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ALMOST BI-LIPSCHITZ EMBEDDINGS AND ALMOST HOMOGENEOUS SETS

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ABSTRACT. This paper is concerned with embeddings of homogeneous spaces into Euclidean spaces. We show that any homogeneous metric space can be embedded into a Hilbert space using an almost bi-Lipschitz mapping (bi-Lipschitz to within logarithmic corrections). The image of this set is no longer homogeneous, but 'almost homogeneous'. We therefore study the problem of embedding an almost homogeneous subset X of a Hilbert space H into a finite-dimensional Euclidean space. In fact we show that if X is a compact subset of a Banach space and $X - X$ is almost homogeneous then, for N sufficiently large, a prevalent set of linear maps from X into $\mathbb{R}^N$ are almost bi-Lipschitz between X and its image. We are then able to use the Kuratowski embedding of (X, d) into $L^\infty(X)$ to prove a similar result for compact metric spaces.

1. Introduction

In this paper we investigate abstract embeddings between metric spaces, Hilbert spaces, and finite-dimensional Euclidean spaces. Historically (starting with Bouligand in 1928), attention has been on bi-Lipschitz embeddings. By weakening this to almost bi-Lipschitz embeddings we are able to obtain a number of new results.

A metric space (X, d) is said to be (M, s)-*homogeneous* (or simply *homogeneous*) if any ball of radius r can be covered by at most $M(r/\rho)^s$ smaller balls of radius ρ. Since any subset of $\mathbb{R}^N$ is homogeneous and homogeneity is preserved under bi-Lipschitz mappings, it follows that (X, d) must be homogeneous if it is to admit a bi-Lipschitz embedding into some $\mathbb{R}^N$ (cf. comments in Hajłasz, 2003). The *Assouad dimension* of X, $d_{\rm A}(X)$, is the infimum of all s such that (X, d) is (M, s)-homogeneous for some $M \geq 1$.

Assouad (1983) showed that (X, d) is homogeneous if and only if the snowflake spaces (X, d^α) with $0 < \alpha < 1$ admit bi-Lipschitz embeddings into some $\mathbb{R}^N$ (where N depends on α). However, there are examples due to Laakso (2002; see also Lang & Plaut, 2001) of homogeneous spaces that do not admit a bi-Lipschitz embedding into any $\mathbb{R}^N$, nor even into an infinite-dimensional Hilbert space. This paper starts with a simple result, based on Assouad's argument, that any homogeneous metric space admits an *almost bi-Lipschitz* embedding into an infinite-dimensional Hilbert space.

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The class of γ-almost L-bi-Lipschitz mappings $f:(X,d)\to(\tilde{X},\tilde{d})$ (or *almost bi-Lipschitz mappings* for short) consists of all those maps for which there exists a $\gamma\geq 0$ and an $L>0$ such that

$$\frac{1}{L}\frac{d(x,y)}{\operatorname{slog}(d(x,y))^{\gamma}}\leq\tilde{d}(f(x),f(y))\leq L\,d(x,y) \tag{1.1}$$

for all $x,y\in X$ such that $x\neq y$. Here $\operatorname{slog}(x)$ is the 'symmetric logarithm' of x, defined as

$$\operatorname{slog}(x):=\log(x+x^{-1}),$$

and so an almost bi-Lipschitz map is bi-Lipschitz to within logarithmic corrections.

Although the bi-Lipschitz image of a homogeneous set is homogeneous, this is not true for almost bi-Lipschitz images; they are, however, *almost homogeneous*: we say that (X,d) is (α,β)-almost (M,s)-homogeneous if

$$\mathcal{N}_X(r,\rho)\leq M\left(\frac{r}{\rho}\right)^{s}\operatorname{slog}(r)^{\beta}\operatorname{slog}(\rho)^{\alpha} \tag{1.2}$$

for all $0<\rho<r<\infty$. The *Assouad (α,β)-dimension* of X, $d_{\rm A}^{\alpha,\beta}(X)$, is the infimum of all s such that X is (α,β)-almost (M,s)-homogeneous for some $M\geq 1$.

Olson (2002) showed that given a compact $X\subset\mathbb{R}^N$ with $d_{\rm A}(X-X)=d$ then almost every projection of rank $k>d$ provides an almost bi-Lipschitz embedding of X into $\mathbb{R}^k$. In this paper we show a similar result for compact subsets X of a Hilbert space: If the set of differences[1] $X-X$ is almost homogeneous with $d_{\rm A}^{\alpha,\beta}(X-X)=d$ then 'most' linear maps into Euclidean spaces $\mathbb{R}^k$ of with $k>d$ provide almost bi-Lipschitz embeddings of X. More explicitly, if $k>d$ then the set of almost bi-Lipschitz embeddings into $\mathbb{R}^k$ is prevalent in the space of all linear maps into $\mathbb{R}^k$, in the sense of Hunt, Sauer & Yorke (1992). We then extend this result to subsets of Banach spaces.

