we will also suppose that J_h satisfies the following approximation-of-identity properties

$$\|\phi - J_h\phi\|_{L^2} \leq Ch^\beta\|\phi\|_{\dot{H}^\beta}, \quad \text{and} \quad \|\phi - J_h\phi\|_{\dot{H}^{-\beta}} \leq Ch^\beta\|\phi\|_{L^2}, \quad \beta \in (0, 1]. \tag{2.10}$$

We will also require J_h to satisfy some boundedness properties. We verify in Appendix B that these properties hold for local spatial averages. They also hold for spectral projection, that is, projection onto finitely many lowest Fourier modes (see Remark B.1). To state these boundedness properties, we will adopt the following notation. For β_1 and β_2 non-negative integers we let $D^\beta := \partial_1^{\beta_1}\partial_2^{\beta_2}$ where $\beta_1 + \beta_2 = \beta$, while if $\beta_j \geq 0$ are real then $D^\beta := \partial_1^{\lfloor\beta_1\rfloor}\partial_2^{\lfloor\beta_2\rfloor}\Lambda^{\beta-\lfloor\beta_1\rfloor-\lfloor\beta_2\rfloor}$. Here $\lfloor\beta\rfloor$ represents the greatest integer less or equal β. Finally, if $\beta \in (-2, 0)$, then $D^\beta := \Lambda^\beta$, i.e., the Riesz potential.

Now, given $\alpha \geq 1$, let $\epsilon(\alpha)$ be as in Proposition B.1.1 (v) when $\alpha \in [1, 2)$ and identically 0 otherwise. Let $C_\alpha > 0$ be a sufficiently large constant, depending possibly on α, and define

$$C_I(\alpha, h) := \begin{cases} C_\alpha\left(\frac{2\pi}{h}\right), & \alpha < 1, \\ C_\alpha\left(\frac{2\pi}{h}\right)^{2+|\alpha|-\epsilon(\alpha)}, & \alpha \geq 1. \end{cases} \tag{2.11}$$

We assume that

$$\|J_h\phi\|_{\dot{H}^\rho(\mathbb{T}^2)} \leq C_I(\beta, h)h^{-(\rho-\beta)}\|\phi\|_{\dot{H}^\beta(\mathbb{T}^2)}, \quad (\rho, \beta) \in [0, \infty) \times [0, 2), \tag{2.12}$$

$$\|J_h\phi\|_{\dot{H}^\rho(\mathbb{T}^2)} \leq Ch^{-\rho}(h^\beta\|\phi\|_{\dot{H}^\beta} + \|\phi\|_{L^2(\mathbb{T}^2)}), \quad (\rho, \beta) \in [0, \infty) \times (-2, 0], \tag{2.13}$$

$$\|J_h\phi\|_{\dot{H}^\rho(\mathbb{T}^2)} \leq C_I(|\rho|, h)h^{-(\rho-\beta)}\|\phi\|_{\dot{H}^\beta(\mathbb{T}^2)}, \quad (\rho, \beta) \in (-2, 0) \times (-\infty, 0], \tag{2.14}$$

$$\|J_hD^\beta\phi\|_{\dot{H}^\rho(\mathbb{T}^2)} \leq C_I(|\rho|, h)h^{-(\rho+\beta-\beta')}\|\phi\|_{\dot{H}^{\beta'}(\mathbb{T}^2)}, \quad (\rho, \beta, \beta') \in (-2, 0) \times (-2, \infty) \times (-\infty, \beta], \tag{2.15}$$

$$\|J_hD^\ell\phi\|_{\dot{H}^\rho(\mathbb{T}^2)} \leq C_I(|\rho|, h)h^{-1-\rho-\ell}\|\phi\|_{L^1(\mathbb{T}^2)}, \quad (\rho, \ell) \in (-2, 0) \times \mathbb{Z}_{\geq 0}, . \tag{2.16}$$

We again emphasize that the above properties are consistent with those satisfied by the projection onto local spatial averages (see (B.11) and (B.12) in Appendix B). Furthermore, we again point out that they are also consistent with those satisfied by the spectral projection, up to possibly different constants (See Remarks 2.3 and B.1). For clarity of exposition, our analysis will be performed with the constants detailed above, though the conclusions are also true for J_h given by spectral projection.

Remark 2.2. *We are able to prove other boundedness properties in Appendix B in addition to the ones shown above. While our analysis requires us only to invoke properties (2.8)-(2.16), the additional boundedness properties asserted in Proposition B.2.2 may find use in other applications.*

Remark 2.3. *In the case where J_h is given by the Littlewood-Paley spectral projection, i.e., projection onto Fourier modes $\lesssim 2^{1/h}$, then we replace $C_I(\alpha,h)$ everywhere above by $C_S(\alpha,h)$ according to the rule*

$$C_I(\alpha,h)h^r \mapsto C_S(\alpha,h)h^r := \begin{cases} C, & r\geq 0,\\ Ch^r, & r<0. \end{cases} \quad \text{and} \quad \widetilde{C}_S := C.$$

Note that $\alpha=\alpha(p)$ implicitly. One may thus refer to operators J_h with constants C_I as "Type I operators" and those with prefactors C_S as "Spectral Type I operators." Observe that in general we have $C_S \lesssim C_I$, so all Spectral Type I operators are automatically Type I operators. We further observe that the Type II operators defined in [3], see also [6], using nodal-point measurements of the velocity field in physical space do not satisfy the above bounds.

Remark 2.4. *Note that in the estimates we perform below, the constant $C>0$ appearing in (2.11) may change line-to-line when invoking the above properties. Nevertheless, it can be fixed to be sufficiently large in the statement of the theorems where such constants appear.*

2.3. **Time-averaged Interpolant Observables.** Suppose $\phi=\phi(x,t)$. We define the time-averaged general interpolant operator, J_h^δ, by

$$(J_h^\delta\phi)(x,t) := \frac{1}{\delta}\int_{t-2\delta}^{t-\delta}(J_h\phi)(x,s)\ ds \tag{2.17}$$

Due to the time-averaging, one must also control errors that arise from temporal deviations of the time-average from the instantaneous value value. Indeed, observe that by the mean value theorem and by commuting ∂_τ with J_h we have

$$\phi - J_h^\delta\phi = (\phi - J_h\phi) + \frac{1}{\delta}\int_{t-2\delta}^{t-\delta}\int_s^t J_h\partial_\tau\phi(x,\tau)\ d\tau\ ds. \tag{2.18}$$

We will make crucial use of (2.18) when we perform the a priori estimates.

Remark 2.5. *It may seem more natural to represent blurred-in-time measurements at time t by an average of the form*

$$(I_h^\delta\phi)(x,t) := \frac{1}{\delta}\int_{t-\delta/2}^{t+\delta/2}(J_h\phi)(x,s)\,ds.$$

However, in this case the corresponding a feedback term obtained by using $I_h^\delta(\eta)$ in place of $J_h^\delta(\eta)$ in (1.2) would violate causality by introducing an integral over times in the future. We emphasize that the same interpolant operator must be used in the feedback as used for the measurements in order to maintain the property that $g=\theta$ for $t\in(-\delta,0]$ implies $\eta=\theta$ for all times $t>0$ in the future. Therefore, the best we could do is insert the measurement $I_h^\delta(\phi)$ into the model delayed in time by $\delta/2$. This approach was taken in [7] and [8] for the Lorenz equations. In the present work, an additional delay has been inserted into the definition of $J_h^\delta\phi$ to make the analysis more convenient. This allows the feedback control to be treated as a time-dependent force, thereby transforming what would have been partial integro-differential equations into merely partial differential equations. While any additional delay would achieve *the same effect, for simplicity we choose its order to be $\delta/2$ which is the same as the delay already dictated by causality.*

2.4. **Calculus inequalities.** We will make use of the following bound for the fractional Laplacian, which can be found for instance in [16, 20, 34].

Proposition 2.4.1. *Let $p \geq 2$, $0 \leq \gamma \leq 2$, and $\phi \in C^\infty(\mathbb{T}^2)$. Then*

$$\frac{2}{p}\|\Lambda^{\gamma/2}(\phi^{p/2})\|_{L^2}^2 \leq \int_{\mathbb{T}^2} |\phi|^{p-2}(x)\phi(x)\Lambda^\gamma\phi(x)\ dx.$$

We will also make use of the following calculus inequality for fractional derivatives (cf. [36, 37] and references therein):

Proposition 2.4.2. *Let $\phi, \psi \in C^\infty(\mathbb{T}^2)$, $\beta > 0$, and $p \in (1,\infty)$. Then we have that*

$$\|\Lambda^\beta(\phi\psi)\|_{L^p} \leq C\|\psi\|_{L^{p_1}}\|\Lambda^\beta\phi\|_{L^{p_2}} + C\|\Lambda^\beta\psi\|_{L^{p_3}}\|\phi\|_{L^{p_4}},$$

where $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$, and $p_2, p_3 \in (1,\infty)$, for a sufficiently large constant $C > 0$ that depends only on σ, p, p_i.

Finally, we will frequently apply the following interpolation inequality, which is a special case of the Gagliardo-Nirenberg interpolation inequality and can be proven with Plancherel's theorem and the Cauchy-Schwarz inequality:

Proposition 2.4.3. *Let $\phi \in \dot{H}^\beta_{per}(\mathbb{T}^2)$ and $0 \leq \alpha \leq \beta$. Then*

$$\|\Lambda^\alpha\phi\|_{L^2} \leq C\|\Lambda^\beta\phi\|_{L^2}^{\frac{\alpha}{\beta}}\|\phi\|_{L^2}^{1-\frac{\alpha}{\beta}}, \tag{2.19}$$

where C depends on α, β.

2.5. **Well-posedness and Global Attractor of the SQG equation.** Let us recall the following well-posedness results of the SQG equation. In [18] it was shown that global strong solutions exist and that weak solutions are unique in the class of strong solutions.

Proposition 2.5.1 (Global existence)**.** *Let $1 < \gamma \leq 2$, and $\sigma > 2-\gamma$. Given $T > 0$, suppose that $\theta_0 \in V_\sigma$ and f satisfies*

$$f \in L^2(0,T;V_{\sigma-\gamma/2}) \cap L^1(0,T;L^p_{per}(\mathbb{T}^2)),$$

where $1-\sigma \leq 2/p < \gamma - 1$. Then there is a weak solution θ of (1.1) such that

$$\theta \in L^\infty(0,T;V_\sigma) \cap L^2(0,T;V_{\sigma+\gamma/2}).$$

Proposition 2.5.2 (Uniqueness)**.** *Let $T > 0$ and $1 < \gamma \leq 2$. Suppose that $\theta_0 \in L^2_{per}(\mathbb{T}^2) \cap \mathcal{Z}$ and $f \in L^2(0,T;V_{-\gamma/2})$. Then for $p \geq 1, q > 0$ satisfying*

$$\frac{1}{p} + \frac{\gamma}{2q} = \frac{\gamma-1}{2},$$

there is at most one solution to (1.1) such that $\theta \in L^q(0,T;L^p_{per}(\mathbb{T}^2))$.

Let us recall the following estimates for the reference solution θ (cf. [20, 34, 44]).

Proposition 2.5.3. *Let $\gamma \in (0,2]$, $\sigma > 2-\gamma$, and $\theta_0 \in V_\sigma, f \in V_{\sigma-\gamma/2} \cap L^p_{per}(\mathbb{T}^2)$. Then there exists a constant $C > 0$ such that for any $p \geq 2$ satisfying $1-\sigma < 2/p < \gamma - 1$, we have*

$$\|\theta(t)\|_{L^p} \leq \left(\|\theta_0\|_{L^p} - \frac{1}{C}F_{L^p}\right)e^{-C\kappa t} + \frac{1}{C}F_{L^p}, \quad F_{L^p} := \frac{1}{\kappa}\|f\|_{L^p}. \tag{2.20}$$

Moreover, if $\theta_0 \in L^2_{per}(\mathbb{T}^2)$ and $f \in V_{-\gamma/2}$, then any weak solution θ of (1.1) satisfies

$$\|\theta(t)\|^2_{L^2} \leq \left(\|\theta_0\|^2_{L^2} - F^2_{H^{-\gamma/2}}\right)e^{-\kappa t} + F^2_{H^{-\gamma/2}}, \quad F_{H^{-\gamma/2}} := \frac{1}{\kappa}\|f\|_{H^{-\gamma/2}}. \tag{2.21}$$

It was shown in [34] for the subcritical range $1 < \gamma \leq 2$, that equation (1.1) has an absorbing ball in V_σ and corresponding global attractor $\mathcal{A} \subset V_\sigma$ when $\sigma > 2-\gamma$. In other words, there is a bounded set $\mathcal{B} \subset V_\sigma$ characterized by the property that for any $\theta_0 \in V_\sigma$, there exists $t_0 > 0$ depending on $\|\theta_0\|_{H^\sigma}$ such that $S(t)\theta_0 \in \mathcal{B}$ for all $t \geq t_0$. Here $\{S(t)\}_{t\geq 0}$ denotes the semigroup of the corresponding dissipative equation.

Proposition 2.5.4 (Global attractor)**.** *Suppose that $1 < \gamma \leq 2$ and $\sigma > 2-\gamma$. Let $f \in V_{\sigma-\gamma/2} \cap L^p_{per}(\mathbb{T}^2)$, where $1-\sigma < 2/p < \gamma-1$. Then (1.1) has an absorbing ball $\mathcal{B}_{H^\sigma}$ given by*

$$\mathcal{B}_{H^\sigma} := \{\theta_0 \in \dot{H}^\sigma_{per} : \|\theta_0\|_{H^\sigma} \leq \Theta_{H^\sigma}\}, \tag{2.22}$$

for some $\Theta_{H^\sigma} < \infty$. Moreover, the solution operator $S = S_f$ of (1.1) given by $S(t)\theta_0 = \theta(t)$ for $t \geq 0$ defines a semigroup in the space V_σ and possesses a global attractor $\mathcal{A} \subset V_\sigma$, i.e., $\mathcal{A}$ is a compact, connected subset of V_σ satisfying the following properties

(1) $\mathcal{A}$ is the maximal bounded invariant set;
(2) $\mathcal{A}$ attracts all bounded subsets in V_σ in the topology of $\dot{H}^\sigma_{per}$.

3. Standing Hypotheses and Statements of main theorems

We will work under the following assumptions for the remainder of the paper.

Standing Hypotheses. *Assume the following:*

(H1) $1 < \gamma < 2$;
(H2) $\sigma \in (2-\gamma, \gamma]$;
(H3) $p \in [1,\infty]$ *such that* $1-\sigma < 2/p < \gamma - 1$;
(H4) $f \in V_{\sigma-\gamma/2} \cap L^p$, *time-independent;*
(H5) $\theta_{-2\delta} \in \mathcal{B}_{H^\sigma}$;
(H6) $g \in C((-2\delta, 0]; V_{\max\{\sigma,\gamma/2\}}) \cap L^2((-2\delta,0]; V_{\sigma+\gamma/2})$;
(H7) $0 < h < \pi/4$.

Observe that $(H1)$ expresses the subcritical range of dissipation, while $(H2)-(H5)$ ensure that we are in a regime of global strong solutions for (1.1) and that the global attractor exists.
Also observe that since $\gamma < 2$, the range for σ in $(H2)$ covers the natural spatial regularity class for strong solutions, e.g. H^γ.
On the other hand, from $(H1)-(H5)$, Propositions 2.5.3 and 2.5.4 imply that

$$\Theta_{L^2} := \sup_{t>-2\delta}\|\theta(t)\|_{L^2} < \infty \quad \text{and} \quad \Theta_{L^p} := \sup_{t>-2\delta}\|\theta(t)\|_{L^p} < \infty. \tag{3.1}$$

In particular, it immediately follows from (2.8) that

$$\sup_{t>-2\delta}\|J_h^\delta\theta(t)\|_{L^q}^2 \leq C_J\Theta_{L^q}^2, \quad q\in[1,\infty], \tag{3.2}$$

and from (2.12) that

$$\sup_{t>-2\delta}\|J_h^\delta\theta(t)\|_{H^\sigma} \leq C_J\Theta_{H^\sigma}, \tag{3.3}$$

for some constant $C_J > 0$. Also, for $1 \leq q \leq \infty$ and $\alpha \in \mathbb{R}$, let us define

$$\Gamma_{L^q} := \sup_{t\in(-2\delta,0]}\|g(t)\|_{L^q} \quad \text{and} \quad \Gamma_{H^\alpha} := \sup_{t\in(-2\delta,0]}\|g(t)\|_{H^\alpha}. \tag{3.4}$$

Then for p given by $(H3)$, the Sobolev embedding theorem and $(H6)$ imply

$$\Gamma_{L^p} < \infty, \quad \Gamma_{H^\sigma} < \infty \quad \text{and} \quad \Gamma_{H^{\gamma/2}} < \infty. \tag{3.5}$$

Finally, we give exact mathematical statements of our main results.

Theorem 1. *Let θ be the unique global strong solution of (1.1) corresponding to initial data $\theta_{-2\delta}$ having zero mean over $\mathbb{T}^2$. Then under the Standing Hypotheses, for all $T > 0$, there exists a unique strong solution $\eta \in L^\infty(0,T;\dot{H}^\sigma_{per}(\mathbb{T}^2)) \cap L^2(0,T;\dot{H}^{\sigma+\gamma/2}_{per}(\mathbb{T}^2))$ satisfying (1.2) with $\eta(\cdot\,,0) = g(\cdot\,,0)$.*

Theorem 2. *Under the hypotheses of Theorem 1, there exists constants $c_0, c_0' > 0$ such that if h, μ satisfy*

$$\frac{1}{c_0'}\left(\frac{\Theta_{L^p}}{\kappa}\right)^{\gamma/(\gamma-1-2/p)} \leq \frac{\mu}{\kappa} \leq \frac{1}{c_0}h^{-\gamma}, \tag{3.6}$$

and $\delta > 0$ is chosen sufficiently small, depending on h, then the solution η given by (1.2) satisfies

$$\|\eta(t)-\theta(t)\|_{L^2}^2 \leq O(e^{-\lambda_0\mu(t-2\delta)}), \quad t > 2\delta, \tag{3.7}$$

for some constant $\lambda_0 \in (0,1)$.

Remark 3.1. *Note that the condition that $\delta > 0$ be sufficiently small can be described precisely by simultaneously satisfying (4.7) and (5.7) below.*

Remark 3.2. *As we pointed out in Remark 2.3, since Spectral Type I operators satisfy all the properties of Type I operators, both Theorem 1 and 2 are also valid for Spectral Type I operators. In particular, they are valid when J_h is given by projection onto finitely many Fourier modes.*

Remark 3.3. *The relationship between the full three-dimensional quasi-geostrophic equations and the SQG equation implies that being able to approximate θ by η, as in the conclusion of Theorem 2, is the same as synchronizing the corresponding three-dimensional solutions in which the potential vorticity is identically zero and the vertical motion eliminated. Therefore, in a way analogous to the discussion in [33], our theorem provides an example where time-averaged data collected on a two-dimensional surface is sufficient to obtain synchronization in a three-dimensional domain.*

Before we move on to the a priori analysis, we will set forth the following convention for constants.

