Real Analysis Exchange

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A SIMPLE EXAMPLE CONCERNING THE UPPER BOX-COUNTING DIMENSION OF A CARTESIAN PRODUCT

Abstract

We give a simple example of two countable sets X and Y of real numbers such that their upper box-counting dimension satisfies the strict inequality $\dim_{\mathrm{b}}(X \times Y) < \dim_{\mathrm{b}}(X) + \dim_{\mathrm{b}}(Y)$.

1 Introduction

Given a metric space X with metric d_X let $\dim_{\mathrm{b}}(X)$ denote the upper box-counting dimension of X, defined by

$$\dim_{\mathrm{b}}(X) = \limsup_{\epsilon \to 0} \frac{\log N(X,\epsilon)}{-\log \epsilon},$$

where $N(X,\epsilon)$ denotes the minimum number of balls of radius ϵ required to cover X, see Falconer [2], Robinson [3], or Tricot [5], for example. (Note that some authors refer to this as the 'fractal dimension', see [1], for example.)

Let the metric space $X \times Y$ be the Cartesian product of X and Y along with a metric $d_{X\times Y}$ assumed to be equivalent to $d_X + d_Y$. It was shown by Tricot [5] that in general

$$\dim_{\mathrm{b}}(X \times Y) \le \dim_{\mathrm{b}}(X) + \dim_{\mathrm{b}}(Y). \tag{1}$$

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Robinson and Sharples [4] gave a somewhat involved example of two generalised Cantor sets X and Y of real numbers for which the inequality in (1) is strict. In this paper we give another, but significantly simpler, example of two countable subsets of real numbers for which the same strict inequality holds.

2 The Example

For convenience we use the notation

$$\operatorname{sll} t = \sin\log\log t \quad \text{and} \quad \operatorname{cll} t = \cos\log\log t.$$

We show that the two sets

$$X = \{\, f(n) : n \in \mathbb{N} \text{ and } n \geq 25 \,\} \cup \{\, 0 \,\}, \quad \text{where} \quad f(t) = t^{-8-\operatorname{sll} t},$$

and

$$Y = \{\, g(n) : n \in \mathbb{N} \text{ and } n \geq 25 \,\} \cup \{\, 0 \,\}, \quad \text{where} \quad g(t) = t^{-8+\operatorname{sll} t},$$

satisfy $\dim_{\mathrm{b}}(X \times Y) < \dim_{\mathrm{b}}(X) + \dim_{\mathrm{b}}(Y)$. Specifically, we will show that

$$\dim_{\mathrm{b}}(X) \geq 1/8, \quad \dim_{\mathrm{b}}(Y) \geq 1/8, \quad \text{and} \quad \dim_{\mathrm{b}}(X \times Y) < 1/4.$$

We begin with a preliminary lemma that gives upper and lower bounds for certain coverings of subsets of X and Y.

Lemma 1. *Choose $r < 5^{-20}$ and let t_1 be such that $r = t_1^{-9-\operatorname{sll} t_1}$. Define*

$$B = \{ f(n) :\ 25 \leq n < t_1 \}$$

Then $t_1 - 26 \leq N(B, r/2) \leq t_1 - 24$.

Proof. First note that $t_1 = r^{-1/(9+\operatorname{sll} t_1)} > \left(5^{20}\right)^{1/(9+\operatorname{sll} t_1)} \geq 5^2 = 25$. Since

$$f'(t) = -t^{-9-\operatorname{sll} t}(8 + \operatorname{sll} t + \operatorname{cll} t) < 0 \tag{2}$$

the sequence $f(n)$ is decreasing, so we can bound the distance between points in B by considering $|f(n+1) - f(n)|$. To bound this we write

$$|f(n+1) - f(n)| = |f'(n) + \tfrac{1}{2} f''(\xi)|$$

for some $\xi \in (n, n+1)$, using Taylor's Theorem. Since $\xi > n \geq 25$ certainly

$$\begin{aligned} f''(\xi) &= \xi^{-10-\operatorname{sll}\xi}\left\{(9 + \operatorname{sll}\xi + \operatorname{cll}\xi)(8 + \operatorname{sll}\xi + \operatorname{cll}\xi) - \frac{\operatorname{cll}\xi - \operatorname{sll}\xi}{\log\xi}\right\} \\ &\leq 112\,\xi^{-10-\operatorname{sll}\xi} \leq 5\xi^{-9-\operatorname{sll}\xi} \leq 5n^{-9-\operatorname{sll} n} \end{aligned}$$

and $f''(\xi) \geq 40\,\xi^{-10-\operatorname{sll}\xi} > 0$. We therefore obtain the upper bound

$$|f(n+1) - f(n)| = |f'(n) + \tfrac{1}{2}f''(\xi)| \leq 13\,n^{-9-\operatorname{sll}n}.$$