There is an unfortunate gap here. An almost homogeneous metric space has an almost bi-Lipschitz image that is an almost homogeneous subset of a Hilbert space. However, our embedding theorem for a subset X of a Hilbert space requires that not X itself, but the set $X-X$ of differences is almost homogeneous.

By using the Kuratowski isometric embedding of (X,d) into $L^\infty(X)$ we can assign a meaning to "$X-X$" even when X is a metric space. With this interpretation, we can also show that if (X,d) is a compact metric space then the assumption that $X-X$ is almost homogeneous is sufficient to ensure that (X,d) can be embedded into a Euclidean space in an almost bi-Lipschitz way.

In Section 2 we state some elementary properties of the (α,β)-Assouad dimension and show that any almost homogeneous metric space (X,d) can be embedded into a Hilbert space in an almost bi-Lipschitz way; that such almost bi-Lipschitz images of almost homogeneous spaces are again almost homogeneous is shown in Section 3. Section 4 treats the local versions of homogeneity and almost homogeneity. Section 5 contains our main result on embedding a subset X of a Hilbert space with

[1] The introduction of a condition on the dimension of the set $X-X$ of differences, rather than on X itself, is common in the literature on abstract embeddings. The proof of Mañé's 1981 embedding theorem requires the Hausdorff dimension of $X-X$ to be finite, a condition not ensured by the finiteness of $d_{\rm H}(X)$. Foias & Olson (1996) and Hunt & Kaloshin (1999) treat the upper box-counting dimension which is unusual in having the property that $d_{\rm F}(X)<\infty$ implies that $d_{\rm F}(X-X)<\infty$. [Recall that $d_{\rm F}(X)=\limsup_{\epsilon\to 0}\log N(X,\epsilon)/(-\log\epsilon)$, where $N(X,\epsilon)$ is the minimum number of balls of radius ϵ needed to cover X.]

$X - X$ almost homogeneous, while in Section 6 we consider what is possible for such subsets knowing only properties of X. In Section 7 we generalise our main theorem to treat subsets of Banach spaces, and hence give a result for compact metric spaces. In Section 8 we explore the relationship between $d_{\mathrm{A}}^{\alpha,\beta}(X)$ and $d_{\mathrm{A}}^{\alpha,\beta}(X-X)$. After Section 9, where we give an example of a homogeneous set that cannot be bi-Lipschitz embedded into any $\mathbb{R}^k$ using any *linear* map, we finish with some interesting open problems.

2. Almost homogeneous metric spaces

As discussed above, we will say that a metric space (X, d) is (α, β)-almost (M, s)-homogeneous (or simply almost homogeneous) if any ball of radius r can be covered by at most[2]

$$\mathcal{N}_X(r,\rho) \leq M \left(\frac{r}{\rho}\right)^s \operatorname{slog}(r)^\beta \operatorname{slog}(\rho)^\alpha \tag{2.1}$$

balls of radius ρ (with $\rho < r$), for some $M \geq 1$ and $s \geq 0$, where $\operatorname{slog}(x) = \log(x + x^{-1})$.

We now give some simple properties of the function slog.

Lemma 2.1. *Given $L > 0$ and $\gamma \geq 0$, there exist constants $A_L, B_L, a_\gamma, b_\gamma, \sigma \in (0, \infty)$ independent of x such that*

(p1) $|\log(x)| \leq \operatorname{slog}(x) \leq \log 2 + |\log(x)|$, *in particular* $\operatorname{slog}(2^k) \leq (1 + |k|) \log 2$,
(p2) $A_L \operatorname{slog}(x) \leq \operatorname{slog}(Lx) \leq B_L \operatorname{slog}(x)$,
(p3) $a_\gamma \operatorname{slog}(x) \leq \operatorname{slog}(x \operatorname{slog}(x)^\gamma) \leq b_\gamma \operatorname{slog}(x)$,

for all $x \geq 0$, and

(p4) *if* $2^{-(k+1)} \leq x \leq 2^{-k}$ *then* $\operatorname{slog}(x) \geq \sigma \operatorname{slog}(2^{-k})$.

Proof. (p1) is elementary. For (p2) consider the quotient function $g : (0, \infty) \to (0, \infty)$ defined by

$$g(x) = \frac{\operatorname{slog}(Lx)}{\operatorname{slog}(x)}.$$

Let $a_L = \inf\{\, g(x) : x \in (0, \infty)\,\}$ and $b_L = \sup\{\, g(x) : x \in (0, \infty)\,\}$. Since

$$\lim_{x\to 0} g(x) = 1, \quad \lim_{x\to\infty} g(x) = 1, \quad \text{and} \quad 0 < g(x) < \infty \text{ for } x \in (0, \infty),$$