Remark 3.4. *In the estimates that follow below, c and C will generically denote positive constants, which depend only on other non-dimensional scalar quantities, and may change line-to-line in the estimates. We emphasize that in the estimates we perform below, the constants c and C may change in magnitude from line-to-line, but as the equations were fully non-dimensionalized from the beginning they will never carry any physical dimensions.*

4. A priori estimates

4.1. **Initial value problem and Proof of Theorem 1.** We recouch (1.2) as a sequence of initial value problems over consecutive time intervals. Once we have defined the setting properly, we may immediately prove Theorem 1 by appealing to Propositions 2.5.1 and 2.5.2.

Observe that owing to the delay in the interpolant operator, J_h^δ, we must initialize the averaging process. By $(H1)-(H5)$ and Proposition 2.5.4, we may assume that θ is the strong solution of (1.1) with initial data starting at $t=-2\delta$ such that $\theta_{-2\delta}\in\mathcal{B}_{H^\sigma}$.

For any $k\geq -2$ set

$$I_{-2}:=\emptyset,\quad I_{-1}:=(-2\delta,0],\quad \text{and}\quad \delta_k:=k\delta,\quad I_k:=(\delta_k,\delta_{k+1}],\quad \text{for } k\geq 0. \tag{4.1}$$

Let $\eta^{(-1)}(\cdot\,,t)=g(\cdot\,,t)$ for $t\in I_{-1}$. Then we may express a solution, η, of

$$\partial_t\eta+\kappa\Lambda^\gamma\eta+v\cdot\nabla\eta=f-\mu J_h^\delta(\eta-\theta),\quad v=\mathcal{R}^\perp\eta,\quad \eta(x,t)\big|_{t\in I_{-1}}=g(x,t). \tag{4.2}$$

as the sum

$$\eta(x,t):=\sum_{k\geq -1}\eta^{(k)}(x,t)\chi_{I_k}(t),$$

where for each $k\geq 0$, $\eta^{(k)}$ satisfies:

$$\begin{aligned}&\partial_t\eta^{(k)}+\kappa\Lambda^\gamma\eta^{(k)}+v^{(k)}\cdot\nabla\eta^{(k)}=f-\mu J_h^\delta(\eta^{(k)}-\theta),\quad t\in I_k,\\ &v^{(k)}=\mathcal{R}^\perp\eta^{(k)},\quad \eta^{(k)}(x,t)\big|_{t\in I_{k-1}\cup I_{k-2}}=\eta^{(k-1)}(x,t).\end{aligned} \tag{4.3}$$

Hence, over each interval I_k we may view the term, $J_h^\delta\eta^{(k)}$, in (4.3) as a smooth, time-dependent forcing term and (4.3) as an initial value problem over I_k with initial data $\eta_0(x)=\eta(x,\delta_k)$. The proof of Theorem 1 follows readily.

Proof of Theorem 1. We proceed by induction on k. For $k=0$, from $(H6)$ we have that $\eta(\cdot\,,0)=g(\cdot\,,0)\in V_\sigma$. Since we assume the Standing Hypotheses, we have that $J_h^\delta g=J_h^\delta\eta^{(0)}, J_h^\delta\theta\in L^2(0,T;V_{\sigma-\gamma/2})\cap L^1(0,T;L^p_{per}(\mathbb{T}^2))$ holds for all $T>0$ (by (2.8) and (2.12)), so that we may apply Proposition 2.5.1 and 2.5.2 to deduce existence and uniqueness of a strong solution, $\eta^{(0)}$, over I_0 to (4.3). Suppose unique strong solutions to (4.3) exist for all $\ell=0,\dots,k$. Consider (4.3) over I_{k+1}. Observe that by hypothesis $\eta^{(k+1)}(\cdot\,,\delta_{k+1})=\eta^{(k)}(\cdot\,,\delta_{k+1})\in V_\sigma$ and $J_h^\delta\eta^{(k+1)}, J_h^\delta\theta\in L^2(\delta_{k-1},\delta_{k-1}+T;V_{\sigma-\gamma/2})\cap L^1(\delta_{k-1},\delta_{k-1}+T;L^p_{per}(\mathbb{T}^2))$ hold once again by (2.8) and (2.12). Therefore, we apply Proposition 2.5.1 and 2.5.2 to guarantee existence and uniqueness of a strong solution $\eta^{(k+1)}$ to (4.3) over I_{k+1}, completing the proof. □

In the remainder of section 4 we establish uniform-in-time estimates for η in L^2, L^p, and H^σ. As we will see, the synchronization property will rely crucially on these uniform estimates. To obtain uniform H^σ estimates, we perform a bootstrap from L^2 to L^p, then from L^p to H^σ. Once we have collected the requisite uniform bounds, we proceed to section 5 and the proof of Theorem 2.

4.2. **Uniform L^2 estimates.** In this section, we will ultimately obtain L^2 estimates for the solution η of (4.2) that are uniform in time. In this work, any bound of this type shall be referred to as a "good" bound. The main result in this section is the "good" bound stated as Proposition 4.2.1 below. We emphasize that the structure of the analysis in sections 4.2.2, 4.2.3, and 4.2.4 will be mimicked in section 5 when we establish the synchronization property.

We begin by introducing some notation that will be convenient when expressing the necessary bounds in our proofs. Let

$$\widetilde{R}_{L^2}^2 := \frac{\kappa^2}{\mu^2} F_{H^{-\gamma/2}}^2 + C_J \Theta_{L^2}^2, \quad R_{L^2}^2 := \frac{\kappa}{\mu} F_{H^{-\gamma/2}}^2 + C_J \Theta_{L^2}^2, \quad M_{L^2}^2 := \Gamma_{2,1}^2 + 8R_{L^2}^2 \tag{4.4}$$

where $\Gamma_{2,k}$ is the function of $\delta > 0$ given by

$$\Gamma_{2,-1} := \Gamma_{L^2} \quad \text{and} \quad \Gamma_{2,k}^2 := \Gamma_{2,k-1}^2 + C\frac{\delta\mu^2}{\kappa}\left(\Gamma_{2,k-1}^2 + \widetilde{R}_{L^2}^2\right) \quad \text{for} \quad k \geq 0. \tag{4.5}$$

Note that $\Gamma_{2,k}$ and consequently $M_{L^2}^2$ are increasing functions of δ. Therefore, any upper bounds given by the constants defined in (4.4) and (4.5) for a particular $\delta = \delta_0$ continue to hold when $\delta < \delta_0$. We shall immediately make use of this property to show that the hypotheses on δ in Proposition 4.2.1 stated below are not vacuous.

Proposition 4.2.1. *There exist constants $c_0, c_1 > 0$, with c_1 depending on c_0, such that if h, μ satisfy*

$$\frac{\mu}{\kappa} \leq \frac{1}{c_0} h^{-\gamma}, \tag{4.6}$$

and δ is chosen such that

$$\delta \leq \frac{1}{c_1}\frac{h^\gamma}{\kappa} \min\left\{1, h^\gamma \frac{8R_{L^2}^2}{(M_{L^2}^2 + \widetilde{R}_{L^2}^2)}, \left(\frac{h}{2\pi}\right)\frac{1}{(1+\kappa^{-1}h^{\gamma-2})}\frac{R_{L^2}}{(1 + M_{L^2}^2 + R_{L^2}^2)}\right\} \tag{4.7}$$

as well as

$$\delta \leq \frac{1}{c_1}\left(\frac{h}{2\pi}\right)\min\left\{\left(\frac{\mu h^\gamma}{\kappa}\right)^{1/2}\frac{h^2}{M_{L^2}}, \frac{h^\gamma}{\kappa}\right\} \tag{4.8}$$

where $\widetilde{R}_{L^2}^2$, $R_{L^2}^2$ and $M_{L^2}^2$ are given in (4.4), then

$$\|\eta(t)\|_{L^2}^2 \leq \left(\Gamma_{2,1}^2 - 8R_{L^2}^2\right) e^{-(\mu/2)(t-2\delta)} + 8R_{L^2}^2 \quad \textit{for} \quad t \geq 2\delta \tag{4.9}$$

and

$$\frac{\kappa}{4}\int_{I_k} \|\eta(s)\|_{H^{\gamma/2}}^2\, ds \leq \Gamma_{2,1}^2 + 8R_{L^2}^2 \leq M_{L^2}^2 \quad \textit{for} \quad k \geq 2. \tag{4.10}$$

Observe that both sides of the inequalities given by (4.7) and (4.8) depend on δ. This is, as already mentioned, because $M_{L^2}^2$ depends on δ. However, since $M_{L^2}^2$ appears in the denominator of the right-hand side and is an increasing function of δ, it is easy to see that there must be a $\delta > 0$ which satisfies both these inequalities.

To prove Proposition 4.2.1, we employ three preliminary lemmas. First, in section 4.2.1 we establish bounds in L^2 which are uniform in each time interval I_k, but ultimately depend on k. Throughout this work we will refer to any bounds that depend on k as "rough" bounds. Such bounds are insufficient on their own but needed in order to close estimates later. Then in section 4.2.2, we establish time-derivative estimates to control the temporal oscillations that emanate from the feedback term (see section 4.2.2). The third lemma is is a non-local Gronwall inequality that ensures uniform bounds provided that the window of time-averaging is sufficiently small; its proof is deferred to Appendix A. This Gronwall inequality will be used again to establish the synchronization property in section 5. We finally prove Proposition 4.2.1 in section 4.2.4.

Remark 4.1. *We will often exchange the quantity μ for the quantity $\kappa h^{-\gamma}$ via the relation (4.6), in order to emphasize that δ and μ ultimately depend only on h (and Θ_{L^p}) alone.*

4.2.1. *Rough L^2 estimates.* We will first establish the following "rough" a priori bound. We omit most of the details, though they can easily be gleaned from the proof of Proposition 4.2.1. An alternative form of Lemma 4.2.1 is given by Corollary 4.2.2 stated below, which will be convenient to use in the proof of Proposition 4.2.1 later.

Lemma 4.2.1. *Let $F_{H^{-\gamma/2}}, \Theta_{L^2}, \widetilde{R}_{L^2}$ be given by (2.21), (3.1), (4.4), respectively. There exists a constant $C_0 > 0$, independent of k, such that*

$$\|\eta(t)\|_{L^2}^2 + \kappa \int_{\delta_k}^t \|\eta(s)\|_{H^{\gamma/2}}^2 \, ds \leq \widetilde{M}_{L^2}^2(k,t), \quad t \in I_k, \quad k \geq 0, \tag{4.11}$$

where

$$\widetilde{M}_{L^2}^2(k,t) := \|\eta(\delta_k)\|_{L^2}^2 + C_0 \frac{\delta\mu^2}{\kappa} \left[\widetilde{R}_{L^2}^2 + \left(\sup_{s\in I_{k-2}\cup I_{k-1}} \|\eta(s)\|_{L^2}^2 \right) \right]. \tag{4.12}$$

Proof. Suppose $t \in I_k$ for some $k \geq 0$. We perform standard energy estimates to obtain

$$\frac{d}{dt}\|\eta\|_{L^2}^2 + \kappa\|\Lambda^{\gamma/2}\eta\|_{L^2}^2 \leq \kappa F_{H^{-\gamma/2}}^2 + C\frac{\mu^2}{\kappa}\left(\|J_h^\delta\theta\|_{L^2}^2 + \|J_h^\delta\eta\|_{L^2}^2\right). \tag{4.13}$$

Observe that by the Cauchy-Schwarz inequality and (2.8) we have

$$\|J_h^\delta\eta(t)\|_{L^2}^2 \leq C\left(\sup_{s\in I_{k-2}\cup I_{k-1}} \|\eta(s)\|_{L^2}^2 \right), \quad t \in I_k.$$

Returning to (4.13) and applying these facts along with (3.2), we obtain

$$\frac{d}{dt}\|\eta\|_{L^2}^2 + \kappa\|\eta\|_{H^{\gamma/2}}^2 \leq \left(\kappa F_{H^{-\gamma/2}}^2 + C\frac{\mu^2}{\kappa}\Theta_{L^2}^2\right) + C\frac{\mu^2}{\kappa}\left(\sup_{s\in I_{k-2}\cup I_{k-1}} \|\eta(s)\|_{L^2}^2 \right). \tag{4.14}$$

Finally, by integrating (4.14) over $[\delta_k, t]$ for $t \in I_k$ we arrive at

$$\begin{aligned}&\|\eta(t)\|_{L^2}^2 + \kappa \int_{\delta_k}^t \|\eta(s)\|_{H^{\gamma/2}}^2 \, ds \\ &\qquad \leq \|\eta(\delta_k)\|_{L^2}^2 + \delta\frac{\mu^2}{\kappa}\left[\left(\frac{\kappa^2}{\mu^2} F_{H^{-\gamma/2}}^2 + C\Theta_{L^2}^2 \right) + C\left(\sup_{s\in I_{k-2}\cup I_{k-1}} \|\eta(s)\|_{L^2}^2 \right) \right],\end{aligned} \tag{4.15}$$

which can be simplified to (4.11) using (4.4), as desired. □

Corollary 4.2.2. *Let $k > 0$. Suppose that for each $0 \leq \ell \leq k$, there exists $M_\ell > 0$ such that*

$$\|\eta(t)\|_{L^2}^2 + \kappa \int_{\delta_k}^t \|\eta(s)\|_{H^{\gamma/2}}^2 \, ds \leq M_\ell, \quad t \in (-2\delta, \delta_{\ell+1}].$$

Then there exists a constant $C_0 > 0$, independent of k, such that

$$\|\eta(t)\|_{L^2}^2 + \kappa \int_{\delta_{k+1}}^t \|\eta(s)\|_{H^{\gamma/2}}^2 \, ds \leq M_k + C_0 \frac{\delta\mu^2}{\kappa}\left(\widetilde{R}_{L^2}^2 + M_k\right), \quad t \in I_{k+1}.$$

While δ can be chosen in these bounds so that the size of $\delta\mu^2/\kappa$ is small, this alone does not suffice to obtain uniform-in-time bounds for $\|\eta(t)\|_{L^2}$ upon iteration in k, which will be crucial in establishing the synchronization property. Nevertheless, these "rough" bounds will be useful in order to close our estimates and achieve uniform bounds later.

4.2.2. *Control of temporal oscillations at fixed spatial scale.* We recall from (2.18) that we will require estimates for the time-derivative, $\partial_t\eta$, but only over length scales $\gtrsim h$, where h measures the spatial resolution of the observables.

Lemma 4.2.3. *Let $k > 0$. Suppose there exists $M_\ell > 0$ such that*

$$\sup_{t\in(-2\delta,\delta_\ell]} \|\eta(t)\|_{L^2} \leq M_{\ell-1} \qquad \text{for each} \qquad 0 \leq \ell \leq k+2. \tag{4.16}$$

Let $c_0 > 0$ be any constant such that

$$\frac{\mu h^\gamma}{\kappa} \leq \frac{1}{c_0}. \tag{4.17}$$

Then there exists a constant $C_0 > 0$, depending on c_0, but independent of k, such that

$$\|(J_h\partial_t\eta)(t)\|_{H^{-\gamma/2}}^2 \leq C_0 \left(\frac{2\pi}{h}\right)^2 \frac{\kappa^2}{h^\gamma}\left(1 + \frac{1}{\kappa}\frac{1}{h^{2-\gamma}}\right)^2 \left(1 + M_k^2 + R_{L^2}^2\right)^2 \tag{4.18}$$

holds for all $t \in (-2\delta, \delta_{k+1}]$, and

$$\begin{aligned}\|(J_h\partial_t\eta)(t)\|_{H^{-\gamma/2}}^2 \leq & C_0 \left(\frac{2\pi}{h}\right)^2 \kappa^2 \|\eta(t)\|_{H^{\gamma/2}}^2 + C_0 \left(\frac{2\pi}{h}\right)^2 \frac{M_{k+1}^2}{h^{4-\gamma}} \|\eta(t)\|_{L^2}^2 \\ & + C_0 \left(\frac{2\pi}{h}\right)^2 \frac{\kappa^2}{h^\gamma}\left(M_k^2 + R_{L^2}^2\right)\end{aligned} \tag{4.19}$$

holds for all $t \in I_{k+1}$.

Proof. By $(H1)$ we have $\gamma/2 < 1$. Therefore, by (2.11), see also (B.16), we have

$$C_I(\gamma/2, h) = C\left(\frac{2\pi}{h}\right).$$

Now, applying J_h to (1.2), using the fact that v is divergence free, and then taking the $H^{-\gamma/2}$-norm we have

$$\begin{aligned}\|J_h\partial_t\eta\|_{H^{-\gamma/2}} \leq \kappa \|J_h\Lambda^\gamma\eta\|_{H^{-\gamma/2}} + \|J_h\nabla\cdot(v\eta)\|_{H^{-\gamma/2}} \\ + \|J_h f\|_{H^{-\gamma/2}} + \mu\|J_h^\delta\eta\|_{H^{-\gamma/2}} + \mu\|J_h^\delta\theta_j\|_{H^{-\gamma/2}}.\end{aligned} \tag{4.20}$$

By $(H1), (H7)$, (2.15), (2.21), (3.1), and (4.16) we may estimate

$$\begin{aligned}
\kappa\|J_h\Lambda^\gamma\eta(t)\|_{H^{-\gamma/2}} &\leq C\left(\frac{2\pi}{h}\right)\kappa h^{-\gamma/2}\|\eta(t)\|_{L^2}\\
&\leq C\left(\frac{2\pi}{h}\right)\kappa h^{-\gamma/2}M_k, \quad t\in(-2\delta,\delta_{k+1}],\\
\kappa\|J_h\Lambda^\gamma\eta(t)\|_{H^{-\gamma/2}} &\leq C\left(\frac{2\pi}{h}\right)\kappa\|\eta(t)\|_{H^{\gamma/2}}, \quad t\in I_{k+1}\\
\|J_hf\|_{H^{-\gamma/2}} &\leq C\left(\frac{2\pi}{h}\right)\kappa F_{H^{-\gamma/2}},\\
\mu\|J_h^\delta\eta(t)\|_{H^{-\gamma/2}} &\leq C\left(\frac{2\pi}{h}\right)\mu h^{\gamma/2}\left(\sup_{s\in(-2\delta,\delta_{k+1}]}\|\eta(s)\|_{L^2}\right)\\
&\leq C\left(\frac{2\pi}{h}\right)\mu h^{\gamma/2}M_k, \quad t\in(-2\delta,\delta_{k+2}],\\
\mu\|J_h^\delta\theta(t)\|_{H^{-\gamma/2}} &\leq C\left(\frac{2\pi}{h}\right)\mu h^{\gamma/2}\Theta_{L^2}, \quad t>-2\delta.
\end{aligned}$$

For the quadratic term apply (2.16), the Cauchy–Schwarz inequality and the fact that $\mathcal{R}^\perp$ is a bounded operator in L^2 to estimate

$$\|J_h\nabla\cdot(v\eta)\|_{H^{-\gamma/2}} \leq C\left(\frac{2\pi}{h}\right)h^{-2+\gamma/2}\|v\eta\|_{L^1} \leq C\left(\frac{2\pi}{h}\right)h^{-2+\gamma/2}M_k^2, \quad t\in(-2\delta,\delta_{k+1}],$$

and

$$\|J_h\nabla\cdot(v\eta)\|_{H^{-\gamma/2}} \leq C\left(\frac{2\pi}{h}\right)h^{-2+\gamma/2}M_{k+1}\|\eta(t)\|_{L^2}, \quad t\in I_{k+1}.$$

Upon collecting these estimates, returning to (4.20), we apply (3.1), and (4.4) to obtain

$$\|(J_h\partial_t\eta)(t)\|_{H^{-\gamma/2}} \leq C\left(\frac{2\pi}{h}\right)\left(1+\frac{\mu h^\gamma}{\kappa}\right)^2\frac{\kappa}{h^{\gamma/2}}\left(1+\frac{1}{\kappa}\frac{1}{h^{2-\gamma}}\right)(1+M_k+R_{L^2})^2,$$

for $t\in(-2\delta,\delta_{k+1}]$, as well as

$$\begin{aligned}
\|(J_h\partial_t\eta)(t)\|_{H^{-\gamma/2}} &\leq C\left(\frac{2\pi}{h}\right)\kappa\|\eta(t)\|_{H^{\gamma/2}} + C\left(\frac{2\pi}{h}\right)\frac{M_{k+1}}{h^{2-\gamma/2}}\|\eta(t)\|_{L^2}\\
&\quad + C\left(\frac{2\pi}{h}\right)\frac{\kappa}{h^{\gamma/2}}\left(1+\frac{\mu h^\gamma}{\kappa}\right)^2(M_k+R_{L^2}),
\end{aligned}$$

for $t\in I_{k+1}$. Note that in collecting the terms we have used the fact that all constants and variables have been non-dimensionalized so that, for example, terms such as $1+1/(\kappa h^{2-\gamma})$ and $1+M_k+R_{L^2}$ make sense. Thus, upon squaring both sides of these inequalities, then applying Young's inequality and (4.17), we arrive at (4.18) and (4.19). □

4.2.3. *Growth during initial transient period.* Due to the delay, we must quantify bounds over the initial transient period during which the feedback effects from large scales can amplify the solution. Consider the definition of $\Gamma_{2,k}$ for $k=-1,0,1,\ldots$ given by (4.5). Observe that

$$\Gamma_{2,k-1}\leq\Gamma_{2,k}, \quad k\geq 0. \tag{4.21}$$

By (3.5), Lemma 4.2.1, and Corollary 4.2.2 we have

$$\|\eta(t)\|_{L^2}^2 + \kappa \int_{\delta_k}^t \|\eta(s)\|_{H^{\gamma/2}}^2\, ds \leq \Gamma_{2,k}^2, \quad t \in I_k, \quad k=-1,0,1.$$

It then follows from (4.21) that

$$\|\eta(t)\|_{L^2}^2 + \kappa \int_{\delta_k}^t \|\eta(s)\|_{H^{\gamma/2}}^2\, ds \leq \Gamma_{2,1}^2 \leq \Gamma_{2,1}^2 + \rho, \quad t \in I_k, \quad k=-1,0,1, \tag{4.22}$$

for any $\rho \geq 0$.