Since $f'(n) < -6n^{-9-\operatorname{sll}n}$ by (2), we also obtain the lower bound

$$|f'(n) + \tfrac{1}{2}f''(\xi)| = |f'(n)| - \tfrac{1}{2}f''(\xi) \geq 6\,n^{-9-\operatorname{sll}n} - 5\,n^{-9-\operatorname{sll}n} = n^{-9-\operatorname{sll}n}.$$

It follows that exactly one $r/2$-ball is required to cover each of the points in B. Therefore

$$N(B, r/2) = \operatorname{card}\{\, n \in \mathbb{N} : 25 \leq n < t_1 \,\}$$

and the lemma follows. $\square$

The slow fluctuation in these upper and lower bounds allows us to prove our main result.

Theorem 2. $\dim_{\mathrm{b}}(X) \geq 1/8$, $\dim_{\mathrm{b}}(Y) \geq 1/8$, *and* $\dim_{\mathrm{b}}(X \times Y) < 1/4 \leq \dim_{\mathrm{b}}(X) + \dim_{\mathrm{b}}(Y)$.

PROOF. First we bound the dimension of X; the bound for Y follows similarly. Let $r < 5^{-20}$ and let t_1 be such that $r = t_1^{-9-\operatorname{sll}t_1}$. Let

$$B = \{\, f(n) : 25 \leq n < t_1 \} \quad \text{and} \quad C = \{\, f(n) : n \geq t_1 \,\}$$

so that $X = B \cup C$. Taking $r \to 0$ along a sequence such that $\operatorname{sll}t_1 = -1$ we can use the result of the lemma to obtain the lower bound

$$N(X, r/2) \geq N(B, r/2) \geq t_1 - 26 \geq r^{-1/(9+\operatorname{sll}t_1)} - 26 \geq r^{-1/8} - 26$$

and therefore $\dim_{\mathrm{b}}(X) \geq 1/8$. The lower bound on $\dim_{\mathrm{b}}(Y)$ follows similarly.

To deal with the product set $X \times Y$, notice that since $C \subseteq [0, f(t_1)]$ it follows that

$$N(C, r/2) \leq \frac{f(t_1)}{r/2} = 2t_1 = 2r^{-1/(9+\operatorname{sll}t_1)}.$$

Lemma 1 provides an estimate on $N(B, r/2)$ from above, so we obtain

$$N(X, r/2) \leq N(B, r/2) + N(C, r/2) \leq K_1 r^{-1/(9+\operatorname{sll}t_1)}.$$

Defining t_2 so that $r = t_2^{-9+\operatorname{sll}t_2}$ a similar argument guarantees that

$$N(Y, r/2) \leq K_2 r^{-1/(9-\operatorname{sll}t_2)}.$$

Therefore

$$N(X \times Y, r/2) \leq N(Y, r/2)N(X, r/2) \leq K_1 K_2 \left(\frac{1}{r}\right)^{\frac{1}{9-\mathrm{sll}\, t_1} + \frac{1}{9+\mathrm{sll}\, t_2}}.$$

Now, since $t_1^{9+\mathrm{sll}\, t_1} = t_2^{9-\mathrm{sll}\, t_1}$, taking logarithms once yields

$$\frac{\log t_1}{\log t_2} = \frac{9 - \mathrm{sll}\, t_2}{9 + \mathrm{sll}\, t_1} \leq 5/4$$

and taking logarithms again shows that $|\log\log t_1 - \log\log t_2| \leq \log(5/4)$. It follows that $N(X \times Y, r/2) \leq K_1 K_2 \,(2/r)^c$, where

$$c = \max\left\{\frac{1}{9 - \sin\theta_1} + \frac{1}{9 + \sin\theta_2} : \ |\theta_1 - \theta_2| \leq \log(5/4)\right\} < 1/4 :$$

clearly $c \leq 2 \times 1/8 = 1/4$, and equality cannot hold since this would require $\sin\theta_1 = 1$ and $\sin\theta_2 = -1$, which is impossible since $|\theta_1 - \theta_2| < \pi$. It follows that

$$\dim_{\mathrm{b}}(X \times Y) \leq c < 1/4 \leq \dim_{\mathrm{b}}(X) + \dim_{\mathrm{b}}(Y),$$

which finishes the proof. □

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