then both a_L and b_L are finite positive constants. The proof of (p3) is similar. For (p4) set $x = 2^{-r}$ with $k \leq r \leq k + 1$. Since $\operatorname{slog}(x) = \log(x + 1/x) \geq \log 2$ and $\operatorname{slog}(2^{-r}) \geq |\log 2^{-r}| = |r| \log 2$ from (p1), then $\operatorname{slog}(x) \geq (1 + |r|)/2$. Therefore, the estimate

$$\frac{\operatorname{slog}(2^{-k})}{\operatorname{slog}(x)} \leq \frac{(1 + |k|) \log 2}{(1 + |r|)/2} \leq 4 \log 2$$

gives (p4) with $\sigma = 1/(4 \log 2)$. $\square$

[2]For bounded metric spaces (2.1) could be replaced by

$$\mathcal{N}_X(r,\rho) \leq M' \left(\frac{r}{\rho}\right)^s \log(\mathrm{e} + \rho^{-1})^\gamma,$$

(in terms of our current definition we would have $M' \geq M$ and $\gamma = \alpha + \beta$) while for compact spaces the factor of e in the logarithm could also be dropped by considering only $\rho \leq r \leq \epsilon$ for some $\epsilon > 0$ (see Section 4). However, (2.1) allows us to treat general metric spaces.

We define the Assouad (α,β)-dimension of X, $d_{\rm A}^{\alpha,\beta}(X)$, to be the infimum of all s for which X is (α,β)-almost (M,s)-homogeneous. When $\alpha=\beta=0$ we recover the standard definition of a homogeneous space and the usual Assouad dimension.

We note here that it is straightforward to show that the Assouad (α,β)-dimension satisfies the minimal properties we would ask for in a dimension, namely that

$$X\subseteq Y \quad\Rightarrow\quad d_{\rm A}^{\alpha,\beta}(X)\le d_{\rm A}^{\alpha,\beta}(Y),\qquad d_{\rm A}^{\alpha,\beta}(X\cup Y)=\max(d_{\rm A}^{\alpha,\beta}(X),d_{\rm A}^{\alpha,\beta}(Y)),$$

and $d_{\rm A}^{\alpha,\beta}(\mathcal{O})=n$ if $\mathcal{O}$ is an open subset of $\mathbb{R}^n$. Furthermore,

$$\alpha_1\ge\alpha_2 \quad\text{and}\quad \beta_1\ge\beta_2 \quad\Rightarrow\quad d_{\rm A}^{\alpha_1,\beta_1}(X)\le d_{\rm A}^{\alpha_2,\beta_2}(X). \tag{2.2}$$

We now show that if (X,d) is almost homogeneous then it can be embedded into an infinite-dimensional Hilbert space in an almost bi-Lipschitz way. Key to this result is the following proposition, which although not given explicitly in this form, essentially occurs in Assouad's paper. Indeed, it is the main ingredient in his proof of the existence of bi-Lipschitz maps between (X,d^α) and $\mathbb{R}^N$.

Proposition 2.2. *Let (X,d) be an (α,β)-almost (M,s)-homogeneous metric space and distinguish a point $a\in X$. Then there are constants $A,B,C>0$ such that for every $j\in\mathbb{Z}$ there exists a map $\phi_j:(X,d)\to\mathbb{R}^{M_j}$, where $M_j=C(1+|j|)^{\alpha+\beta}$, with $\phi_j(a)=0$, and for every $x_1,x_2\in X$*

(a1) *$2^{-(j+1)}<d(x_1,x_2)\le 2^{-j}$ implies that $\|\phi_j(x_1)-\phi_j(x_2)\|\ge A$, and*
(a2) *$\|\phi_j(x_1)-\phi_j(x_2)\|\le BM_j\min[1,2^jd(x_1,x_2)]$.*

Proof. The proof follows exactly the steps in Assouad's original paper (see also the lecture notes of Heinonen (2003) for an account that is easier to follow) which we outline very briefly here: if N_j is a maximal 2^{-j} net in (X,d), then for every $x\in X$

$$\begin{aligned}\operatorname{card}\left(N_j\cap B(x,12\cdot 2^{-j})\right) &\le \mathcal{N}_X(12\cdot 2^{-j},2^{-j-1})\\ &\le 24M\operatorname{slog}(12\cdot 2^{-j})^\alpha\operatorname{slog}(2^{-j-1})^\beta\\ &\le C(1+|j|)^{\alpha+\beta}\end{aligned}$$

where the constant C is a product of M and the constants appearing in Proposition 2.1. Thus, there exists a 'colouring map' $\kappa_j:N_j\to\{e_1,\dots,e_{M_j}\}$, where $\{e_1,\dots,e_{M_j}\}$ is the standard basis of $\mathbb{R}^{M_j}$, such that $\kappa_j(a)\ne\kappa_j(b)$ if $d(a,b)<12\cdot 2^{-j}$. Let

$$\tilde\phi_j(x)=\sum_{a_i\in N_j}\max\left\{(2-2^jd(x,a_i)),0\right\}\kappa_j(a_i).$$