As we will see, the choice of ρ will be dictated by the estimates (4.37) and (4.41) below. In anticipation of this, consider the third definition of (4.4) given by

$$M_{L^2}^2 := \Gamma_{2,1}^2 + 8R_{L^2}^2. \tag{4.23}$$

Then (4.22) implies

$$\|\eta(t)\|_{L^2}^2 + \frac{\kappa}{2} \int_{\delta_k}^t \|\eta(s)\|_{H^{\gamma/2}}^2\, ds \leq M_{L^2}^2, \quad t \in I_k, \quad k=-1,0,1. \tag{4.24}$$

Therefore, the conclusion of Proposition 4.2.1 is that there is a choice of ρ such that the bound given by (4.24) propagates beyond the initial transient period, provided that δ is chosen small enough. In particular, Proposition 4.2.1 provides a more precise version of (4.24), which not only allows this bound to propagate through all times $t > 2\delta$, but in such a way that it eventually "forgets" the initializing function, g, as well.

We are now ready to prove Proposition 4.2.1.

4.2.4. *Proof of Proposition 4.2.1.* We proceed by induction on k. As we shall see shortly, by Lemma A.0.1 (ii), it suffices to show for $k \geq 2$ and $t \in I_k$ that

$$\begin{aligned}\|\eta(t)\|_{L^2}^2 &+ \frac{\kappa}{2} \int_{\delta_k}^t e^{-(\mu/2)(t-s)} \|\eta(t)\|_{H^{\gamma/2}}^2\, ds \\ &\leq \left(\|\eta(\delta_k)\|_{L^2}^2 - 8R_{L^2}^2\right) e^{-(\mu/2)(t-\delta_k)} + 8R_{L^2}^2.\end{aligned} \tag{4.25}$$

We proceed in three steps. Step I proves the base case when $k=2$ while Step II provides the induction step thereby completing the induction. Finally, Step III uses (4.25) along with Lemma A.0.1 (ii) to obtain (4.9) and (4.10) which finishes the proof.

I. Base case. Let $k=2$ and suppose $t \in I_2$. By Corollary 4.2.2 and (4.24) we have

$$\|\eta(t)\|_{L^2}^2 \leq \Gamma_{2,1}^2 + C\frac{\delta\mu^2}{\kappa}\left(\Gamma_{2,1}^2 + \widetilde{R}_{L^2}^2\right) = \Gamma_{2,2}^2, \quad t \in I_2.$$

It then follows from (4.24) and the second condition of (4.7) that

$$\|\eta(t)\|_{L^2}^2 \leq \Gamma_{2,2}^2 \leq M_{L^2}^2, \quad t \in (-2\delta, 3\delta]. \tag{4.26}$$

Multiply (4.3) by η, integrate over $\mathbb{T}^2$, and apply (2.18) to obtain

$$\frac{1}{2}\frac{d}{dt}\|\eta\|_{L^2}^2 + \kappa\|\Lambda^{\gamma/2}\eta\|_{L^2}^2 + \mu\|\eta\|_{L^2}^2 = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4$$

where

$$\mathcal{I}_1 = \int f\eta\, dx, \qquad \mathcal{I}_2 = \mu\int(\eta - J_h\eta)\eta\, dx, \qquad \mathcal{I}_4 = \mu\int(J_h^\delta\theta)\eta\, dx$$

and

$$\mathcal{I}_3 = \frac{\mu}{\delta}\int\int_{t-2\delta}^{t-\delta}\int_s^t [J_h\partial_\tau\eta(\tau)]\eta(t)\ d\tau ds dx.$$

Observe that by (2.10), Cauchy-Schwarz inequality, Young's inequality, and (3.2) we have

$$\mathcal{I}_1 \le \frac{1}{\kappa}\|\Lambda^{-\gamma/2}f\|_{L^2}^2 + \frac{\kappa}{4}\|\Lambda^{\gamma/2}\eta\|_{L^2}^2,$$

$$\mathcal{I}_2 \le C\mu h^{\gamma/2}\|\Lambda^{\gamma/2}\eta\|_{L^2}\|\eta\|_{L^2} \le \frac{\kappa}{8}\|\Lambda^{\gamma/2}\eta\|_{L^2}^2 + Ch^\gamma\frac{\mu^2}{\kappa}\|\eta\|_{L^2}^2$$

and

$$\mathcal{I}_4 \le \mu\|J_h^\delta\theta\|_{L^2}\|\eta\|_{L^2} \le C\mu\Theta_{L^2}^2 + \frac{\mu}{4}\|\eta\|_{L^2}^2.$$

Further estimating $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_4$ using (4.6), (3.2) and (4.4) gives

$$\frac{1}{2}\frac{d}{dt}\|\eta\|_{L^2}^2 + \frac{5\kappa}{8}\|\Lambda^{\gamma/2}\eta\|_{L^2}^2 + \frac{\mu}{2}\|\eta\|_{L^2}^2 \le \mu R_{L^2}^2 + \mathcal{I}_3. \tag{4.27}$$

To estimate $\mathcal{I}_3$, apply Fubini's theorem, Parseval's theorem, the Cauchy-Schwarz inequality, (2.8), and Young inequalities in the following sequence of estimates

$$\begin{aligned}
\mathcal{I}_3 &\le \frac{\mu}{\delta}\int_{t-2\delta}^{t-\delta}\int_s^t\left|\int (J_h\partial_\tau\eta(x,\tau))(\eta(x,t))\ dx\right| d\tau\ ds \\
&\le C\mu\frac{1}{\delta}\int_{t-2\delta}^{t-\delta}(t-s)\left(\int_s^t\|J_h\partial_\tau\eta(\tau)\|_{H^{-\gamma/2}}^2\ d\tau\right)^{1/2} ds\|\eta(t)\|_{H^{\gamma/2}} \\
&\le C\mu\left(\frac{1}{\delta}\int_{t-2\delta}^{t-\delta}(t-s)\int_{t-2\delta}^t\|J_h\partial_\tau\eta(\tau)\|_{H^{-\gamma/2}}^2\ d\tau\ ds\right)^{1/2}\|\eta(t)\|_{H^{\gamma/2}} \\
&\le \frac{1}{2}\left(C\frac{\delta\mu^2}{\kappa}\int_{t-2\delta}^t\|J_h\partial_s\eta(s)\|_{H^{-\gamma/2}}^2\ ds\right) + \frac{\kappa}{8}\|\eta(t)\|_{H^{\gamma/2}}^2.
\end{aligned} \tag{4.28}$$

Let

$$\mathcal{S}(t) := C\frac{\delta\mu^2}{\kappa}\int_{t-2\delta}^t\|J_h\partial_s\eta(s)\|_{H^{-\gamma/2}}^2\ ds. \tag{4.29}$$

Observe that $\mathcal{S}(t) = \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2(t)$, where for $\ell \ge 0$, we have defined

$$\mathcal{S}_\ell(t) := C\delta\frac{\mu^2}{\kappa}\int_{\delta_\ell}^t\|J_h\partial_s\eta(s)\|_{H^{-\gamma/2}}^2\ ds \qquad \text{and} \qquad \mathcal{S}_\ell := \mathcal{S}_\ell(\delta_{\ell+1}). \tag{4.30}$$

Returning to (4.27) and applying (4.28) and (4.6), we have

$$\frac{d}{dt}\|\eta(t)\|_{L^2}^2 + \kappa\|\eta(t)\|_{H^{\gamma/2}}^2 + \mu\|\eta(t)\|_{L^2}^2 \le \frac{2}{c_0}\frac{\kappa}{h^\gamma}R_{L^2}^2 + \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2(t). \tag{4.31}$$

To obtain bounds on $\mathcal{S}_0$ and $\mathcal{S}_1$ define

$$O_1(\delta^2) := C\delta^2\frac{\kappa^3}{h^{3\gamma}}\left(\frac{2\pi}{h}\right)^2\left(1 + \frac{1}{\kappa}\frac{1}{h^{2-\gamma}}\right)^2\left(1 + M_{L^2}^2 + R_{L^2}^2\right)^2 \tag{4.32}$$

so that, upon simplifying (4.18) with (4.6), we obtain from Lemma 4.2.3 and (4.26) that

$$\max\{\mathcal{S}_0, \mathcal{S}_1\} \le O_1(\delta^2). \tag{4.33}$$

To bound $\mathcal{S}_2(t)$ for $t \in I_2$, observe that by Lemma 4.2.3 and (4.26) we have

$$\mathcal{S}_2(t) \leq C_1(h)\delta \int_{\delta_2}^t \|\eta(s)\|_{L^2}^2 \, ds + C_2(h)\delta \int_{\delta_2}^t \|\eta(s)\|_{H^{\gamma/2}}^2 \, ds + O_2(\delta^2). \tag{4.34}$$

where, upon simplifying (4.19) with (4.6), we have defined

$$C_1(h) := C\kappa \left(\frac{2\pi}{h}\right)^2 \frac{M_{L^2}^2}{h^{4+\gamma}}, \qquad C_2(h) := C\frac{\kappa^3}{h^{2\gamma}}\left(\frac{2\pi}{h}\right)^2$$

and

$$O_2(\delta^2) := C\delta^2 \frac{\kappa^3}{h^{3\gamma}}\left(\frac{2\pi}{h}\right)^2 \left(M_{L^2}^2 + R_{L^2}^2\right).$$

Combining (4.33) and (4.34) then gives

$$\mathcal{S}(t) \leq C_1(h)\delta \int_{\delta_2}^t \|\eta(s)\|_{H^{\gamma/2}}^2 \, ds + C_2(h)\delta \int_{\delta_2}^t \|\eta(s)\|_{L^2}^2 \, ds + O_1(\delta^2) + O_2(\delta^2). \tag{4.35}$$

Observe that since $O_2(\delta^2) \leq O_1(\delta^2)$, it follows from the third condition on δ in (4.7) that

$$O_1(\delta^2) + O_2(\delta^2) \leq \frac{2}{c_0}\frac{\kappa}{h^\gamma} R_{L^2}^2.$$

Thus, upon returning to (4.28), we have

$$I_3 \leq C_1(h)\delta \int_{\delta_2}^t \|\eta(s)\|_{H^{\gamma/2}}^2 \, ds + C_2(h)\delta \int_{\delta_2}^t \|\eta(s)\|_{L^2}^2 \, ds + \frac{2}{c_0}\frac{\kappa}{h^\gamma}R_{L^2}^2 + \frac{\kappa}{8}\|\eta(t)\|_{H^{\gamma/2}}^2.$$

By applying the resulting bounds on $\mathcal{S}(t)$ in (4.31), we have for $t \in I_2$ that

$$\begin{aligned}
&\frac{d}{dt}\|\eta\|_{L^2}^2 + \mu\|\eta\|_{L^2}^2 + \kappa\|\eta\|_{H^{\gamma/2}}^2 \\
&\qquad \leq \frac{4}{c_0}\frac{\kappa}{h^\gamma}R_{L^2}^2 + C_1(h)\delta \int_{\delta_2}^t \|\eta(s)\|_{L^2}^2 \, ds + C_2(h)\delta \int_{\delta_2}^t \|\eta(s)\|_{H^{\gamma/2}}^2 \, ds.
\end{aligned} \tag{4.36}$$

Now observe that (4.8) ensures that (A.2) holds in Lemma A.0.1 with

$$a = \mu, \quad b = \kappa, \quad A = C_1, \quad B = C_2, \quad F = \frac{4}{c_0}\frac{\kappa}{h^\gamma}R_{L^2}^2.$$

Applying Lemma A.0.1 (i) then gives

$$\begin{aligned}
&\|\eta(t)\|_{L^2}^2 + \frac{\kappa}{2}\int_{\delta_2}^t e^{-(\mu/2)(t-s)}\|\eta(s)\|_{H^{\gamma/2}}^2 \, ds \\
&\qquad \leq \left(\|\eta(\delta_2)\|_{L^2}^2 - 8R_{L^2}^2\right) e^{-(\mu/2)(t-\delta_2)} + 8R_{L^2}^2, \quad t \in I_2,
\end{aligned} \tag{4.37}$$

which finishes the proof of the base case.

II. Induction Step. Suppose $k \geq 2$ and for each $\ell = 2, \dots, k$ and $t \in I_\ell$ that

$$\|\eta(t)\|_{L^2}^2 + \frac{\kappa}{2}\int_{\delta_\ell}^t e^{-(\mu/2)(t-s)}\|\eta(s)\|_{H^{\gamma/2}}^2 \, ds \leq \left(\|\eta(\delta_\ell)\|_{L^2}^2 - 8R_{L^2}^2\right) e^{-(\mu/2)(t-\delta_\ell)} + 8R_{L^2}^2 \tag{4.38}$$

We show the bound corresponding to $\ell = k+1$ holds for $t \in I_{k+1}$.

As already demonstrated, our choice of δ has been chosen so that the hypotheses of Lemma A.0.1 hold for the differential inequality (4.36). These hypotheses are also satisfied for the modified inequality obtained by replacing δ_2 by δ_ℓ for $\ell = 2, \dots, k$ which we write as

$$\begin{aligned}\frac{d}{dt}\|\eta\|_{L^2}^2 + \mu\|\eta\|_{L^2}^2 + \kappa\|\eta\|_{H^{\gamma/2}}^2 \\ \le \frac{4}{c_0}\frac{\kappa}{h^\gamma}R_{L^2}^2 + C_1(h)\delta\int_{\delta_\ell}^t \|\eta(s)\|_{L^2}^2\, ds + C_2(h)\delta\int_{\delta_\ell}^t \|\eta(s)\|_{H^{\gamma/2}}^2\, ds \end{aligned} \tag{4.39}$$

for $t \in I_\ell$. Now, dropping the integral in (4.38) and rewriting the last term yields

$$\|\eta(t)\|_{L^2}^2 \le \|\eta(\delta_\ell)\|_{L^2}^2 e^{-(\mu/2)(t-\delta_\ell)} + 8R_{L^2}^2\int_{\delta_\ell}^t \frac{2}{\mu}e^{-(\mu/2)(t-s)}ds \quad \text{for} \quad t \in I_\ell$$

so that by iterating part (ii) of Lemma A.0.1 for $\ell = 2, \dots, k$ we obtain

$$\|\eta(t)\|_{L^2}^2 \le \|\eta(\delta_2)\|_{L^2}^2 e^{-(\mu/2)(t-\delta_2)} + 8R_{L^2}^2\int_{\delta_2}^t \frac{2}{\mu}e^{-(\mu/2)(t-s)}ds \quad \text{for} \quad t \in (\delta_2, \delta_{k+1}].$$

Since $\|\eta(\delta_2)\|_{L^2}^2 \le \Gamma_{2,1}^2$ by (4.22), we immediately obtain (4.9) and in particular that

$$\|\eta(t)\|_{L^2}^2 \le \Gamma_{2,1}^2 + 8R_{L^2}^2 = M_{L^2}^2, \quad t \in (\delta_2, \delta_{k+1}]. \tag{4.40}$$

By Corollary 4.2.2 it follows that

$$\|\eta(t)\|_{L^2}^2 \le M_{L^2}^2 + C\frac{\delta\mu^2}{\kappa}\left(M_{L^2}^2 + \widetilde{R}_{L^2}^2\right), \quad t \in I_{k+1}.$$

Thus, by the second condition in (4.7) we have

$$\|\eta(t)\|_{L^2}^2 + \frac{\kappa}{4}\int_{\delta_{k+1}}^t \|\eta(s)\|_{H^{\gamma/2}}\, ds \le M_{L^2}^2, \quad t \in I_{k+1}. \tag{4.41}$$

Now proceed exactly as in the base case, this time making use of the bounds (4.40) and (4.41). Indeed, we may derive (4.31) as before. Then, since $t \in I_{k+1}$, we may split the time integral over three regions:

$$\int_{t-2\delta}^t \le \int_{I_{k-1}} + \int_{I_k} + \int_{\delta_{k+1}}^t .$$

Over I_{k-1} and I_k, Lemma 4.2.3 and (4.41) imply (4.33) for $\mathcal{S}_{k-1}$ and $\mathcal{S}_k$. Over I_{k+1}, we have (4.41), so that Lemma 4.2.3 implies (4.34) for $\mathcal{S}_{k+1}(t)$. We then deduce (4.35) for $t \in I_{k+1}$, which leads to the differential inequality (4.39) with $\ell = k+1$. Applying Lemma A.0.1 (i) as before then yields

$$\begin{aligned}\|\eta(t)\|_{L^2}^2 + \frac{\kappa}{2}\int_{\delta_{k+1}}^t e^{-(\mu/2)(t-s)}\|\eta(s)\|_{H^{\gamma/2}}^2\, ds \\ \le \left(\|\eta(\delta_{k+1})\|_{L^2}^2 - 8R_{L^2}^2\right)e^{-(\mu/2)(t-\delta_{k+1})} + 8R_{L^2}^2\end{aligned} \tag{4.42}$$

for $t \in I_{k+1}$ thus completing the induction.

III. Finish the Proof. We have already obtained (4.9) for all values of k by iterating Lemma A.0.1 (ii) as part of the induction step. To obtain (4.10) drop the first term in (4.25) and keep the integral. Consequently, we may then deduce that

$$\frac{\kappa}{2}e^{-(\mu/2)(t-\delta_k)}\int_{\delta_k}^t \|\eta(s)\|_{H^{\gamma/2}}^2\,ds \le \left(\|\eta(\delta_k)\|_{L^2}^2 - 8R_{L^2}^2\right)e^{-(\mu/2)(t-\delta_k)} + 8R_{L^2}^2, \quad t\in I_k.$$

Since the first condition in (4.7) and (4.6) together imply $e^{(\mu/2)\delta}\le 2$, it follows from (4.4) and (4.22) that

$$\frac{\kappa}{4}\int_{I_k}\|\eta(s)\|_{H^{\gamma/2}}^2\,ds \le \|\eta(\delta_k)\|_{L^2}^2 + 8R_{L^2}^2(e^{(\mu/2)\delta}-1) \le M_{L^2}^2. \tag{4.43}$$

This completes the proof. ☐

Remark 4.2. *We point out that the energy estimates in L^p and H^σ will not proceed along these lines, the reason being that even if one were to do so, the resulting bounds would still not be independent of h. So long as these bounds are uniform-in-time, however, we will be able to use them strengthen the topology of convergence in which the synchronization takes place via interpolation. We will thus be content with rather modest bounds in L^p and H^σ.*

4.3. **L^2 to L^p uniform bounds.** We will prove the following "good" bound:

Proposition 4.3.1. *Let $F_{L^p}, \Theta_{L^p}, M_{L^2}$ be given by (2.20), (3.1), (4.4), respectively. Define*

$$\widetilde{R}_{L^p}^p(h) := F_{L^p}^p + \Theta_{L^p}^p + \widetilde{C}(h,p)^p M_{L^2}^p, \tag{4.44}$$

where

$$\widetilde{C}(h,p)^p := 1 + h^{-(p-2)}. \tag{4.45}$$

Let $c_0>0$ be any constant. Suppose that

$$\frac{\mu h^\gamma}{\kappa} \le \frac{1}{c_0}. \tag{4.46}$$

Then there exists a constant $C_0>0$, depending on c_0, such that

$$\|\eta(t)\|_{L^p}^p \le \left(\Gamma_{L^p}^p - \left(\frac{C_0}{h^\gamma}\right)^p \widetilde{R}_{L^p}^p\right)e^{-\kappa t} + \left(\frac{C_0}{h^\gamma}\right)^p \widetilde{R}_{L^p}^p, \quad t\ge 0,$$

In particular,

$$\|\eta(t)\|_{L^p} \le \widetilde{M}_{L^p}, \quad t\ge 0,$$

where

$$\widetilde{M}_{L^p}(h)^p := \Gamma_{L^p}^p + \left(\frac{C_0}{h^\gamma}\right)^p \widetilde{R}_{L^p}^p(h). \tag{4.47}$$

Proof. Observe that by (3.5), we have

$$\|\eta(t)\|_{L^p} \le \Gamma_{L^p}, \quad t\in I_{-1}.$$

For $t\ge 0$, the evolution of $\|\eta(t)\|_{L^p}$ is obtained by multiplying (4.2) by $\eta|\eta|^{p-2}$, integrating over $\mathbb{T}^2$, applying Proposition 2.4.1, Hölder's inequality, Young's inequality, and (2.20) to obtain

$$\frac{1}{p}\frac{d}{dt}\|\eta\|_{L^p}^p+\frac{2\kappa}{p}\|\Lambda^{\gamma/2}(\eta^{p/2})\|_{L^2}^2\leq C^p\frac{\kappa}{p}F_{L^p}^p+C^p\frac{\kappa}{p}\left(\frac{\mu}{\kappa}\right)^p\left(\|J_h^\delta\eta\|_{L^p}^p+\|J_h^\delta\theta\|_{L^p}^p\right)+\frac{\kappa}{2p}\|\eta\|_{L^p}^p \tag{4.48}$$

Applying Hölder's inequality, the Fubini-Tonelli theorem, (2.9) with $q=2$, and Proposition 4.2.1 we have

$$\|J_h^\delta\eta\|_{L^p}^p\leq\frac{1}{\delta}\int_{t-2\delta}^{t-\delta}\|J_h\eta(s)\|_{L^p}^p ds\leq C^ph^{2-p}M_{L^2}^p. \tag{4.49}$$

Similarly, by (3.2), $\|J_h^\delta\theta\|_{L^p}^p\leq C_J^p\Theta_{L^p}^p$. Upon defining

$$\langle\eta^{p/2}\rangle_{\mathbb{T}^2}=\frac{1}{4\pi^2}\int_{\mathbb{T}^2}\eta^{p/2}dx$$

observe that

$$\|\eta\|_{L^p}^p-(4\pi^2)^{-1}\|\eta\|_{L^{p/2}}^p=\|\eta^{p/2}-\langle\eta^{p/2}\rangle_{\mathbb{T}^2}\|_{L^2}^2\leq C(2\pi)^\gamma\|\Lambda^{\gamma/2}(\eta^{p/2})\|_{L^2}^2. \tag{4.50}$$

Note that the constant $(2\pi)^\gamma$ carries the units of L^γ; however, as $L=2\pi$ throughout this paper we avoid keeping track of the dimensions in this case, and simply denote the prefactor $C(2\pi)^\gamma$ by C. By interpolation, Young's inequality, and Hölder's inequality we have

$$\|\eta\|_{L^{p/2}}^p\leq\|\eta\|_{L^p}^{\frac{p(p-2)}{p-1}}\|\eta\|_{L^1}^{\frac{p}{p-1}}\leq C^p\left(\frac{p-2}{p-1}\right)^{p-2}M_{L^2}^p+\pi^2\|\eta\|_{L^p}^p. \tag{4.51}$$

Upon combining (4.49), (4.50), (4.51), (4.45) and returning to (4.48), we arrive at

$$\frac{d}{dt}\|\eta\|_{L^p}^p+\kappa\|\eta\|_{L^p}^p\leq C^p\frac{\kappa}{h^{\gamma p}}\left(\frac{\mu h^\gamma}{\kappa}\right)^p\left(F_{L^p}^p+\widetilde{C}(h,p)^pM_{L^2}^p+\Theta_{L^p}^p\right).$$

An application of (4.46) and Gronwall's inequality completes the proof. □

4.4. **Uniform H^σ-estimates.** As in the previous section, we obtain "good" H^σ-bounds without appealing to time-derivative estimates.

Proposition 4.4.1. *Let M_{L^2} be given by (4.23) and let $\Theta_{H^\sigma},\widetilde{M}_{L^p}$ be given by (2.22), (4.47), respectively. Define*

$$\widetilde{\Xi}_{L^p}(h):=\left(\frac{\widetilde{M}_{L^p}(h)}{\kappa}\right)^{\frac{\sigma}{\gamma-1-2/p}}, \tag{4.52}$$

as well as

$$F_{H^{\sigma-\gamma/2}}:=\frac{1}{\kappa}\|f\|_{H^{\sigma-\gamma/2}}\quad\text{and}\quad R_{H^\sigma}^2:=F_{H^{\sigma-\gamma/2}}^2+\Theta_{L^2}^2. \tag{4.53}$$

Let $c_0>0$ be the constant given in Proposition 4.2.1. Suppose that

$$\frac{\mu h^\gamma}{\kappa}\leq\frac{1}{c_0}. \tag{4.54}$$

Then there exists a constant $C_0>0$, depending on c_0, such that

$$\|\eta(t)\|_{H^\sigma}^2\leq\Gamma_{H^\sigma}^2e^{-\kappa t}+C_0\left[\left(\widetilde{\Xi}_{L^p}^{\frac{2\sigma+\gamma}{\sigma}}+\frac{1}{h^{2\gamma}}\right)M_{L^2}^2+\frac{1}{h^{2\gamma}}R_{H^\sigma}^2\right]\left(1-e^{-\kappa t}\right),$$

holds for $t \geq 0$ and $\sigma \leq \gamma/2$, and

$$\|\eta(t)\|_{H^\sigma}^2 \leq \Gamma_{H^\sigma}^2 e^{-\kappa t} + C_0\left[\left(\widetilde{\Xi}_{L^p}^{\frac{2\sigma+\gamma}{\sigma}} + \frac{1}{h^{2\sigma+\gamma}}\right)M_{L^2}^2 + \frac{1}{h^{2\sigma+\gamma}} + \frac{1}{h^{2\sigma+\gamma}}R_{H^\sigma}^2\right]\left(1-e^{-\kappa t}\right),$$

holds for $t \geq 0$ and $\sigma > \gamma/2$.

Proof. Suppose $t \geq 0$. We multiply (4.2) by $\Lambda^{2\sigma}\eta$ and integrate over $\mathbb{T}^2$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|\eta\|_{H^\sigma}^2 + \kappa\|\eta\|_{H^{\sigma+\gamma/2}}^2\\ &= -\int v\cdot\nabla\eta\Lambda^{2\sigma}\eta\, dx + \int f\Lambda^{2\sigma}\eta\, dx + \mu\int J_h^\delta\eta\Lambda^{2\sigma}\eta\, dx + \mu\int J_h^\delta\theta\Lambda^{2\sigma}\eta\, dx\\ &= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4. \end{aligned} \tag{4.55}$$

We estimate $\mathcal{J}_1$ with Hölder's inequality, interpolation, and Young's inequality as in [18, 33], and invoke (4.52) to obtain

$$|\mathcal{J}_1| \leq C\|\Lambda^{\sigma+\gamma/2}\eta\|_{L^2}^{\frac{2\sigma-(\gamma-1-2/p)}{\sigma}}\|\Lambda^{\gamma/2}\eta\|_{L^2}^{\frac{\gamma-1-2/p}{\sigma}}\|\eta\|_{L^p} \leq \frac{\kappa}{10}\|\eta\|_{H^{\sigma+\gamma/2}}^2 + C\widetilde{\Xi}_{L^p}^2(\kappa\|\eta\|_{H^{\gamma/2}}^2).$$

Note that $(H1)$, $(H2)$ and $(H3)$ are needed for the interpolation. We interpolate once more to obtain

$$\|\eta\|_{H^{\gamma/2}} \leq C\|\eta\|_{H^{\sigma+\gamma/2}}^{\frac{\gamma/2}{\sigma+\gamma/2}}\|\eta\|_{L^2}^{\frac{\sigma}{\sigma+\gamma/2}}.$$

Thus, by Young's inequality, we have

$$C\widetilde{\Xi}_{L^p}^2(\kappa\|\eta\|_{H^{\gamma/2}}^2) \leq C\kappa\|\eta\|_{H^{\sigma+\gamma/2}}^{\frac{\gamma}{\sigma+\gamma/2}}(\widetilde{\Xi}_{L^p}^2\|\eta\|_{L^2}^{\frac{2\sigma}{\sigma+\gamma/2}}) \leq \frac{\kappa}{10}\|\eta\|_{H^{\sigma+\gamma/2}}^2 + C\kappa\widetilde{\Xi}_{L^p}^{2+\gamma/\sigma}M_{L^2}^2.$$

For $\mathcal{J}_2$, we make the familiar estimate through Parseval's theorem, the Cauchy-Schwarz inequality, and then (4.53) to obtain

$$|\mathcal{J}_2| \leq \kappa F_{H^{\sigma-\gamma/2}}^2 + \frac{\kappa}{10}\|\eta\|_{H^{\sigma+\gamma/2}}^2.$$

For $\mathcal{J}_3$ and $\mathcal{J}_4$, we consider two cases: $\sigma \leq \gamma/2$ and $\sigma > \gamma/2$.

Case: $\sigma \leq \gamma/2$: It follows from Fubini's theorem, Hölder's inequality, (2.8), and the Poincarè inequality that

$$\begin{aligned}\left|\int J_h^\delta\eta\Lambda^{2\sigma}\eta\, dx\right| &\leq \frac{1}{\delta}\int_{t-2\delta}^{t-\delta}\|J_h\eta(s)\|_{L^2}\|\eta(t)\|_{H^{2\sigma}}\, ds\\ &\leq \left(\sup_{s\in I_{k-2}\cup I_{k-1}}\|\eta(s)\|_{L^2}\right)\|\eta(t)\|_{H^{\sigma+\gamma/2}}\\ &\leq CM_{L^2}\|\eta(t)\|_{H^{\sigma+\gamma/2}}\end{aligned}$$

Thus, by Young's inequality we have

$$|\mathcal{J}_3| \leq C\frac{\mu^2}{\kappa}M_{L^2}^2 + \frac{\kappa}{10}\|\eta\|_{H^{\sigma+\gamma/2}}^2.$$

Similarly, since $\theta_{-2\delta} \in \mathcal{B}_{L^2}$ by $(H5)$, by (2.22) we have

$$|\mathcal{J}_4| \leq C\frac{\mu^2}{\kappa}\Theta_{L^2}^2 + \frac{\kappa}{10}\|\eta\|_{H^{\sigma+\gamma/2}}^2.$$

Therefore, upon returning to (4.55), then applying the estimates for $\mathcal{J}_1$ through $\mathcal{J}_4$ and the Poincaré inequality gives

$$\frac{d}{dt}\|\eta\|_{H^\sigma}^2 + \kappa\|\eta\|_{H^\sigma}^2 \leq 8\kappa F_{H^{\sigma-\gamma/2}}^2 + C\kappa\widetilde{\Xi}_{L^p}^{2+\gamma/\sigma}M_{L^2}^2 + C\frac{\mu^2}{\kappa}\left(M_{L^2}^2 + \Theta_{L^2}^2\right).$$

Then the Gronwall inequality implies

$$\begin{aligned}
&\|\eta(t)\|_{H^\sigma}^2 + \int_0^t e^{-\kappa(t-s)}\|\eta(s)\|_{H^{\sigma+\gamma/2}}^2\,ds \\
&\qquad \leq \Gamma_{H^\sigma}^2 e^{-\kappa t} + C\left[\left(\widetilde{\Xi}_{L^p}^{2+\gamma/\sigma} + \frac{1}{h^{2\gamma}}\right)M_{L^2}^2 + \frac{1}{h^{2\gamma}}R_{H^\sigma}^2\right]\left(1-e^{-\kappa t}\right),
\end{aligned}$$

as desired.

Case: $\sigma > \gamma/2$: Observe that by Fubini's theorem, Plancherel's theorem, Hölder's inequality, (2.12), Proposition 4.2.1, and Young's inequality we have

$$\begin{aligned}
|\mathcal{J}_3| &\leq \mu\|J_h^\delta\eta(t)\|_{H^{\sigma-\gamma/2}}\|\eta(t)\|_{H^{\sigma+\gamma/2}} \\
&\leq \mu\left(\frac{1}{\delta}\int_{t-2\delta}^{t-\delta}\|J_h\eta(s)\|_{H^{\sigma-\gamma/2}}\,ds\right)\|\eta(t)\|_{H^{\sigma+\gamma/2}} \\
&\leq C\mu h^{-(\sigma-\gamma/2)}\left(\sup_{s\in I_{k-2}\cup I_{k-1}}\|\eta(s)\|_{L^2}\right)\|\eta(t)\|_{H^{\sigma+\gamma/2}} \\
&\leq C\frac{\mu}{h^{2\sigma}}\frac{\mu h^\gamma}{\kappa}M_{L^2}^2 + \frac{\kappa}{10}\|\eta(t)\|_{H^{\sigma+\gamma/2}}^2.
\end{aligned}$$

Similarly, since $\theta_{-2\delta} \in \mathcal{B}_{L^2}$ by $(H5)$, by (2.22) we have

$$|\mathcal{J}_4| \leq C\frac{\mu}{h^{2\sigma}}\frac{\mu h^\gamma}{\kappa}\Theta_{L^2}^2 + \frac{\kappa}{10}\|\eta(t)\|_{H^{\sigma+\gamma/2}}^2.$$

Therefore, upon returning to (4.55), then applying the estimates for $\mathcal{J}_1$ through $\mathcal{J}_4$ and the Poincaré inequality gives

$$\frac{d}{dt}\|\eta\|_{H^\sigma}^2 + \kappa\|\eta\|_{H^\sigma}^2 \leq 8\kappa F_{H^{\sigma+\gamma/2}}^2 + C\kappa\widetilde{\Xi}_{L^p}^{2+\gamma/\sigma}M_{L^2}^2 + C\frac{\mu^2}{h^{2\sigma-\gamma}\kappa}\left(M_{L^2}^2 + \Theta_{L^2}^2\right),$$

Then the Gronwall inequality and (4.54) implies

$$\begin{aligned}
&\|\eta(t)\|_{H^\sigma}^2 + \int_0^t e^{-\kappa(t-s)}\|\eta(s)\|_{H^{\sigma+\gamma/2}}^2\,ds \\
&\qquad \leq \Gamma_{H^\sigma}^2 e^{-\kappa t} + C\left[F_{H^{\sigma-\gamma/2}}^2 + \left(\widetilde{\Xi}_{L^p}^{\frac{2\sigma+\gamma}{\sigma}} + \frac{1}{h^{2\sigma+\gamma}}\right)M_{L^2}^2 + \frac{1}{h^{2\sigma+\gamma}}\Theta_{L^2}^2\right]\left(1-e^{-\kappa t}\right),
\end{aligned}$$

as desired. □

5. Proof of Theorem 2

We are left to establish the synchronization of η to the reference solution θ. We point out that the uniform L^2 bounds will be used in a crucial way to establish suitable control on the time derivative and guarantee synchronization in a rather weak topology, i.e., the $H^{-1/2}$ topology. We then make use of the uniform L^p and H^σ-bounds in order to strengthen the regularity of the convergence of the synchronization by interpolation.

Consider the difference $\zeta := \eta - \theta$, where $\theta \in \mathcal{B}_{H^\sigma}$ and η is the unique strong solution of (4.2). Observe that (3.5) ensures that ζ is defined for $t \in I_{-1}$. The evolution of ζ is given by:

$$\begin{cases} \partial_t\zeta + \kappa\Lambda^\gamma\zeta + w\cdot\nabla\zeta + w\cdot\nabla\theta + u\cdot\nabla\zeta = -\mu J_h^\delta\zeta, \\ w = \mathcal{R}^\perp\zeta, \quad \zeta(t) = g(t) - \theta(t), \quad t \in (-2\delta, 0]. \end{cases} \tag{5.1}$$

It will be convenient to work at the regularity level of the stream function of ζ. Thus, we define

$$\psi := -\Lambda^{-1}\zeta. \tag{5.2}$$

5.1. **Synchronization.** Our main claim is the following.