Note that $2^{2-j}<d(x_1,x_2)\le 2^{3-j}$ implies $\tilde\phi_j(x_1)$ is orthogonal to $\tilde\phi_j(x_2)$. It is then straightforward to show that the map $\phi_j(x)=\tilde\phi_{j+3}(x)-\tilde\phi_{j+3}(a)$ satisfies the properties given in the statement of the proposition. □

Theorem 2.3. *Let (X,d) be an (α,β)-almost (M,s)-homogeneous metric space and H an infinite-dimensional separable Hilbert space. Then, for every $\gamma>\alpha+\beta+\frac12$, there exists a map $f:X\to H$ and a constant L such that*

$$\frac{1}{L}\frac{d(x,y)}{\operatorname{slog}(d(x,y))^\gamma}\le\|f(s)-f(t)\|\le L\,d(x,y),$$

i.e., f is γ-almost bi-Lipschitz.

Proof. Let $\{e_j\}_{j\in\mathbb{Z}}$ be an orthonormal set of vectors in some Hilbert space. Let $\delta > 1/2$ and define $f : (X, d) \to \bigoplus_{j=1}^{\infty} \mathbb{R}^{M_j} \otimes e_j \simeq H$ by

$$f(x) = \sum_{j=-\infty}^{\infty} \frac{2^{-j}}{(1+|j|)^{\delta} M_j}\, \phi_j(x) \otimes e_j, \tag{2.3}$$

where the maps ϕ_j are those of Proposition 2.2. Since $f(a) = 0$, then the upper bound on $\|f(s) - f(t)\|$ that we now prove will also show convergence of the series (2.3) defining f. Let (x_1, x_2) be a pair of distinct points of X. Thus, there exists $l \in \mathbb{Z}$ such that $2^{-(l+1)} < d(x_1, x_2) \leq 2^{-l}$. Note that for such a pair of points $\|\phi_l(x_1) - \phi_l(x_2)\| \geq A$. We have

$$\begin{aligned} \|f(x_1) - f(x_2)\|^2 &= \sum_{j=-\infty}^{\infty} \frac{2^{-2j}}{(1+|j|)^{2\delta}} \frac{\|\phi_j(x_1) - \phi_j(x_2)\|^2}{M_j^2} \\ &\leq \sum_{j=-\infty}^{\infty} \frac{B^2}{(1+|j|)^{2\delta}}\, d(x_1, x_2)^2 \\ &\leq c_1\, d(x_1, x_2)^2, \end{aligned}$$

where the sum converges since $2\delta > 1$.

The lower bound is straightforward, since

$$\begin{aligned} \|f(x_1) - f(x_2)\| &\geq \frac{2^{-l}}{(1+|l|)^{\delta} M_l} \|\phi_l(x_1) - \phi_l(x_2)\| \;\geq\; A\, \frac{2^{-l}}{(1+|l|)^{\delta} M_l} \\ &\geq c_2\, \frac{2^{-l}}{(1+|l|)^{\alpha+\beta+\delta}} \;\geq\; c_2 \frac{d(x, y)}{(1+|l|)^{\alpha+\beta+\delta}} \end{aligned}$$

Since $d(x, y) = 2^{-r}$ with $l \leq r < l + 1$ it follows using (p1) from Lemma 2.1 that

$$\frac{1+|l|}{\operatorname{slog}(d(x, y))} = \frac{1+|l|}{\operatorname{slog}(2^{-r})} \geq \frac{1+|l|}{(1+|r|)\log 2} \geq \frac{1}{2\log 2},$$

and so

$$\|f(x_1) - f(x_2)\| \geq c_3 \frac{d(x, y)}{\operatorname{slog}(d(x, y))^{\alpha+\beta+\delta}}.$$

Taking $L = \max(c_1, 1/c_3)$ finishes the proof. □

We note here that if (X, d) is bounded then there exists a k such that $d(x_1, x_2) \leq 2^k$ for all $x_1, x_2 \in X$. In this case the definition of f in (2.3) can be simplified to

$$f(x) = \sum_{j=-k}^{\infty} \frac{2^{-j}}{(1+|j|)^{\delta} M_j}\, \phi_j(x) \otimes e_j \tag{2.4}$$

and will still provide a γ-almost bi-Lipschitz embedding.

3. Almost bi-Lipschitz images of sets

Since we can embed any almost homogeneous metric space into a Hilbert space using an almost bi-Lipschitz map, it is natural to study the effect of such mappings on almost homogeneous spaces. Here we show that almost bi-Lipschitz images of almost homogeneous metric spaces are still almost homogeneous. In particular this implies that it is necessary that X be almost homogeneous if it is to enjoy an almost bi-Lipschitz embedding into some $\mathbb{R}^N$.