Proposition 5.1.1. *Let $\Theta_{H^\sigma}, \Theta_{L^2}, \Theta_{L^p}$ and M_{L^2} be given by (2.22), (3.1) and (4.4). Define*

$$\Xi_{L^p} := \left(\frac{\Theta_{L^p}}{\kappa}\right)^{\frac{\gamma/2}{\gamma-1-2/p}}, \qquad \Psi := 4\sqrt{2}M_{L^2}, \tag{5.3}$$

$$\widetilde{C}_1(h) := \kappa^3\left(\frac{2\pi}{h}\right)^2\left(\frac{1}{h^{1+3\gamma}} + \frac{1}{\kappa^2}\frac{1}{h^{4+\gamma}}\right)(1 + M_{L^2}^2 + \Theta_{L^2}^2) \tag{5.4}$$

and

$$\widetilde{C}_2(h) := \left(\frac{2\pi}{h}\right)^2\frac{1}{h^{4+\gamma}}\left(M_{L^2}^2 + \Theta_{L^2}^2\right). \tag{5.5}$$

There exist constants $c_0, c_0', c_1, c_2 \geq 1$ such that if h, μ satisfies

$$\frac{1}{c_0'}\Xi_{L^p}^2 \leq \frac{\mu}{\kappa} \leq \frac{1}{c_0}h^{-\gamma}. \tag{5.6}$$

and δ is chosen to satisfy

$$\frac{1}{\kappa}\frac{\delta^2\widetilde{C}_1(h) + \delta\widetilde{C}_2(h)}{\Xi_{L^p}^2} \leq \frac{c_0'}{c_1} \quad \text{and} \quad \delta \leq \frac{1}{c_2^{1/2}}\min\left\{c_1^{1/2}, \frac{h^\gamma}{\kappa}, \frac{\kappa^{1/2}}{\widetilde{C}_2(h)^{1/2}}\right\}. \tag{5.7}$$

then

$$\|\psi(t)\|_{H^{1/2}}^2 \leq \Psi^2 e^{-(\mu/4)(t-\delta)}, \quad t \geq 2\delta. \tag{5.8}$$

To prove this, we proceed as in section 4.2.4 and make some preparatory estimates.

5.1.1. *Control of temporal oscillations at a fixed spatial scale.*

Lemma 5.1.1. *Let Θ_{L^2} and M_{L^2} be given by (3.1), (4.23), respectively. Let $c_0 > 0$ be a constant. Suppose that*

$$\frac{\mu h^\gamma}{\kappa} \leq \frac{1}{c_0}, \tag{5.9}$$

Then there exists a constant $C_0>0$, depending on c_0, such that

$$\begin{aligned}\|\partial_t J_h\zeta(t)\|_{H^{-\gamma/2}}^2 &\leq C_0\left(\frac{2\pi}{h}\right)^2\frac{\kappa^2}{h^{1+\gamma}}\left(\frac{1}{\delta}\int_{t-2\delta}^{t-\delta}\|\psi(s)\|_{H^{1/2}}^2\,ds\right)\\ &\quad+C_0\kappa^2\left(\frac{2\pi}{h}\right)^2\left(M_{L^2}^2+\Theta_{L^2}^2\right)\left(\frac{1}{h^{\gamma+1}}+\frac{1}{\kappa^2}\frac{1}{h^{4-\gamma}}\right)\|\psi\|_{H^{1/2}}^2 \qquad (5.10)\\ &\quad+C_0\left(\frac{2\pi}{h}\right)^2\left(M_{L^2}^2+\Theta_{L^2}^2\right)h^{-(4-\gamma)}\|\psi\|_{H^{(\gamma+1)/2}}^2,\quad \textit{for}\quad t>-2\delta.\end{aligned}$$

Proof. Let $t>-2\delta$. Applying J_h to (5.1) and taking the $H^{-\gamma/2}$-norm yields

$$\begin{aligned}\|\partial_t J_h\zeta\|_{H^{-\gamma/2}} &\leq \kappa\|J_h\Gamma^\gamma\zeta\|_{H^{-\gamma/2}}+\mu\|J_hJ_h^\delta\zeta\|_{H^{-\gamma/2}}\\ &\quad+\|J_h\nabla\cdot(w\zeta)\|_{H^{-\gamma/2}}+\|J_h\nabla\cdot(w\theta)\|_{H^{-\gamma/2}}+\|J_h\nabla\cdot(u\zeta)\|_{H^{-\gamma/2}}.\end{aligned}$$

Observe that by $(H1)$, we have $\gamma/2<1$, so that by (B.16), we have

$$C_I(\gamma/2,h)=C\left(\frac{2\pi}{h}\right).$$

By (2.14), (3.1), (5.2), the Cauchy-Schwarz inequality, and (5.9) we have

$$\begin{aligned}\kappa\|J_h\Lambda^\gamma\zeta(t)\|_{H^{-\gamma/2}} &\leq C\kappa\left(\frac{2\pi}{h}\right)h^{\gamma/2-\gamma-1/2}\|\Lambda^\gamma\zeta\|_{H^{-\gamma-1/2}}\\ &\leq C\kappa\left(\frac{2\pi}{h}\right)h^{-(\gamma+1)/2}\|\psi\|_{H^{1/2}},\\ \mu\|J_hJ_h^\delta\zeta(t)\|_{H^{-\gamma/2}} &\leq \frac{\mu}{\delta}\int_{t-2\delta}^{t-\delta}\|J_h\zeta(s)\|_{H^{-\gamma/2}}\,ds\\ &\leq C\left(\frac{2\pi}{h}\right)\mu h^{(\gamma-1)/2}\left(\frac{1}{\delta}\int_{t-2\delta}^{t-\delta}\|\zeta(s)\|_{H^{-1/2}}\,ds\right)\\ &\leq C\left(\frac{2\pi}{h}\right)\left(\frac{\mu}{\delta^{1/2}}\right)h^{(\gamma-1)/2}\left(\int_{t-2\delta}^{t-\delta}\|\psi(s)\|_{H^{1/2}}^2\,ds\right)^{1/2}\\ &\leq C\left(\frac{2\pi}{h}\right)\frac{\kappa}{h^{(1+\gamma)/2}}\left(\frac{1}{\delta}\int_{t-2\delta}^{t-\delta}\|\psi(s)\|_{H^{1/2}}^2\,ds\right)^{1/2}.\end{aligned}$$

To estimate the nonlinear terms, we apply (2.16), the Cauchy-Schwarz inequality, (3.1), Proposition 4.2.1, (5.2), interpolation, and Young's inequality. For instance, we have

$$\begin{aligned}\|J_h\nabla\cdot(w\zeta)\|_{H^{-\gamma/2}} &\leq C\left(\frac{2\pi}{h}\right)h^{-2+\gamma/2}\|(\mathcal{R}^\perp\zeta)\zeta\|_{L^1}\\ &\leq C\left(\frac{2\pi}{h}\right)h^{-2+\gamma/2}\|\zeta\|_{L^2}^2\\ &\leq C\left(\frac{2\pi}{h}\right)h^{-2+\gamma/2}(M_{L^2}+\Theta_{L^2})\left(\|\psi\|_{H^{(\gamma+1)/2}}+\|\psi\|_{H^{1/2}}\right).\end{aligned}$$

Similarly

$$\|J_h\nabla\cdot(w\theta)\|_{H^{-\gamma/2}},\|J_h\nabla\cdot(u\zeta)\|_{H^{-\gamma/2}}\leq C\left(\frac{2\pi}{h}\right)h^{-2+\gamma/2}\Theta_{L^2}\left(\|\psi\|_{H^{(\gamma+1)/2}}+\|\psi\|_{H^{1/2}}\right).$$

Therefore, by summing each of these estimates, we arrive at (5.10). as desired. ☐

5.1.2. *Growth during transient period.* We introduce the following notation: Let $\alpha \in (0,1)$ and $\ell \in \mathbb{Z}$, then define

$$\delta_{\alpha\ell} := \alpha\ell\delta.$$

Observe that by the Poincarè inequality, (4.22) implies

$$\|\psi(t)\|_{H^{1/2}}^2 + \kappa\int_{I_k}^t e^{-(\mu/2)(t-s)}\|\psi(s)\|_{H^{1/2}}^2\, ds \le M_{L^2}^2 e^{(\mu/2)\delta} \le 32M_{L^2}^2, \quad t \in I_k, \quad k \ge -1.$$

Clearly, one has

$$M_{L^2}^2 e^{(\mu/2)\delta} \le M_{L^2}^2 e^{(5\mu/2)\delta} e^{-(\mu/2)(t-\delta_{k/2})} \le 32M_{L^2}^2 e^{-(\mu/2)(t-\delta_{k/2})}, \quad t \in I_k, \quad k = -1, 0, 1.$$

Then

$$\|\psi(t)\|_{H^{1/2}}^2 + \frac{\kappa}{2}\int_{\delta_k}^t e^{-(\mu/2)(t-s)}\|\psi(s)\|_{H^{(\gamma+1)/2}}^2\, ds \le \Psi^2 e^{-(\mu/2)(t-\delta_{k/2})}, \quad t \in I_k, \quad k = -1, 0, 1. \tag{5.11}$$

We are now ready to prove the synchronization property.

5.2. Proof of Proposition 5.1.1.

Proof of Proposition 5.1.1. We proceed by induction on k with the base case, $k = 1$, as established by (5.11). Suppose that the following holds:

$$\|\psi(t)\|_{H^{1/2}}^2 + \frac{\kappa}{2}\int_{\delta_k}^t e^{-(\mu/2)(t-s)}\|\psi(s)\|_{H^{(\gamma+1)/2}}^2\, ds \le \Psi^2 e^{-(\mu/2)(t-\delta_{k/2})} \tag{5.12}$$

for $t \in I_\ell$ and $\ell = 0, \dots, k$, where Ψ is given by (5.3). We show that this corresponding bound holds over I_{k+1} as well.

Let $t \in I_{k+1}$, $k \ge 1$. Multiply (5.1) by ψ and integrate over $\mathbb{T}^2$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|\psi\|_{H^{1/2}}^2 + \kappa\|\psi\|_{H^{(\gamma+1)/2}}^2 + \mu\|\psi\|_{H^{1/2}}^2\\ &= \int (u\cdot\nabla\psi)\zeta dx + \mu\int(\zeta - J_h\zeta)\psi dx + \mu\int(J_h\zeta - J_h^\delta\zeta)\psi\, dx\\ &= \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3.\end{aligned} \tag{5.13}$$

Note that we have used the orthogonality property, i.e., $\mathcal{R}^\perp f\cdot\mathcal{R}f = 0$.

We refer to [18, 44] to estimate $\mathcal{K}_1$. In particular, by Hölder's inequality, the Calderòn-Zygmund theorem, and Sobolev embedding, $H^{1/p} \hookrightarrow L^q$, we have

$$|\mathcal{K}_1| \le C\|u\|_{L^p}\|\zeta\|_{L^q}\|\nabla\psi\|_{L^q} \le C\|\theta\|_{L^p}\|\psi\|_{H^{1+1/p}}^2, \tag{5.14}$$

where $1/p + 2/q = 1$. Since $p > 2/(\gamma-1)$ by $(H3)$, by interpolation we have

$$\|\psi\|_{H^{1+1/p}} \le C\|\psi\|_{H^{(\gamma+1)/2}}^{\frac{1+2/p}{\gamma}}\|\psi\|_{H^{\frac{1}{2}}}^{\frac{\gamma-1-2/p}{\gamma}}.$$

Thus, by Young's inequality we obtain

$$|\mathcal{K}_1| \le \frac{\kappa}{6}\|\psi\|_{H^{(\gamma+1)/2}}^2 + C\kappa\Xi_{L^p}^2\|\psi\|_{H^{1/2}}^2.$$

where Ξ_{L^p} is given by (5.3). We estimate $\mathcal{K}_2$ with the Parseval's theorem, the Cauchy-Schwarz inequality, (2.10), (5.2), interpolation, and Young's inequality to get

$$\begin{aligned}|\mathcal{K}_2| &\leq \mu\|\zeta - J_h\zeta\|_{H^{-\gamma/2}}\|\psi\|_{H^{\gamma/2}}\\ &\leq \mu h^{\gamma/2}\|\psi\|_{H^1}\|\psi\|_{H^{\gamma/2}}\\ &\leq \mu h^{\gamma/2}\|\psi\|_{H^{(\gamma+1)/2}}\|\psi\|_{H^{1/2}}\\ &\leq \frac{\kappa}{6}\|\psi\|^2_{H^{(\gamma+1)/2}} + C\frac{\mu^2 h^\gamma}{\kappa}\|\psi\|^2_{H^{1/2}}.\end{aligned}$$

For $\mathcal{K}_3$, similar to (4.28), we estimate

$$\begin{aligned}|\mathcal{K}_3| &\leq C\delta\frac{\mu^2}{\kappa}\int_{t-2\delta}^t \|\partial_s J_h\zeta(s)\|^2_{H^{-\gamma/2}}\,ds + \frac{\kappa}{4}\|\psi\|^2_{H^{\gamma/2}}\\ &\leq C\delta\frac{\mu^2}{\kappa}\int_{t-2\delta}^t \|\partial_s J_h\zeta(s)\|^2_{H^{-\gamma/2}}\,ds + \frac{\kappa}{6}\|\psi\|^2_{H^{(\gamma+1)/2}}.\end{aligned}$$

Returning to (5.13) and combining $\mathcal{K}_1$ through $\mathcal{K}_3$, then applying (5.6) with c_0 and c_0' sufficiently large, we get

$$\frac{d}{dt}\|\psi\|^2_{H^{1/2}} + \kappa\|\psi\|^2_{H^{(\gamma+1)/2}} + \mu\|\psi\|^2_{H^{1/2}} \leq \widetilde{\mathcal{S}}(t), \tag{5.15}$$

where

$$\widetilde{\mathcal{S}}(t) := C\delta\frac{\kappa}{h^{2\gamma}}\int_{t-2\delta}^t \|J_h\partial_s\psi(s)\|^2_{H^{-\gamma/2}}\,ds.$$

Observe that $\widetilde{\mathcal{S}}(t) \leq \widetilde{\mathcal{S}}_{k-1} + \widetilde{\mathcal{S}}_k + \widetilde{\mathcal{S}}_{k+1}(t)$, where

$$\widetilde{\mathcal{S}}_\ell(t) := C\delta\frac{\kappa}{h^{2\gamma}}\int_{\delta_\ell}^t \|J_h\partial_s\psi(s)\|^2_{H^{-\gamma/2}}\,ds \qquad \text{and} \qquad \widetilde{\mathcal{S}}_\ell := \widetilde{\mathcal{S}}_\ell(\delta_{\ell+1}).$$

Let $\ell \in \{k-3, k-2, k-1, k\}$. By the second condition in (5.7), with c_2 chosen large enough, we have $\delta\mu \leq C^{-1}(\ln 4)$, so that Lemma A.0.2 guarantees that

$$\frac{1}{\delta}\int_{I_\ell}\|\psi(s)\|^2_{H^{1/2}}\,ds \leq C\Psi^2 e^{-(\mu/2)(t-\delta_{\ell'}/2)}, \quad \ell' \in (\ell, \ell+N], \quad N = 3, \tag{5.16}$$

as well as

$$\kappa\int_{I_\ell}\|\psi(s)\|^2_{H^{(\gamma+1)/2}}\,ds \leq C\Psi^2 e^{-(\mu/2)(t-\delta_{\ell'}/2)}, \quad \ell' \in (\ell, \ell+N], \quad N = 3. \tag{5.17}$$

Thus, by Lemma 5.1.1 and (5.12), (5.16), and (5.17) we have

$$\begin{aligned}\widetilde{\mathcal{S}}(t) \leq & C\delta^2\widetilde{C}_1(h)\sum_{\ell=k-3}^{k}\frac{1}{\delta}\int_{I_\ell}\|\psi(s)\|_{H^{1/2}}^2\,ds + C\delta\widetilde{C}_2(h)\sum_{\ell=k-1}^{k}\kappa\int_{I_\ell}\|\psi(s)\|_{H^{(\gamma+1)/2}}^2\,ds\\ &+ C\delta\widetilde{C}_1(h)\int_{\delta_{k+1}}^t\|\psi(s)\|_{H^{1/2}}^2\,ds + C\delta\widetilde{C}_2(h)\kappa\int_{\delta_{k+1}}^t\|\psi(s)\|_{H^{(\gamma+1)/2}}^2\,ds\\ \leq & \widetilde{O}(\delta)\Psi^2e^{-(\mu/2)(t-\delta_{k/2})}\\ &+ \widetilde{O}_1(\delta)\int_{\delta_{k+1}}^t\|\psi(s)\|_{H^{1/2}}^2\,ds + \widetilde{O}_2(\delta)\kappa\int_{\delta_{k+1}}^t\|\psi(s)\|_{H^{(\gamma+1)/2}}^2\,ds. \end{aligned} \tag{5.18}$$

where $\widetilde{C}_1(h), \widetilde{C}_2(h)$ are given by (5.4), (5.5) and

$$\widetilde{O}(\delta) := C(\widetilde{O}_1(\delta^2) + \widetilde{O}_2(\delta)), \quad \widetilde{O}_1(\delta) := \delta\widetilde{C}_1(h), \quad \widetilde{O}_2(\delta) := C\delta\widetilde{C}_2(h). \tag{5.19}$$

for some constant $C > 0$.

Returning to (5.15) and combining (5.18) gives

$$\begin{aligned}\frac{d}{dt}\|\psi\|_{H^{1/2}}^2 + \kappa\|\psi\|_{H^{(\gamma+1)/2}}^2 + \mu\|\psi\|_{H^{1/2}}^2 &\\ \leq \widetilde{O}(\delta)\Psi^2e^{-(\mu/2)(t-\delta_{k/2})}&\\ + \widetilde{O}_1(\delta)\int_{\delta_{k+1}}^t\|\psi(s)\|_{H^{1/2}}^2\,ds &+ \widetilde{O}_2(\delta)\left(\kappa\int_{\delta_{k+1}}^t\|\psi(s)\|_{H^{(\gamma+1)/2}}^2\,ds\right).\end{aligned}$$

Hence, provided that c_1, c_2 are chosen sufficiently large with c_2 depending on c_1, it follows from (5.7) that Lemma A.0.1 (i) applies over $t \in I_{k+1}$ with

$$a = \mu, \quad b = \kappa, \quad A = C(\delta\widetilde{C}_1(h) + \widetilde{C}_2(h)), \quad B = C\widetilde{C}_2(h), \quad F = \widetilde{O}(\delta)\Psi^2e^{-(\mu/2)(t-\delta_{k/2})}.$$

In particular, Lemma A.0.1 (i) implies

$$\begin{aligned}\|\psi(t)\|_{H^{1/2}}^2 + \frac{\kappa}{2}\int_{\delta_{k+1}}^t e^{-(\mu/2)(t-s)}\|\psi(s)\|_{H^{(\gamma+1)/2}}\,ds&\\ \leq \|\psi(\delta_{k+1})\|_{H^{1/2}}^2e^{-(\mu/2)(t-\delta_{k+1})} + \widetilde{O}(\delta)\Psi e^{-(\mu/2)(t-\delta_{k/2})}(t-\delta_{k+1}).&\end{aligned}$$

By (5.12), we have

$$\|\psi(\delta_{k+1})\|_{H^{1/2}}^2e^{-(\mu/2)(t-\delta_{k+1})} \leq \Psi^2e^{-(\mu/2)(\delta_{k+1}-\delta_{k/2})}e^{-(\mu/2)(t-\delta_{k+1})} = \Psi^2e^{-(\mu/2)(t-\delta_{k/2})}, \quad t \in I_{k+1}.$$

Also, we have

$$\widetilde{O}(\delta)\Psi e^{-(\mu/2)(t-\delta_{k/2})}(t-\delta_{k+1}) \leq \delta\widetilde{O}(\delta)\Psi e^{-(\mu/2)(t-\delta_{k/2})}.$$

Since

$$e^{-(\mu/2)(t-\delta_{k/2})} = e^{-(\mu/4)\delta}e^{-(\mu/2)(t-\delta_{(k+1)/2})}$$

It follows that

$$\begin{aligned}\|\psi(t)\|_{H^{1/2}}^2 + \frac{\kappa}{2}\int_{\delta_{k+1}} e^{-(\mu/2)(t-s)}\|\psi(s)\|_{H^{(\gamma+1)/2}}^2\,ds&\\ \leq \Psi^2\left(1+\delta\widetilde{O}(\delta)\right)e^{-(\mu/4)\delta}e^{-(\mu/2)(t-\delta_{(k+1)/2})}, \quad t \in I_{k+1}.&\end{aligned} \tag{5.20}$$

Observe that (5.7) with c_1 chosen sufficiently large ensures $1+\delta\widetilde{O}(\delta)\leq e^{(\mu/4)\delta}$. This establishes (5.12) for $k+1$. Through Lemma A.0.1 (ii), we may iterate this bound to deduce (5.8), as desired. □

5.3. **Proof of Theorem 2.** Under the Standing Hypotheses, Theorem 1 guarantees a unique, global strong solution η of (4.2). Let c_0 denote the maximum among all the constants, c_0, c_0', appearing in Propositions 4.2.1 and 5.1.1. Then let c_1, c_2 denote the maximum among all the c_1, c_2 appearing in those propositions as well (possibly choosing c_2 larger). Suppose that μ, h satisfies

$$\frac{1}{c_0'}\Xi_{L^p}^2\leq\frac{\mu}{\kappa}\leq\frac{1}{c_0}h^{-\gamma}. \tag{5.21}$$

Choose δ so that (4.7), (5.7) are satisfied, and is chosen smaller than

$$\frac{1}{c_1}\left(\frac{h}{2\pi}\right)\min\left\{\frac{h^{\gamma/2}}{(c_0')^{1/2}}\Xi_{L^p}\frac{h^2}{M_{L^2}},\frac{h^\gamma}{\kappa}\right\}.$$

Then (5.21) implies that (4.8) holds as well. Thus, upon applying Propositions 4.2.1 and 5.1.1, η satisfies

$$\|\eta(t)-\theta(t)\|_{H^{-1/2}}\leq O(e^{-(\mu/4)(t-2\delta)}),\quad t>2\delta.$$

Observe that Propositions 2.5.4, 4.3.1, and 4.4.1 then imply that

$$\sup_{t>-2\delta}\|\eta(t)-\theta(t)\|_{H^\sigma}\leq\widetilde{M}_{H^\sigma}(h)+\Theta_{H^\sigma},$$

where

$$\widetilde{M}_{H^\sigma}(h):=\Gamma_{H^\sigma}^2+C_0\left[\left(\widetilde{\Xi}_{L^p}^{\frac{2\sigma+\gamma}{\sigma}}+\frac{1}{h^{2\sigma+\gamma}}\right)M_{L^2}^2+\frac{1}{h^{2\sigma+\gamma}}+\frac{1}{h^{2\sigma+\gamma}}R_{H^\sigma}^2\right],$$

for some sufficiently large constant $C_0>0$. Therefore, for each $\sigma'<\sigma$, by interpolation, there exists a constant $\lambda_0=\lambda_0(\sigma')\in(0,1)$ such that

$$\|\eta(t)-\theta(t)\|_{H^{\sigma'}}\leq O(e^{-\lambda_0\mu(t-2\delta)}),\quad t>2\delta.$$

Choosing $\sigma'=0$, yields the desired convergence in L^2.

□

5.4. **Concluding remarks.** Depending on the type of measurement, the size of the averaging window that effectively blurs the observations in time may be quite different. For example, radiometers and hot-wire anemometers may produce data with averages in the microsecond range. Velocities obtained from mechanical weather-vane anemometers may be averaged with respect to a time window measured in seconds, while velocity data obtained from the Lagrangian trajectories of buoys placed in the ocean is likely to include time averages measured in hours if not days. Observations of temperatures are similar. As we saw, it is important for our analysis that the size of the time-averaging window is not too large. Intuitively speaking, the length of the averaging window should be smaller than any dynamically relevant timescales in the flow. Numerical computations involving the Lorenz system [7] show that synchronization occurs when the averaging window is of size $\delta=0.25$ which, poetically speaking, is about ten times smaller than the time it takes to travel around one wing of the butterfly. In the case of the fluids, we conjecture that the averaging window should be at least ten times smaller than the turnover time of the smallest physically relevant eddy. Alternatively, the largest averaging window such that our data assimilation algorithm leads to full recovery of the observed solution could be interpreted as a definition of the smallest physically relevant time scale.

We reiterate that a main motivation to consider a more realistic representation of physical observations is the reason for considering time averages. The additional δ delay introduced into equations (1.2) helps close the estimates in the analysis while being of the same magnitude as the $\delta/2$ delay dictated by causality considerations in the feedback controller (see Remark 2.5). In practice, such a delay may also be used to advance an initial condition already obtained by data assimilation for a short time into the future to increase the stability of further predictions. However, this idea must be left for a different study.

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Appendix A.

To obtain the uniform estimates, we invoked a non-local Gronwall inequality, which ensured such bounds provided that the non-local term was sufficiently small.

Lemma A.0.1. *Let Φ, Ψ, F be non-negative, locally integrable functions on $(t_0, t_0+\delta]$ for some $t_0 \in \mathbb{R}$ and $\delta > 0$ such that*

$$\frac{d}{dt}\Phi + a\Phi + b\Psi \leq F + A\delta \int_{t_0}^{t} \Phi(s)\, ds + B\delta \int_{t_0}^{t} \Psi(s)\, ds, \quad t \in (t_0, t_0+\delta), \tag{A.1}$$

for some $a, b, A, B > 0$. Suppose that δ, a, c satisfy

$$\delta\left(e^{(a/2)\delta} - 1\right) \leq \frac{a}{4}\min\left\{\frac{a}{A}, \frac{b}{B}\right\}, \tag{A.2}$$

where we use the convention that $a/A = \infty$, $b/B = \infty$ if $A = 0$, $B = 0$, respectively. Then the following hold:

(i) For all $t \in (t_0, t_0+\delta]$:

$$\Phi(t) + \frac{b}{2}\int_{t_0}^{t} e^{-(a/2)(t-s)}\Psi(s)\, ds \leq e^{-(a/2)(t-t_0)}\Phi(t_0) + \int_{t_0}^{t} e^{-(a/2)(t-s)}F(s)\, ds. \tag{A.3}$$

(ii) If Φ satisfies

$$\Phi(t) \leq e^{-(a/2)(t-\delta_0)}\Phi(\delta_0) + \int_{\delta_0}^{t} e^{-(a/2)(t-s)}F(s)\, ds, \quad t \in (\delta_0, t_0], \tag{A.4}$$

for some $\delta_0 < t_0$, then (A.4) persists over $t \in (t_0, t_0+\delta]$.

Proof. Multiplying by the factor $e^{(a/2)\kappa t}$, then integrating over $[t_0, t]$, we obtain

$$\begin{aligned}\Phi(t) + \frac{a}{2}\int_{t_0}^t e^{-(a/2)(t-s)}\Phi(s)\ ds + b\int_{t_0}^t e^{-(a/2)(t-s)}\Psi(s)\ ds\\ \leq & e^{-(a/2)(t-t_0)}\Phi(t_0) + \int_{t_0}^t e^{-(a/2)(t-s)}F(s)\ ds\\ & + A\delta\int_{t_0}^t e^{-(a/2)\kappa(t-s)}\int_{t_0}^s \Phi(\tau)\ d\tau\ ds + B\delta\int_{t_0}^t e^{-(a/2)\kappa(t-s)}\int_{t_0}^s \Psi(\tau)\ d\tau\ ds,\end{aligned}$$

Observe that

$$\frac{a}{2}\int_{t_0}^t e^{-(a/2)(t-s)}\Phi(s)\ ds \geq \frac{a}{2}e^{-(a/2)(t-t_0)}\int_{t_0}^t \Phi(s)\ ds$$

$$A\delta\int_{t_0}^t e^{-(a/2)(t-s)}\int_{t_0}^s \Phi(\tau)\ d\tau\ ds \leq \frac{2A\delta}{a}\left(1-e^{-(a/2)(t-t_0)}\right)\int_{t_0}^t \Phi(\tau)\ d\tau.$$

Similarly

$$\frac{b}{2}\int_{t_0}^t e^{-(a/2)(t-s)}\Psi(s)\ ds \geq \frac{b}{2}e^{-(a/2)(t-t_0)}\int_{t_0}^t \Psi(s)\ ds$$

$$B\delta\int_{t_0}^t e^{-(a/2)(t-s)}\int_{t_0}^s \Psi(\tau)\ d\tau\ ds \leq \frac{2B\delta}{b}\left(1-e^{-(a/2)(t-t_0)}\right)\int_{t_0}^t \Psi(s)\ ds.$$

It follows that

$$\begin{aligned}\frac{a}{2}\int_{t_0}^t e^{-(a/2)(t-s)}\Phi(s)\ ds - c\delta\int_{t_0}^t e^{-(a/2)(t-s)}\int_{t_0}^s \Phi(\tau)\ d\tau\ ds\\ \geq \frac{a}{2}\left[1-\frac{4A\delta}{a^2}\left(e^{(a/2)(t-t_0)}-1\right)\right]e^{-(a/2)(t-t_0)}\int_{t_0}^t \Phi(s)\ ds \geq 0,\end{aligned}$$

provided that the first condition in (A.2) holds. This also holds with b, B, ψ, replacing a, A, Φ, respectively, provided the second condition in (A.2) holds. This implies (A.3).

Now assume that (A.1) holds over $(t_0, t_0+\delta)$ and that (A.4) holds over $[\delta_0, t_0]$, for some $\delta_0 > 0$. Then applying (A.4) at t_0 to (A.3) we have

$$\Phi(t) \leq \Phi(\delta_0)e^{-(a/2)(t-\delta_0)} + \int_{\delta_0}^{t_0} e^{-(a/2)(t-s)}F(s)\ ds + \int_{t_0}^t e^{-(a/2)(t-s)}F(s)\ ds,$$

which simplifies to (A.4), as desired. □

We also made use of the following lemma in order to control feedback effects that enter the present instant through a past time interval and ultimately, ensure synchronization (see (5.15)).

Lemma A.0.2. *Let $\ell \geq -1$ and $N > 0$. Let $\delta > 0$ and define $\delta_\ell := \ell\delta$ and $I_\ell := (\delta_\ell, \delta_{\ell+1}]$. Let Φ, Ψ be non-negative, locally integrable functions. Suppose that for some $\ell \geq -1$, there exist constants $a, b, \Phi_0 > 0$, independent of ℓ, N, such that*

$$\Phi(t) + b\int_{\delta_\ell}^t e^{-(a/2)(t-s)}\Psi(s)\ ds \leq \Phi_0 e^{-(a/2)(t-\delta_{\ell/2})}, \quad t \in I_\ell. \tag{A.5}$$

If δ satisfies

$$\delta < \frac{c}{a}, \tag{A.6}$$

for some constant $c > 0$, then there exists a constant $C_N > 0$ such that

$$\frac{1}{\delta}\int_{I_\ell} \Phi(s)\ ds \leq C_N \Phi_0 e^{-(a/2)(t-\delta_{\ell'/2})} \quad \ell' \in (\ell, \ell+N]. \tag{A.7}$$

and

$$b\int_{I_\ell} \Psi(s)\ ds \leq C_N \Phi_0 e^{-(a/2)(t-\delta_{\ell'/2})}, \quad \ell' \in (\ell, \ell+N]. \tag{A.8}$$

Proof. Observe that by the mean value theorem

$$\begin{aligned}\int_{I_\ell} \Phi(s)\ ds &\leq \Phi_0 \int_{\delta_\ell}^{\delta_{\ell+1}} e^{-(a/2)(s-\delta_{\ell/2})}\ ds \\ &= \Phi_0 e^{(a/2)\delta_{\ell/2}} \frac{2}{a}\left(e^{-(a/2)\delta_\ell} - e^{-(a/2)\delta_{\ell+1}}\right) \\ &= \Phi_0 e^{(a/2)\delta_{\ell/2}} e^{-(a/2)\delta_{\ell+1}} \frac{2}{a}\left(e^{(a/2)\delta} - 1\right) \\ &= \Phi_0 e^{-(a/2)\delta_{\ell/2}} e^{-(a/2)\delta(1-\theta)}\delta.\end{aligned}$$

for some $0 < \theta < 1$, depending on δ.

By assumption on ℓ', ℓ, and the fact that $t \leq \delta_{\ell'+1}$, we have

$$\begin{aligned} e^{-(a/2)\delta_{\ell/2}} &= e^{-(a/2)(t-\delta_{\ell'/2})} e^{-(a/2)\delta_{\ell/2}} e^{(a/2)(t-\delta_{\ell'/2})} \\ &\leq e^{-(a/2)(t-\delta_{\ell'/2})} e^{(a/2)(\delta_{\ell'/2}-\delta_{\ell/2})} e^{(a/2)\delta} \\ &\leq e^{(a/2)\delta(1+N/2)} e^{-(a/2)(t-\delta_{\ell'/2})}. \end{aligned} \tag{A.9}$$

Thus, by letting $C_N := e^{(c/2)(1+N/2)}$, (A.6) and (A.9) imply (A.7).

On the other hand, observe that

$$b\int_{\delta_\ell}^{t} e^{-(a/2)(t-s)} \Psi(s)\ ds \geq e^{-(a/2)(t-\delta_\ell)} b\int_{\delta_\ell}^{t} \Psi(s)\ ds.$$

Upon application of (A.5), we have

$$b\int_{\delta_\ell}^{t} \Psi(s)\ ds \leq e^{(a/2)(t-\delta_\ell)} \Phi_0 e^{-(a/2)(t-\delta_{\ell/2})} = \Phi_0 e^{-(a/2)\delta_{\ell/2}}, \quad t \in I_\ell.$$

Thus, by (A.9) we have

$$b\int_{I_\ell} \Psi(s)\ ds \leq C_N e^{-(a/2)(t-\delta_{\ell'/2})},$$

and we are done. □

Appendix B.

B.1. **Partition of unity.** Let us briefly recall the partition of unity constructed in [3] and used in [33]. To this end, we define for $\phi \in L^1(\mathbb{T}^2)$

$$\langle \phi \rangle := \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \phi(x)\, dx. \tag{B.1}$$

Let $N > 0$ be a perfect square integer and partition Ω into $4N$ squares of side-length $h = \pi/\sqrt{N}$. Let $\mathcal{J} = \{0, \pm 1, \pm 2, \dots, \pm(\sqrt{N}-1), -\sqrt{N}\}^2$ and for each $\alpha \in \mathcal{J}$, define the semi-open square

$$Q_\alpha = [ih, (i+1)h) \times [jh, (j+1)h), \quad \text{where} \quad \alpha = (i,j) \in \mathcal{J}.$$

Let $\mathcal{Q}$ denote the collection of all Q_α, i.e.

$$\mathcal{Q} := \{Q_\alpha\}_{\alpha \in \mathcal{J}}.$$

Suppose that $N \geq 9$ and $\epsilon = h/10$. For each $\alpha = (i,j) \in \mathcal{J}$, let us also define the augmented squares, $\hat{Q}_\alpha$ and $Q_\alpha(\epsilon)$, by

$$\hat{Q}_\alpha := [(i-1)h, (i+2)h] \times [(j-1)h, (j+2)h] \quad \text{and} \quad Q_\alpha(\epsilon) := Q_\alpha + B(0,\epsilon). \tag{B.2}$$

so that $Q_\alpha \subset Q_\alpha(\epsilon) \subset \hat{Q}_\alpha$ for each $\alpha \in \mathcal{J}$, and the "core," $C_\alpha(\epsilon)$, by

$$C_\alpha(\epsilon) := Q_\alpha(\epsilon) \setminus \bigcup_{\alpha' \neq \alpha} Q_{\alpha'}(\epsilon) \neq \varnothing, \quad \alpha \in \mathcal{J}.$$

Then there exists a collection of functions $\{\psi_\alpha\}$ satisfying the properties in Proposition B.1.1. Note that we will use the convention that when β a positive integer, then $D^\beta = \partial_1^{\beta_1}\partial_2^{\beta_2}$, where $\beta_1 + \beta_2 = \beta$ and $\beta_j \geq 0$ are integers, while if $\beta > 0$ is not an integer then $D^\beta = \partial_1^{[\beta_1]}\partial_2^{[\beta_2]}\Lambda^{\beta - [\beta]}$, where $[\beta] = [\beta_1] + [\beta_2]$, and finally, if $\beta \in (-2, 0)$, then $D^\beta = \Lambda^\beta$.