Lemma 3.1. *Let (X,d) be an (α,β)-almost (M,s)-homogeneous metric space and $\phi:(X,d)\to(\tilde X,\tilde d)$ a γ-almost L-bi-Lipschitz map. Then $(\phi(X),\tilde d)$ is an almost homogeneous metric space with $d_{\rm A}^{\alpha+\gamma,\beta+\gamma}(X)\le d_{\rm A}^{\alpha,\beta+\gamma}(\phi(X))\le d_{\rm A}^{\alpha,\beta}(X)$.*

Proof. Increase L if necessary so that

$$L^2 b^\gamma(\log 2)^\gamma\ge 1, \tag{3.1}$$

where here and in the rest of the proof $b=b_\gamma$, where b_γ is the constant occurring in (p3) in Lemma 2.1; clearly ϕ remains γ-almost L-bi-Lipschitz under this assumption. Take $s>d_{\rm A}^{\alpha,\beta}(X)$, $0<\rho<r<\infty$, and consider an arbitrary ball $B_{\tilde X}(\phi(x),r)$ of radius r in $\phi(X)$. Now, we have

$$B_{\tilde X}(\phi(x),r)\subseteq\phi\left\{B_X(x,Lrb^\gamma\mathrm{slog}(Lrb^\gamma)^\gamma)\right\},$$

since using (p3) in Lemma 2.1

$$\frac{1}{L}\frac{Lrb^\gamma\mathrm{slog}(Lrb^\gamma)^\gamma}{\mathrm{slog}(Lrb^\gamma\mathrm{slog}(Lrb^\gamma)^\gamma)^\gamma}\ge\frac{rb^\gamma\mathrm{slog}(Lrb^\gamma)^\gamma}{[b\,\mathrm{slog}(Lrb^\gamma)]^\gamma}=r.$$

By our choice of L in (3.1) and since $\rho<r$ we have $0<\rho/L<Lrb^\gamma\mathrm{slog}(Lrb^\gamma)^\gamma$ and so we can cover $B_X(x,Lrb^\gamma\mathrm{slog}(Lrb^\gamma)^\gamma)$ by

$$\begin{aligned}&\mathcal N_X(Lrb^\gamma\mathrm{slog}(Lrb^\gamma)^\gamma,\rho/L)\\&\qquad\le M\left(\frac{Lrb^\gamma\,\mathrm{slog}(Lrb^\gamma)^\gamma}{\rho/L}\right)^s\mathrm{slog}(Lrb^\gamma\,\mathrm{slog}(Lrb^\gamma)^\gamma)^\beta\,\mathrm{slog}(\rho/L)^\alpha\\&\qquad\le c_1\left(\frac{r}{\rho}\right)^s\mathrm{slog}(r)^\beta\mathrm{slog}(\rho)^\alpha\end{aligned}$$

balls of radius ρ/L (in X) where c_1 depends on M, L and the constants appearing in Lemma 2.1. Denote these balls by $B_X(x_i,\rho/L)$. Since

$$\phi\left\{B_X(x_i,\rho/L)\right\}\subseteq B_{\tilde X}(\phi(x_i),\rho)$$

and $B_{\tilde X}(\phi(x),r)$ was arbitrary, it follows that

$$\mathcal N_{\phi(X)}(r,\rho)\le c_1\left(\frac{r}{\rho}\right)^s\mathrm{slog}(r)^\beta\mathrm{slog}(\rho)^\alpha$$

for any $0<\rho<r<\infty$. Thus $\phi(X)$ is $(\alpha,\beta+\gamma)$-almost (c_1,s)-homogeneous. Taking the infimum over $s>d_{\rm A}^{\alpha,\beta}(X)$ yields $d_{\rm A}^{\alpha,\beta+\gamma}(\phi(X))\le d_{\rm A}^{\alpha,\beta}(X)$.

By considering similarly the inverse map $\phi^{-1}:\phi(X)\to X$ one obtains the lower bound $d_{\rm A}^{\alpha,\beta+\gamma}(\phi(X))\ge d_{\rm A}^{\alpha+\gamma,\beta+\gamma}(X)$. □

Combined with Lemma 3.1, the embedding result of Proposition 2.3 shows that any almost homogeneous metric space (X,d) has an almost bi-Lipschitz image $f(X)$ that is an almost homogeneous subset of a Hilbert space.

We end by noting since almost bi-Lipschitz maps are, in fact, Lipschitz then for any almost bi-Lipschitz map ϕ the upper box-counting ('fractal') dimension satisfies $d_{\rm F}(\phi(X))\le d_{\rm F}(X)$. Moreover, it is not difficult to prove the following:

Lemma 3.2. *Let (X,d) be a metric space and $\phi:(X,d)\to(\tilde X,\tilde d)$ an almost bi-Lipschitz map. Then $d_{\rm F}(\phi(X))=d_{\rm F}(X)$.*

4. Aside: Compact spaces and local versions of (almost) homogeneity

In this section we briefly discuss the local definitions of homogeneity and almost homogeneity, and the dimensions associated with them. While they agree for compact spaces, they are distinct in general.