Proposition B.1.1. *Let $N \geq 9$, $h := L/\sqrt{N}$, and $\epsilon := h/10$. The collection $\{\psi_\alpha\}_{\alpha \in \mathcal{J}}$ forms a smooth partition of unity satisfying*

(i) $0 \leq \widetilde{\psi}_\alpha \leq 1$ *and* $\operatorname{spt} \widetilde{\psi}_\alpha \subset (Q_\alpha(\epsilon) + (2\pi\mathbb{Z})^2)$*;*
(ii) $\widetilde{\psi}_\alpha = 1$, *for all* $x \in (C_\alpha(\epsilon) + (2\pi\mathbb{Z})^2)$ *and* $\sum_{\alpha \in \mathcal{J}} \widetilde{\psi}_\alpha(x) = 1$, *for all* $x \in \mathbb{R}^2$*;*
(iii) $c_1 h^{2/p} \leq \|\widetilde{\psi}_\alpha\|_{L^p(\mathbb{T}^2)} \leq c_2 h^{2/p}$, *for all* $p \in [1, \infty)$, *for some constants* $c_1, c_2 > 0$*; in particular* $(h/(2\pi))^2 \leq \langle \widetilde{\psi}_\alpha \rangle \leq c(h/(2\pi))^2$*;, for some constant* $c > 1$*;*
(iv) $\sup_{\alpha \in \mathcal{J}} \|\widetilde{\psi}_\alpha\|_{\dot{H}^\beta(\mathbb{T}^2)} \lesssim h^{1-\beta}$, *for all* $\beta > -1$*;*
(v) $\sup_{\alpha \in \mathcal{J}} \|\widetilde{\psi}_\alpha\|_{\dot{H}^\beta(\mathbb{T}^2)} \lesssim \left(\frac{2\pi}{h}\right)^{1-\beta-\epsilon(|\beta|)} h^{1-\beta}$, *for all* $\beta \in (-2, -1]$, *for some* $\epsilon \in (1,2)$, *where the suppressed constant depends on* β*;*
(vi) $\sup_{\alpha \in \mathcal{J}} \|\Lambda^\beta D^k \psi_\alpha\|_{L^\infty(\mathbb{T}^2)} \lesssim h^{-k-\beta}$, *for all* $\beta \in [0,1)$, $k \geq 0$ *integer.*

Property (iii) was exploited in [33], but only in the case $p = 2$. We observe here, however, that it also holds for any $p \in [1, \infty)$ since $\mathbb{I}_{Q_\alpha} \leq \widetilde{\psi}_\alpha \leq 1$ and $\operatorname{spt} \widetilde{\psi}_\alpha \subset (Q_\alpha(\epsilon) + (2\pi\mathbb{Z})^2)$. On the other hand, property (iv) for $\beta \geq 0$ was sufficient for the purposes in [33]. We will show here that it also holds $\beta \in (-2, 0)$, i.e. property (v), as well as the L^∞ estimate (vi). For this, we will appeal to the following elementary fact:

$$(\Lambda^\beta(\phi(\lambda\cdot)))(x) = \lambda^\beta(\Lambda^\beta\phi)(\lambda x), \quad x \in \mathbb{T}^2, \quad \lambda > 0, \tag{B.3}$$

where we define

$$\phi(\lambda\cdot\,)(x) := \phi(\lambda x).$$

The relation (B.3) can be seen easily by appealing to the Fourier transform. Due to the subtleties of working with periodic functions, we include the details in Lemma B.1.1 below. To this end, let us define

$$\langle \phi, \psi\rangle_{L^2(\Omega)} := \int_\Omega \phi(x)\overline{\psi(x)}\, dx.$$

Let us also denote the Fourier transform on $\mathbb{T}^2$, i.e., for functions which are periodic with period 2π in x, y, by

$$\mathcal{F}(\phi)(\mathbf{k}) = \frac{1}{4\pi^2}\int_{\mathbb{T}^2} e^{-i\mathbf{k}\cdot x}\phi(x)\, dx,$$

and by $\mathcal{F}_\lambda$ the Fourier transform on $\lambda^{-1}\mathbb{T}^2$, for $\lambda > 0$, i.e., for functions which are periodic with period $\lambda^{-1}2\pi$ in x, y, by

$$(\mathcal{F}_\lambda\phi)(\widetilde{\mathbf{k}}) = \frac{\lambda^2}{4\pi^2}\int_{\lambda^{-1}\mathbb{T}^2} e^{-i\widetilde{\mathbf{k}}\cdot x}\phi(x)\, dx, \quad \widetilde{\mathbf{k}} \in \lambda\mathbb{Z}^2.$$

Lemma B.1.1. *Let $\beta \in (-2, 2]$. Then*

(i) $\langle \phi, \psi\rangle_{L^2(\mathbb{T}^2)} = \lambda^2\langle \phi(\lambda\cdot\,), \psi(\lambda\cdot\,)\rangle_{L^2(\lambda^{-1}\mathbb{T}^2)}$, for $\lambda > 0$.

(ii) $\Lambda^\beta\phi(\lambda\cdot\,)(x) = \lambda^\beta(\Lambda^\beta\phi)(\lambda x)$, for $\lambda > 0$, and any $\beta \in \mathbb{R}$, provided that $\phi \in \mathcal{Z}$.

Proof. The first property follows by a change of variables. Now observe that if $\phi \in C^\infty_{per}(\mathbb{T}^2)\cap \mathcal{Z}$, then $\phi(\lambda\cdot\,) \in C^\infty_{per}(\lambda^{-1}\mathbb{T}^2) \cap \mathcal{Z}$ with period $2\pi\lambda^{-1}$ in x, y, where $\mathcal{Z}$ is as in (2.1). Let $\widetilde{\mathbf{k}} = \lambda\mathbf{k}$, for $\mathbf{k} \in \mathbb{Z}^2$. Then

$$\mathcal{F}_\lambda(\Lambda^\beta\phi(\lambda\cdot\,))(\widetilde{\mathbf{k}}) = \frac{\lambda^2}{4\pi^2}\int_{\lambda^{-1}\mathbb{T}^2} e^{i\widetilde{\mathbf{k}}\cdot x}|\widetilde{\mathbf{k}}|^\beta\phi(\lambda x)\, dx = \lambda^\beta\frac{1}{4\pi^2}\int_{\mathbb{T}^2} e^{-i\mathbf{k}\cdot x}|\mathbf{k}|^\beta\phi(x)\, dx = \lambda^\beta\mathcal{F}(\Lambda^\beta\phi)(\mathbf{k}).$$

It follows that for $x \in \lambda^{-1}\mathbb{T}^2$, we have

$$\Lambda^\beta\phi(\lambda\cdot\,)(x) = \sum_{\widetilde{\mathbf{k}}\in\lambda\mathbb{Z}^2} e^{i\widetilde{\mathbf{k}}\cdot x}\mathcal{F}_\lambda(\Lambda^\beta\phi(\lambda\cdot\,)(\widetilde{\mathbf{k}}) = \lambda^\beta\sum_{\mathbf{k}\in\mathbb{Z}^2} e^{i\mathbf{k}\cdot(\lambda x)}\mathcal{F}(\Lambda^\beta\phi)(\mathbf{k}) = \lambda^\beta(\Lambda^\beta\phi)(\lambda x).$$

□

Let us now return to the proof of Proposition B.1.1 $(v) - (vii)$. For this, let

$$\widetilde{\Psi}_\alpha(x) = \widetilde{\psi}_\alpha(hx), \tag{B.4}$$

and $\bar{\Psi}_\alpha = \widetilde{\Psi}_\alpha - \left(\frac{h}{2\pi}\right)^2\int_{h^{-1}\mathbb{T}^2}\widetilde{\Psi}_\alpha(x)\, dx$, so that $\bar{\psi}_\alpha(x) = \bar{\Psi}_\alpha(h^{-1}x)$ and $\langle\bar{\Psi}_\alpha\rangle = 0$. Moreover, observe that $\widetilde{\Psi}_\alpha$ is supported in a square of area $\lesssim 1$.

Proof of Proposition B.1.1 (iv) through (vi).

Proof of (iv) for $\beta \in (-1, 0)$. For convenience, let $\beta > 0$. By Lemma B.1.1 (ii), we have

$$\|\widetilde{\psi}_\alpha\|_{\dot{H}^{-\beta}(\mathbb{T}^2)} = \|\Lambda^{-\beta}(\widetilde{\Psi}_\alpha(h^{-1}\cdot\,))\|_{L^2(\mathbb{T}^2)} = h^\beta\|(\Lambda^{-\beta}\widetilde{\Psi}_\alpha)(h^{-1}\cdot\,)\|_{L^2(\mathbb{T}^2)} = h^{\beta+1}\|\Lambda^{-\beta}\widetilde{\Psi}_\alpha\|_{L^2(h^{-1}\mathbb{T}^2)}. \tag{B.5}$$

It follows from the Hardy-Littlewood-Sobolev inequality that

$$\|\Lambda^{-\beta}\widetilde{\Psi}_\alpha\|_{L^2(h^{-1}\mathbb{T}^2)} \le C\|\widetilde{\Psi}_\alpha\|_{L^{2/(1+\beta)}(h^{-1}\mathbb{T}^2)} \le C.$$

We see now that from (B.5) we have

$$\|\widetilde{\psi}_\alpha\|_{\dot{H}^{-\beta}(\mathbb{T}^2)} \leq Ch^{\beta+1}, \quad \beta \in (0,1),$$

with constant independent of α and h, as desired.

Proof of (v). Let $\beta \in [1,2)$. We estimate by duality. Indeed, let $\chi \in \dot{H}^\beta(\mathbb{T}^2)$ such that $\|\chi\|_{\dot{H}^\beta(\mathbb{T}^2)}$. Then since $\chi \in \mathcal{Z}$, by Parseval's theorem we have

$$\langle \widetilde{\psi}_\alpha, \chi\rangle_{L^2(\mathbb{T}^2)} = \langle \bar{\psi}_\alpha, \chi\rangle_{L^2(\mathbb{T}^2)} = \langle \Lambda^{-\beta}\bar{\psi}_\alpha, \Lambda^\beta\chi\rangle_{L^2(\mathbb{T}^2)}.$$

Let $q > 2/(2-\beta)$, so that $q \in (2,\infty)$, and let $q^* \in (1,2)$ be its Sobolev conjugate, i.e., $1/q = 1/q^* - \beta/2$. Let $\epsilon = 2/q^*$ and q' denote the Hölder conjugate of q. Observe that $1 < q' < 2 < q < \infty$. Then by Hölder's inequality, (B.4), and the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
|\langle \widetilde{\psi}_\alpha, \chi\rangle_{L^2(\mathbb{T}^2)}| &\leq \|\Lambda^{-\beta}\bar{\psi}_\alpha\|_{L^q(\mathbb{T}^2)}\|\Lambda^\beta\chi\|_{L^{q'}(\mathbb{T}^2)} \\
&\leq (2\pi)^{2/q'-1}h^{\beta+2/q}\|\Lambda^{-\beta}\bar{\Psi}_\alpha\|_{L^q(h^{-1}\mathbb{T}^2)}\|\Lambda^\beta\chi\|_{L^2(\mathbb{T}^2)} \\
&\leq C\left(\frac{2\pi}{h}\right)^{1-2/q} h^{1+\beta}\|\bar{\Psi}_\alpha\|_{L^{q^*}(\mathbb{T}^2)}\|\chi\|_{\dot{H}^\beta(\mathbb{T}^2)}, \\
&\leq C\left(\frac{2\pi}{h}\right)^{1+\beta-\epsilon(\beta)} h^{1+\beta}
\end{aligned}$$

where for the last inequality, we made use of the fact that $|\widetilde{\Psi}_\alpha| \lesssim 1$ in $h^{-1}\mathbb{T}^2$ and $\widetilde{\Psi}_\alpha$ is supported in a ball of area ~ 1. Thus

$$\|\widetilde{\psi}_\alpha\|_{\dot{H}^{-\beta}(\mathbb{T}^2)} \leq C\left(\frac{2\pi}{h}\right)^{1+\beta-\epsilon(\beta)} h^{1+\beta}, \quad \beta \in [1,2),$$

as desired.

Proof of (vi). The result is trivial when $\beta = 0$ and $k > 0$ simply by rescaling and observing that $D^k\widetilde{\psi}_\alpha$ is still supported in $Q_\alpha(\epsilon)$.

Suppose that $\beta \in (0,1)$. Now observe that that for $x \in \mathbb{T}^2$, Lemma B.1.1 (ii) implies that

$$\begin{aligned}
(\Lambda^\beta\widetilde{\psi}_\alpha)(x) &= c_\beta \sum_k p.v.\int_{\mathbb{T}^2} \frac{\widetilde{\psi}_\alpha(x) - \widetilde{\psi}_\alpha(y)}{|x-y-2\pi k|^{2+\beta}}\,dy \\
&= \frac{c_\beta}{h^\beta}\sum_k p.v.\int_{h^{-1}\mathbb{T}^2} \frac{\widetilde{\Psi}_\alpha(h^{-1}x) - \widetilde{\Psi}_\alpha(y)}{|\frac{x}{h} - y - \frac{2\pi}{h}k|^{2+\beta}}\,dy = h^{-\beta}(\Lambda^\beta\Psi_\alpha)(h^{-1}x). \qquad \text{(B.6)}
\end{aligned}$$

Since $\|\Delta\Psi_\alpha\|_{L^\infty(h^{-1}\mathbb{T}^2)} \leq C$, this settles the case $\beta = 2$. Since L^∞ is invariant under dilations and $\widetilde{\Psi}_\alpha$ is $2\pi h^{-1}$-periodic in x, y, it suffices to consider

$$\begin{aligned}
\Lambda^\beta\Psi_\alpha(x) &= c_\beta\sum_k p.v.\int_{h^{-1}\mathbb{T}^2}\frac{\widetilde{\Psi}_\alpha(x) - \widetilde{\Psi}_\alpha(y)}{|x-y-\frac{2\pi}{h}k|^{2+\beta}}\,dy \\
&= c_\beta p.v.\int_{\mathbb{R}^2}\frac{\widetilde{\Psi}_\alpha(x) - \widetilde{\Psi}_\alpha(y)}{|x-y|^{2+\beta}}\,dy, \quad x \in h^{-1}\mathbb{T}^2.
\end{aligned}$$

Let us consider two cases: $x \notin 2h^{-1}Q_\alpha$ and $x \in 2h^{-1}Q_\alpha$.

If $x \notin 2h^{-1}Q_\alpha \cap h^{-1}\mathbb{T}^2$, then $\widetilde{\Psi}_\alpha(x) = 0$ and $|x-y| \geq 2$. Thus

$$|\Lambda^\beta \Psi_\alpha(x)| \leq C\|\widetilde{\Psi}\|_{L^\infty} \int_{|y|\geq 2} \frac{dy}{|y|^{2+\beta}} \leq C.$$

If $x \in 2h^{-1}Q_\alpha \cap h^{-1}\mathbb{T}^2$, then $|x-y| \leq 2$ and we have

$$\begin{aligned}\left(\int_{\substack{|x-y|\leq 2\\ |y|\leq 1}} + \int_{\substack{|x-y|\leq 2\\ |y|\geq 1}}\right) &\frac{|\widetilde{\Psi}_\alpha(x) - \widetilde{\Psi}_\alpha(y)|}{|x-y|^{2+\beta}}\, dy \\ &\leq C\|\nabla\Psi_\alpha\|_{L^\infty} \int_{|y|\leq 1} \frac{dy}{|y|^{1+\beta}} + C\|\Psi_\alpha\|_{L^\infty} \int_{|y|\geq \delta} \frac{dy}{|y|^{2+\beta}} \\ &\leq \frac{C}{1-\beta}\|\nabla\Psi_\alpha\|_{L^\infty} + \frac{C}{\beta}\|\Psi_\alpha\|_{L^\infty} \leq C.\end{aligned}$$

Thus $|\Lambda^\beta\Psi_\alpha(x)| \leq C$ for all $x \in h^{-1}\mathbb{T}^2$, which implies $|\Lambda^\beta\psi_\alpha(x)| \leq Ch^{-\beta}$ for all $x \in \mathbb{T}^2$, where C is independent of $\alpha \in \mathcal{J}$. This establishes (v). ☐

To ultimately prove (2.12), (2.13) and (2.14), we will exploit an additional property of the bump functions $\widetilde{\psi}_\alpha$. For this, we will make use of the following short-hand for ϕ localized to the squares $Q_\alpha(\epsilon)$:

$$\phi_\alpha(x) = \phi(x)\mathbb{I}_{Q_\alpha(\epsilon)}(x), \quad x \in \mathbb{T}^2.$$

Lemma B.1.2. *Let $\beta \in (-\infty, 0)$ and $\phi \in \dot{H}^\beta(\mathbb{T}^2)$. Then there exists a constant $C > 0$ such that*

$$|\langle \phi, \widetilde{\psi}_\alpha\rangle_{L^2(\mathbb{T}^2)}| \leq Ch^{1+\beta}\|\phi_\alpha\|_{\dot{H}^\beta(\mathbb{T}^2)} + C\left(\frac{h}{2\pi}\right)^2 h\|\phi\|_{L^2(\mathbb{T}^2)}. \tag{B.7}$$

Proof. Suppose that $\beta \in (-\infty, 0)$. Observe that

$$\langle \phi, \widetilde{\psi}_\alpha\rangle_{L^2(\mathbb{T}^2)} = \langle \phi_\alpha, \widetilde{\psi}_\alpha\rangle_{L^2(\mathbb{T}^2)} = \langle \phi_\alpha, \bar{\psi}_\alpha\rangle_{L^2(\mathbb{T}^2)} + \frac{\widetilde{a}(Q_\alpha)}{4\pi^2}\int_{\mathbb{T}^2} \phi_\alpha(x)\, dx.$$

Then by Parseval's theorem, the Cauchy-Schwarz inequality, and Proposition B.1.1 (iv), we have

$$\begin{aligned}|\langle \phi, \widetilde{\psi}_\alpha\rangle| &\leq \|\phi_\alpha\|_{\dot{H}^\beta(\mathbb{T}^2)}\|\widetilde{\psi}_\alpha\|_{\dot{H}^{|\beta|}(\mathbb{T}^2)} + C\left(\frac{h}{2\pi}\right)^2 h\|\phi\|_{L^2(\mathbb{T}^2)} \\ &\leq Ch^{1-|\beta|}\|\phi_\alpha\|_{\dot{H}^\beta(\mathbb{T}^2)} + C\left(\frac{h}{2\pi}\right)^2 h\|\phi\|_{L^2(\mathbb{T}^2)},\end{aligned} \tag{B.8}$$

as desired.

☐

B.2. **Boundedness properties of volume element interpolants.** For $\phi \in L^1_{loc}(\Omega)$, define

$$\phi_Q = \frac{1}{a(Q)}\int_Q \phi(x)\, dx \quad \text{and} \quad \widetilde{\phi}_{Q_\alpha} = \frac{1}{\widetilde{a}(Q_\alpha)}\int_{\mathbb{T}^2} \phi(x)\widetilde{\psi}_\alpha(x)\, dx,$$

where $a(Q)$ denotes the area of Q and

$$\widetilde{a}(Q_\alpha) := \int_{\mathbb{T}^2} \widetilde{\psi}_\alpha(x)\ dx. \tag{B.9}$$

Observe that for each $\alpha \in \mathcal{J}$, there exists a constant $c > 0$, independent of h, α, ϵ, such that

$$c^{-1} \leq \frac{\widetilde{a}(Q_\alpha)}{a(Q)}, \frac{a(Q)}{a(\hat{Q}_\alpha)}, \frac{a(Q)}{a(Q_\alpha(\epsilon))} \leq c, \quad Q \in \{Q_\alpha, Q_\alpha(\epsilon), \hat{Q}_\alpha\}. \tag{B.10}$$

We define the smooth volume element interpolant by

$$\mathcal{I}_h(\phi) := \sum_{\alpha\in\mathcal{J}} \widetilde{\phi}_{Q_\alpha} \widetilde{\psi}_\alpha, \tag{B.11}$$

and the "shifted" smooth volume element interpolant by

$$I_h(\phi) := \sum_{\alpha\in\mathcal{J}} \widetilde{\phi}_{Q_\alpha} \bar{\psi}_\alpha, \quad \bar{\psi}_\alpha = \widetilde{\psi}_\alpha - \langle \widetilde{\psi}_\alpha \rangle. \tag{B.12}$$

We will make use of the following elementary fact for a "square-type" function. Let $\mathcal{A}$ be a finite index set and $\{A_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathbb{T}^2$ be a countable collection of sets such that for each $x \in \mathbb{T}^2$, $\sup_{x\in\mathbb{T}^2} \#\{\alpha \in \mathcal{A} : x \in Q_\alpha\} < \infty$. Define

$$(S\phi)(x) := \left(\sum_{\alpha\in\mathcal{A}} (\phi_\alpha(x))^2\right)^{1/2}, \quad \phi_\alpha(x) := \phi(x)\mathbb{I}_{A_\alpha}(x).$$

Lemma B.2.1. *Let $\phi \in L^1(\mathbb{T}^2)$. There exists a constant $C > 0$ such that*

$$|S\phi(x)| \leq C|\phi(x)|, \quad a.e. \quad x \in \mathbb{T}^2, \tag{B.13}$$

and

$$\sum_{\alpha\in\mathcal{A}} \left(\int \phi_\alpha(x)\ dx\right)^2 \leq \left(\int S\phi(x)\ dx\right)^2. \tag{B.14}$$

Proof. Let $N := \sup_{x\in\mathbb{T}^2} \#\{\alpha \in \mathcal{A} : x \in A_\alpha\}$. Since $N < \infty$, we have that for each $x \in \mathbb{T}^2$, there are at most N sets A_α such that $x \in A_\alpha$. It follows that for each $x \in \mathbb{T}^2$, there exists an integer $C(x) > 0$ such that $C(x) \leq N$. In particular, we have

$$\sum_\alpha |\phi_\alpha(x)|^2 = C(x)|\phi(x)|^2 \leq N|\phi(x)|^2.$$

On the other hand, by Fubini's theorem, and the Cauchy-Schwarz inequality we have that

$$\begin{aligned}\sum_\alpha \left(\int \phi_\alpha(x)\ dx\right)^2 &= \sum_\alpha \iint \phi_\alpha(x)\phi_\alpha(y)\ dxdy \\ &\leq \iint (S\phi)(x)(S\phi)(y)\ dxdy = \left(\int (S\phi)(x)\ dx\right)^2.\end{aligned}$$

This completes the proof. □

We immediately obtain the following corollary.

Corollary B.2.2. *Let $K \in L^1_{loc}(\mathbb{R}^2)$ such that $K \geq 0$. Let $\phi \in L^1(\mathbb{T}^2)$ such that $K * \phi \in L^2(\mathbb{T}^2)$.*

$$\sum_{\alpha\in\mathcal{A}} \|K * \phi_\alpha\|_{L^2}^2 \leq C\|K * \phi\|_{L^2}^2.$$

In particular, for $\beta \in (-2, 0)$, we have

$$\sum_{\alpha\in\mathcal{J}} \|\phi_\alpha\|_{\dot{H}^\beta}^2 \leq C\|\phi\|_{\dot{H}^\beta}^2, \tag{B.15}$$

where $(\mathcal{A}, \{A_\alpha\})$ is given by $(\mathcal{J}, \{Q_\alpha(\epsilon)\})$ as in (B.2).

Proof. Observe that

$$\begin{aligned}
\|K * \phi_\alpha\|_{L^2}^2 &= \int (K * \phi_\alpha)(x)^2\, dx \\
&= \int \left(\int K(x-y)\phi_\alpha(y)\, dy\right)^2 dx \\
&\leq \int\int\int K(x-y)K(x-y')|\phi_\alpha(y)||\phi_\alpha(y')|\, dydy'dx.
\end{aligned}$$

Therefore, by the non-negativity of K, the Cauchy-Schwarz inequality, and (B.13) of Lemma B.2.1, we have

$$\begin{aligned}
\sum_{\alpha\in\mathcal{A}} \|K * \phi_\alpha\|_{L^2}^2 &\leq \int\int\int K(x-y)K(x-y')(S\phi)(y)(S\phi)(y')\, dydy'dx \\
&\leq C^2 \int\int\int K(x-y)K(x-y')\phi(y)\phi(y')\, dydy'dx \\
&\leq C^2\|K * \phi\|_{L^2}^2.
\end{aligned}$$

It then follows as a special case that (B.15) holds. Indeed, the Riesz potential, Λ^β, $\beta \in (-2, 0)$, has kernel $K(x) \sim |x|^{-2+\beta}$, which is locally integrable and non-negative. □

Proposition B.2.1. *Let J_h be given by either (B.11) or (B.12). Given $\alpha \geq 1$, let $\epsilon(\alpha)$ be as in Proposition B.1.1 (v) when $\alpha \in [1, 2)$, and identically 0 otherwise. Let $C > 0$ and define*

$$C_I(\alpha, h) := \begin{cases} C\left(\frac{2\pi}{h}\right), & \alpha < 1, \\ C\left(\frac{2\pi}{h}\right)^{2+\alpha-\epsilon(\alpha)}, & \alpha \geq 1. \end{cases} \tag{B.16}$$

There exists a constant $C > 0$, depending on α, such that:

(1) If $(\rho, \beta) \in [0, \infty) \times [0, 2)$, then

$$\|J_h\phi\|_{\dot{H}^\rho(\mathbb{T}^2)} \leq C_I(\beta, h)h^{\beta-\rho}\|\phi\|_{\dot{H}^\beta(\mathbb{T}^2)}. \tag{B.17}$$

(2) If $(\rho, \beta) \in [0, \infty) \times (-2, 0]$, then

$$\|J_h\phi\|_{\dot{H}^\rho(\mathbb{T}^2)} \leq Ch^{-\rho}(h^\beta\|\phi\|_{\dot{H}^\beta} + \|\phi\|_{L^2(\mathbb{T}^2)}). \tag{B.18}$$

(3) If $(\rho, \beta) \in (-2, 0) \times (-\infty, 0]$, then

$$\|J_h\phi\|_{\dot{H}^\rho(\mathbb{T}^2)} \leq C_I(|\rho|, h)h^{\beta-\rho}\|\phi\|_{\dot{H}^\beta(\mathbb{T}^2)}. \tag{B.19}$$

Proof. We will prove the lemma for the case J_h given by (B.11). The case when J_h is given by (B.12) is similar.

Let $(\rho,\beta)\in[0,\infty)\times[0,2)$. Then by Proposition B.1.1 (iv) and (v), we have

$$\begin{aligned}\|J_h\phi\|^2_{\dot H^\rho(\mathbb{T}^2)} &\le C\sum_\alpha \widetilde{\phi}^2_{Q_\alpha}\|\widetilde{\psi}_\alpha\|^2_{\dot H^\rho(\mathbb{T}^2)}\\ &\le Ch^{-2-2\rho}\sum_\alpha |\langle\phi,\widetilde{\psi}_\alpha\rangle|^2\\ &\le \widetilde{C}(\beta,h)^2h^{-2-2\rho}h^{2+2\beta}\sum_\alpha\|\phi\|^2_{\dot H^\beta(\mathbb{T}^2)}\\ &\le \widetilde{C}(\beta,h)^2h^{-2-2\rho}h^{2+2\beta}\left(\frac{2\pi}{h}\right)^2\|\phi\|^2_{\dot H^\beta(\mathbb{T}^2)},\end{aligned}$$

where the constant $\widetilde{C}$ is defined as

$$\widetilde{C}(\alpha,h):=\begin{cases}C, & \alpha<1,\\ C\left(\frac{2\pi}{h}\right)^{1+|\alpha|-\epsilon(\alpha)}, & \alpha\ge 1,\end{cases}\tag{B.20}$$

where $\epsilon(\alpha)>0$ is chosen according to Proposition B.1.1 (v) and $C>0$ is some constant, depending on α.

Hence, by (B.16) we have

$$\|J_h\phi\|_{\dot H^\rho(\mathbb{T}^2)}\le C_I(\beta,h)h^{\beta-\rho}\|\phi\|_{\dot H^\beta(\mathbb{T}^2)},\tag{B.21}$$

where $C_I(\beta,h)$ is defined by (B.16), as desired.

Next, let $(\rho,\beta)\in[0,\infty)\times(-2,0]$. We estimate as before, except that we apply Lemma B.1.2 and Corollary B.2.2 to obtain

$$\begin{aligned}\|J_h\phi\|^2_{\dot H^\rho(\mathbb{T}^2)} &\le Ch^{-2-2\rho}\sum_\alpha|\langle\phi,\widetilde{\psi}_\alpha\rangle|^2\\ &\le Ch^{2\beta-2\rho}\sum_\alpha\|\phi_\alpha\|^2_{\dot H^\beta(\mathbb{T}^2)}+Ch^{-2\rho}\|\phi\|^2_{L^2(\mathbb{T}^2)}\\ &\le Ch^{2\beta-2\rho}\|\phi\|^2_{\dot H^\beta}+Ch^{-2\rho}\|\phi\|^2_{L^2(\mathbb{T}^2)},\end{aligned}$$

as desired.

Finally, let $(\rho,\beta)\in(-2,0)\times(-\infty,0]$. To prove (B.19), we proceed by duality. Let $\|\chi\|_{\dot H^{|\rho|}(\mathbb{T}^2)}=1$. Since J_h is self-adjoint and $\phi\in\mathcal{Z}$, it follows from Parseval's theorem and (B.17) that

$$\begin{aligned}|\langle J_h\phi,\chi\rangle_{L^2(\mathbb{T}^2)}| &\le \|\phi\|_{\dot H^\beta(\mathbb{T}^2)}\|J_h\chi\|_{\dot H^{|\beta|}(\mathbb{T}^2)}\\ &\le \widetilde{C}(|\rho|,h)h^{|\rho|-|\beta|}\left(\frac{2\pi}{h}\right)\|\phi\|_{\dot H^\beta(\mathbb{T}^2)}\|\chi\|_{\dot H^{|\rho|}(\mathbb{T}^2)}.\end{aligned}$$

Thus, we have

$$\|J_h\phi\|_{\dot H^\rho(\mathbb{T}^2)}\le C_I(|\rho|,h)h^{\beta-\rho}\|\phi\|_{\dot H^\beta(\mathbb{T}^2)},$$

as desired. □

Proposition B.2.2. *Let J_h be given by (B.11) or (B.12). Let $C_I(\alpha,h)$ be defined as in (B.16). Define*

$$\widetilde{C}_I := \frac{2\pi}{h}. \tag{B.22}$$

Let $\rho,\beta\in\mathbb{R}$. There exists a constant $C>0$, depending only on ρ,β, such that

(1) If $\rho\geq 0$ and $\beta=\ell$ is an integer, then

$$\|J_hD^\ell\phi\|_{\dot{H}^\rho(\mathbb{T}^2)}\leq Ch^{-(\rho+\ell-\ell')}\|\phi\|_{\dot{H}^{\ell'}(\mathbb{T}^2)},\quad 0\leq\ell'\leq\ell, \tag{B.23}$$

and

$$\|J_hD^\ell\phi\|_{\dot{H}^\rho(\mathbb{T}^2)}\leq Ch^{-1-(\rho+\ell-\ell')}\|D^{\ell'}\phi\|_{L^1(\mathbb{T}^2)},\quad 0\leq\ell'\leq\ell. \tag{B.24}$$

(2) If $\rho\in(-2,0)$, $\beta\in(-2,\infty)$, and $\beta'\in(-\infty,\beta]$, then

$$\|J_hD^\beta\phi\|_{\dot{H}^\rho(\mathbb{T}^2)}\leq C_I(|\rho|,h)h^{-(\rho+\beta-\beta')}\|\phi\|_{\dot{H}^{\beta'}(\mathbb{T}^2)}, \tag{B.25}$$

On the other hand, if $\beta=\ell$ is an integer, then

$$\|J_hD^\ell\phi\|_{\dot{H}^\rho(\mathbb{T}^2)}\leq C_I(|\rho|,h)h^{-1-\rho-\ell}\|\phi\|_{L^1(\mathbb{T}^2)}. \tag{B.26}$$

(3) For $\rho\geq0$ and $\beta\in(0,2)$ a non-integer we have

$$\|J_hD^\beta\phi\|_{\dot{H}^\rho(\mathbb{T}^2)}\leq \widetilde{C}_Ih^{-(\rho+\beta-\beta')}\|\phi\|_{\dot{H}^{\beta'}(\mathbb{T}^2)},\quad 0\leq\beta'\leq\beta. \tag{B.27}$$

Proof. Let $\rho\geq0$. By integrating by parts, Proposition B.1.1 (iv), and the Cauchy-Schwarz inequality we have

$$\begin{aligned}\|J_hD^\ell\phi\|^2_{\dot{H}^\rho(\mathbb{T}^2)}&\leq C\sum_\alpha|\widetilde{(D^\ell\phi)}_{Q_\alpha}|^2\|\widetilde{\psi}_\alpha\|^2_{\dot{H}^\rho(\mathbb{T}^2)}\\&\leq Ch^{-2-2\rho}\sum_\alpha|\langle D^{\ell'}\phi,D^{\ell-\ell'}\widetilde{\psi}_\alpha\rangle_{L^2(\mathbb{T}^2)}|^2\\&\leq Ch^{-2\rho-2(\ell-\ell')}\sum_\alpha\|D^{\ell'}\phi\|^2_{L^2(Q_\alpha(\epsilon))}\\&\leq Ch^{-2(\rho+\ell-\ell')}\|D^{\ell'}\phi\|^2_{L^2(\mathbb{T}^2)},\end{aligned}$$

which proves (B.23).

Similarly, estimating as before and applying Proposition B.1.1 (vi) (instead of (iv)) and Hölder's inequality (instead of Cauchy-Schwarz) we have

$$\begin{aligned}\|J_hD^\ell\phi\|^2_{\dot{H}^\rho(\mathbb{T}^2)}&\leq C\sum_\alpha|\widetilde{(D^\ell\phi)}_{Q_\alpha}|^2\|\widetilde{\psi}_\alpha\|^2_{\dot{H}^\rho(\mathbb{T}^2)}\\&\leq Ch^{-2-2\rho}\sum_\alpha|\langle D^{\ell'}\phi,D^{\ell-\ell'}\widetilde{\psi}_\alpha\rangle_{L^2(\mathbb{T}^2)}|^2\\&\leq Ch^{-2-2\rho-2(\ell-\ell')}\sum_\alpha\|D^{\ell'}\phi\|^2_{L^1(Q_\alpha(\epsilon))}\\&\leq Ch^{-2-2(\rho+\ell-\ell')}\|D^{\ell'}\phi\|^2_{L^1(\mathbb{T}^2)}.\end{aligned}$$

Arguing as before, we ultimately arrive at (B.24).

For $\rho \in (-2, 0)$ and $\beta' \in (-\infty, \beta]$, we proceed by duality. Indeed, let $\chi \in \dot{H}^{|\rho|}(\mathbb{T}^2)$ with $\|\chi\|_{\dot{H}^{|\rho|}(\mathbb{T}^2)} = 1$. Since J_h is self-adjoint, by Parseval's theorem we have

$$|\langle J_h D^\beta \phi, \chi\rangle_{L^2(\mathbb{T}^2)}| = |\langle \phi, D^\beta J_h \chi\rangle_{L^2(\mathbb{T}^2)}|.$$

Then by Parseval's theorem, the fact that $\phi \in \mathcal{Z}$, the Cauchy-Schwarz inequality, the Poincaré inequality, and (B.17) of Proposition B.2.1, we have

$$\begin{aligned}|\langle J_h D^\beta \phi, \chi\rangle_{L^2(\mathbb{T}^2)}| &\leq C\|\phi\|_{\dot{H}^{\beta'}(\mathbb{T}^2)}\|J_h\chi\|_{\dot{H}^{\beta-\beta'}(\mathbb{T}^2)}\\ &\leq C\|\phi\|_{\dot{H}^{\beta'}(\mathbb{T}^2)}\|J_h\chi\|_{\dot{H}^{\beta-\beta'}(\mathbb{T}^2)}\\ &\leq C_I(|\rho|, h)h^{|\rho|-(\beta-\beta')}\|\phi\|_{\dot{H}^{\beta'}(\mathbb{T}^2)}\|\chi\|_{\dot{H}^{|\rho|}(\mathbb{T}^2)},\end{aligned}$$

which implies (B.25).

On the other hand, to prove (B.26), let $\beta = k$ be an integer. Since J_h is self-adjoint, upon integrating by parts, then applying Hölder's inequality we obtain

$$\begin{aligned}|\langle J_h D^k \phi, \chi\rangle_{L^2(\mathbb{T}^2)}| &= |\langle \phi, D^k J_h \chi\rangle_{L^2(\mathbb{T}^2)}|\\ &\leq C\|\phi\|_{L^1(\mathbb{T}^2)}\|D^k J_h\chi\|_{L^\infty(\mathbb{T}^2)}.\end{aligned}$$

Observe that

$$(D^k J_h\chi)(x) = h^{-k}\sum_\alpha \widetilde{\chi}_{Q_\alpha}(D^k\widetilde{\Psi}_\alpha)(h^{-1}x).$$

Now recall that $N = \sup_{x\in\mathbb{T}^2} \#\{\alpha \in \mathcal{J} : x \in Q_\alpha(\epsilon)\} < \infty$. Let $\mathcal{J}'(x) := \{\alpha \in \mathcal{J} : x \in Q_\alpha(\epsilon)\}$. Then it follows from Parseval's theorem, the Cauchy-Schwarz inequality, and Proposition B.1.1 (iv) through (vi) that

$$\begin{aligned}|D^k J_h\chi(x)| &\leq Ch^{-2-k}\sum_{\alpha\in\mathcal{J}'(x)}\|\widetilde{\psi}_\alpha\|_{\dot{H}^\rho(\mathbb{T}^2)}\|\chi\|_{\dot{H}^{|\rho|}(\mathbb{T}^2)}\|D^k\widetilde{\Psi}_\alpha(h^{-1}\cdot\,)\|_{L^\infty(\mathbb{T}^2)}\\ &\leq C_I(|\rho|, h)Nh^{-1-k}h^{-\rho}\|\chi\|_{\dot{H}^\rho(\mathbb{T}^2)}.\end{aligned}$$

Therefore

$$|\langle J_h D^k\phi, \chi\rangle_{L^2(\mathbb{T}^2)}| \leq C_I(|\rho|, h)h^{-1-k-\rho}\|\phi\|_{L^1(\mathbb{T}^2)}\|\chi\|_{\dot{H}^\rho(\mathbb{T}^2)},$$

which implies (B.26), as desired.

Finally, we prove (B.27). Let $\beta \in (0, 2)$ a non-integer. Then by Proposition B.1.1 (iv) and (v), integration by parts, the fact that Λ is self-adjoint, and the Cauchy-Schwarz inequality we have

$$\begin{aligned}\|J_h D^\beta\phi\|^2_{\dot{H}^\rho(\mathbb{T}^2)} &\leq C\sum_\alpha |\widetilde{D^\beta\phi}_{Q_\alpha}|^2\|\widetilde{\psi}_\alpha\|^2_{\dot{H}^\rho(\mathbb{T}^2)}\\ &\leq Ch^{-2-\rho}\sum_\alpha |\langle D^{\beta'}\phi, D^{\beta-\beta'}\widetilde{\psi}_\alpha\rangle_{L^2(\mathbb{T}^2)}|^2\\ &\leq Ch^{-2\rho-2(\beta-\beta')}\left(\frac{2\pi}{h}\right)^2\|\phi\|^2_{\dot{H}^{\beta'}(\mathbb{T}^2)}.\end{aligned}$$

This completes the proof. □

Remark B.1. *We point out that all of the above boundedness properties for J_h hold also when J_h is given by projection onto finitely many Fourier modes, in specific, when J_h is given by the Littlewood-Paley projection. The only difference is in the constants. Indeed, one may* *notice above that this "defect" between the spectral projection and the "volume-elements" projection can be traced to the fact the operator, $\Lambda^\beta, \beta \in (-2,2)$, is a non-local operator; although its input may be compactly supported, the output need not have compact support. Generally speaking, the projection onto Fourier modes up to wave-number $\lesssim 1/h$ satisfies convenient "orthogonality" properties, as captured by the Bernstein inequalities, that are not enjoyed by projection onto local spatial averages. The above boundedness properties then follow immediately from this inequality and the fact that differential operators will commute J_h when it is given as this projection. For this reason, we omit the details, but refer to [33], where the relevant estimates are carried out.*

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