A metric space (X, d) is said to be *locally* (M, s)*-homogeneous* (or simply *locally homogeneous*) if there exists an $\epsilon > 0$ such that any ball of radius $r < \epsilon$ can by covered by at most $M(r/\rho)^s$ smaller balls of radius ρ. The introduction of the constant ϵ for a locally homogeneous space may be interpreted as the small scale beneath which the set may be viewed as homogeneous. In this case M may depend on ϵ which in turn depends on the units of measurement used in the definition of the metric.

Movahedi-Lankarani (1992) defined the metric (or 'Bouligand') dimension

$$d_{\mathrm{B}}(X) = \lim_{\epsilon\to 0}\lim_{t\to\infty}\sup\left\{\frac{\log \mathcal{N}_X(r,\rho)}{\log(r/\rho)} : 0<\rho<r<\epsilon \text{ and } r>t\rho\right\}, \tag{4.1}$$

where $\mathcal{N}_X(r, \rho)$ is the minimum number of balls of radius ρ necessary to cover any ball of radius r. This dimension, $d_{\mathrm{B}}(X)$, is the infimum of all s such that (X, d) is locally (M, s)-homogeneous for some $M \geq 1$.

Here we give a simple example that shows that the concepts of homogeneous and locally homogeneous are indeed different. Let H be a Hilbert space with orthonormal basis given by $\{e_n\}_{n\in\mathbb{N}}$. Define

$$X = \{\, \rho_n e_n : n \in \mathbb{N}\,\} \quad \text{where} \quad \rho_n = 1 - \frac{1}{n}.$$

If (X, d) is (M, s)-homogeneous for some M and s then

$$\mathcal{N}_X(\rho_{2n}, \rho_n) \leq M(\rho_{2n}/\rho_n)^s = M\left(\frac{2n-1}{2n-2}\right)^s \leq M. \tag{4.2}$$

However, each ball $B(0, \rho_{2n})$ contains the n points

$$\{0\} \cup \{\, \rho_k e_k : n < k < 2n\,\}$$

which are mutually more than a distance ρ_n apart. Therefore $\mathcal{N}_X(\rho_{2n}, \rho_n) \geq n$. Taking n large enough shows that (4.2) cannot hold, and so (X, d) is not homogeneous. On the other hand, (X, d) is locally homogeneous for any $\epsilon < 1$.

Note that if (X, d) is compact, then the notions of homogeneous and locally homogeneous are equivalent (see Olson, 2002). Thus $d_{\mathrm{A}}(X) = d_{\mathrm{B}}(X)$ for compact spaces X.

As with homogeneous spaces, there is a similarly distinct notion of *locally* (α, β)*-almost* (M, s)*-homogeneous*. This means there is some $\epsilon > 0$ such that (2.1) holds for all $0 < \rho < r < \epsilon$. Similar arguments to those given in Olson (2002) show that the notions of almost homogeneous and locally almost homogeneous are equivalent when (X, d) is compact. Define the *local Assouad* (α, β)*-dimension* of X, $d_{\mathrm{B}}^{\alpha,\beta}(X)$, to be the infimum of all s such that (X, d) is locally (α, β)-almost (M, s)-homogeneous for some $\epsilon > 0$ and $M \geq 1$.

Let (X, d) be a metric space. In general $d_{\mathrm{B}}^{\alpha,\beta}(X) \leq d_{\mathrm{A}}^{\alpha,\beta}(X)$. Both $d_{\mathrm{A}}^{\alpha,\beta}$ and $d_{\mathrm{B}}^{\alpha,\beta}$ are invariant under a rescaling of the metric. Thus, the metric space $(\tilde{X}, \tilde{d})$ where $\tilde{X} = X$ and $\tilde{d} = \eta d$ for some $\eta > 0$ has $d_{\mathrm{A}}^{\alpha,\beta}(\tilde{X}) = d_{\mathrm{A}}^{\alpha,\beta}(X)$ and $d_{\mathrm{B}}^{\alpha,\beta}(\tilde{X}) = d_{\mathrm{B}}^{\alpha,\beta}(X)$. Note that

$$d_{\mathrm{B}}^{\alpha+\theta\beta,(1-\theta)\beta}(X) \leq d_{\mathrm{B}}^{\alpha,\beta}(X) \leq d_{\mathrm{B}}^{(1-\theta)\alpha,\theta\alpha+\beta}(X)$$

for $0 \leq \theta \leq 1$. Moreover, if X is compact, then

$$d_{\mathrm{F}}(X) \leq d_{\mathrm{A}}^{\alpha,\beta}(X) = d_{\mathrm{B}}^{\alpha,\beta}(X)$$

where $d_{\mathrm{F}}(X)$ denotes the fractal or upper box-counting dimension.

We note here that d_{B} shares with d_{A} the usual properties of dimension discussed in Section 2, along with the monotonicty property in (2.2).

5. Embedding Hilbert subsets X with $X - X$ homogeneous

In this section we prove our main result, in which we take a subspace X of a Hilbert space, assume that $X - X$ is almost homogeneous, and obtain an almost bi-Lipschitz embedding into a finite-dimensional space.

Our argument is essentially a combination of that of Olson (2002), who treated a subset X of a Euclidean space with $d_{\mathrm{A}}(X-X)$ finite, and that of Hunt & Kaloshin (1999), who considered a subset of a Hilbert space with finite upper box-counting ('fractal') dimension. The key to combining these successfully is Lemma 5.3, below.

In line with the treatment in Sauer et al. (1991) and in Hunt & Kaloshin (1999), our main theorem is expressed in terms of prevalence. This concept, which generalises the notion of 'almost every' from finite to infinite-dimensional spaces, was introduced by Hunt, Sauer & Yorke (1992); see their paper for a detailed discussion.

Definition 5.1. A Borel subset S of a normed linear space V is *prevalent* if there exists a compactly supported probability measure μ such that $\mu(S+v) = 1$ for all $v \in V$. In particular, if S is prevalent then S is dense in V.

Note that if we set $Q = \mathrm{supp}(\mu)$ then Q can be thought of as a 'probe set', which consists of 'allowable perturbations' with which, given a $v \in V$, we 'probe' and test whether $v + q \in S$ for almost every $q \in Q$.

Since we will use it below, and for its historical importance, we quote Hunt & Kaloshin's result here, in a form suitable for what follows. Given a set X, we recall here that its upper box-counting ('fractal') dimension is defined as

$$d_{\mathrm{F}}(X) = \limsup_{\epsilon \to 0} \frac{\log N(X,\epsilon)}{-\log \epsilon},$$

where $N(X,\epsilon)$ denotes the minimum number of balls of radius ϵ necessary to cover X; and its thickness exponent, $\tau(X)$, is

$$\tau(X) = \limsup_{\epsilon \to 0} \frac{\log d(X,\epsilon)}{-\log \epsilon}, \tag{5.1}$$

where $d(X,\epsilon)$ is the minimum dimension of all finite-dimensional subspaces, V, of B such that every point of X lies within ϵ of V. We note here for later use that $\tau(X) \leq d_{\mathrm{F}}(X)$.

Theorem 5.2 (Hunt & Kaloshin)**.** *Let X be a compact subset of a Hilbert space H, D an integer with $D > d_{\mathrm{F}}(X-X)$, and $\tau(X)$ the thickness exponent of X. If θ is chosen with*

$$\theta > \frac{D(1+\tau(X)/2)}{D - d_{\mathrm{F}}(X-X)}$$

then for a prevalent set of linear maps $L : B \to \mathbb{R}^D$ there exists a $c > 0$ such that

$$c\|x-y\|^{\theta} \leq |Lx - Ly| \leq \|L\|\|x-y\| \quad \textit{for all} \quad x,y \in X;$$

in particular these maps are injective on X.

We note here that $d_{\mathrm{F}}(X-X) \leq 2d_{\mathrm{F}}(X)$, so that for zero thickness sets with finite box-counting dimension one can choose any $D > 2d_{\mathrm{F}}(X)$ and $\theta > D/(D-2d_{\mathrm{F}}(X))$.

5.1. **Construction of the probability measure μ for a given X.** We now apply the definition of prevalence given a particular compact subset X of our Hilbert space H such that $X - X$ is (α, β)-almost (M, s)-homogeneous.

For some fixed N, let V be the set of linear functions $L : H \to \mathbb{R}^N$. We now construct a compactly supported probability measure μ on V (as required by the definition of prevalence) that is carefully tailored to the particular set X. Key to this is the following result.

Lemma 5.3. *Suppose that X is a compact (α, β)-almost (M, s)-homogeneous subset of H. Then there exists a sequence of nested linear subspaces U_n with $U_n \subseteq U_{n+1}$,*

$$\dim U_n \leq C(1+n)^{\alpha+\beta+1},$$

and

$$\|P_n x\| \geq \frac{1}{8}\|x\| \quad \text{for all } x \in X \text{ with } \|x\| \geq 2^{-n},$$

where P_n is the orthogonal projection onto U_n.

Proof. Consider the collection of shells

$$\Delta_j = \left\{ x \in X : 2^{-(j+1)} \leq \|x\| \leq 2^{-j} \right\}.$$

Since $\Delta_j \subset B(0, 2^{-j})$ it can be covered using

$$\mathcal{N}_X(2^{-j}, 2^{-(j+3)}) \leq 8^s M (\log 2)^2 (1+|j|)^\beta (4+|j|)^\alpha \leq c_2(1+|j|)^{\alpha+\beta} := M_j$$

balls of radius $2^{-(j+3)}$, where c_2 is independent of j. We choose the centres $\left\{u_i^{(j)}\right\}_{i=1}^{M_j}$ of these balls so that $\|u_i^{(j)}\| \geq 2^{-(j+2)}$.

Since X is compact, $X \subset B(0, 2^k)$ for some k sufficiently large, and so

$$\bigcup_{j=-k}^{n} \Delta_j = \left\{ x \in X : \ \|x\| \geq 2^{-n} \right\}.$$

Let P_n be the orthogonal projection onto the linear subspace U_n spanned by the collection $\left\{ u_i^{(j)} : j = -k, \ldots, n \text{ and } i = 1, \ldots, M_j \right\}$. Then the dimension of U_n is bounded by $c_3(1+n)^{\alpha+\beta+1}$ using the same estimate as in (6.1). Moreover, for every $x \in \Delta_j$ there exists $u_i^{(j)}$ such that $x = u_i^{(j)} + v$ where $\|v\| \leq 2^{-(j+3)}$. Since $\|P_n\| = 1$ and $\|P_n u\| = \|u\|$ for $u \in U_n$, then

$$\|P_n x\| = \|P_n(u_i^{(j)} + v)\| \geq \|P_n u_i^{(j)}\| - \|P_n v\| \geq 2^{-(j+2)} - 2^{-(j+3)} \geq \frac{1}{8}\|x\|.$$

□

Applying this lemma to $X - X$ there are subspaces U_k with $\dim U_k \leq d_k := c(1+k)^{\alpha+\beta+1}$ such that $\|P_k z\| \geq \|z\|/8$ for all $z \in X - X$ with $\|z\| \geq 2^{-k}$. Let S_k denote the closed unit ball in U_k; clearly any $\phi \in S_k$ induces a linear functional L_ϕ on H via the definition $L_\phi(u) = (\phi, u)$, where $(\cdot, \cdot)$ is the inner product in H. Let $\zeta > 0$ be fixed and define $C_\zeta = 1/\sum_{k=1}^{\infty} k^{-1-\zeta}$. We now define the probe set

(5.2)

$$Q = \left\{ (l_1, \ldots, l_N) : \ l_n = L_{\phi_n} \text{ where } \phi_n = C_\zeta \sum_{k=1}^{\infty} k^{-1-\zeta} \phi_{nk} \text{ with } \phi_{nk} \in S_k \right\}.$$

We can identify S_j with the unit ball B_{d_j} in $\mathbb{R}^{d_j}$, and we denote by λ_j the probability measure on S_j that corresponds to the uniform probability measure on B_{d_j}. We let μ be the probability measure on Q that results from choosing each ϕ_{nk} randomly with respect to λ_{d_k}. Note that Q is a compact subset of V, and that all elements of Q have Lipschitz constant at most $\sqrt{N}$.

Before proving our main theorem we will prove a key estimate on μ. Although the argument is essentially the same as that in Hunt & Kaloshin (1999) our version is a little more explicit and we include it here for completeness. The estimate relies on the following simple inequality.

Lemma 5.4. *If $x \in \mathbb{R}^j$ and $\eta \in \mathbb{R}$ then*

$$\lambda_j \{ \omega \in B_j : \ |\eta + (\omega \cdot x)| < \epsilon \} \leq c j^{1/2} \epsilon |x|^{-1}.$$

where c is a constant that does not depend on η or j.

Proof. Let $\hat{x} = x/|x|$. This follows immediately from estimate

$$\begin{aligned} \lambda_j \{ \omega \in B_j : \ |\eta + (\omega \cdot x)| < \epsilon \} &\leq \lambda_j \left\{ \omega \in B_j : |\omega \cdot \hat{x}| < \epsilon |x|^{-1} \right\} \\ &= \frac{\Omega_{j-1}}{\Omega_j} 2 \int_0^{\min(\epsilon|x|^{-1},1)} (1-\xi^2)^{(j-1)/2} \, \mathrm{d}\xi \\ &\leq \frac{\Omega_{j-1}}{\Omega_j} 2\epsilon \, |x|^{-1} \end{aligned}$$

where $\Omega_j = \pi^{j/2}\Gamma(j/2+1)$ is the volume of the unit ball in $\mathbb{R}^j$. $\square$

Lemma 5.5. *If $x \in H$ and $f \in V$ then*

$$\mu \{ L \in Q : \ |(L-f)(x)| < \epsilon \} \leq c (d_k^{1/2} k^{1+\zeta} \epsilon \|P_k x\|^{-1})^N$$

for every $k \in \mathbb{N}$ where c is a constant independent of f and k.

Proof. Given $k \in \mathbb{N}$, let $\mathcal{J}$ be the index set $\mathcal{J} = \mathbb{N} \setminus \{k\}$ and define

$$B = \Big(\bigoplus_{j \in \mathcal{J}} B_{d_j} \Big)^N .$$

Given $\alpha = ((\alpha_{nj})_{j\in\mathcal{J}})_{n=1}^N \in B$ fixed, define

$$A_\alpha = \left\{ (\phi_{nk})_{n=1}^N : \ |(\eta_n + k^{-1-\zeta}\phi_{nk})(x)| < \epsilon \text{ for all } n \right\}$$

where

$$\eta_n(x) = C_\zeta \sum_{j\in\mathcal{J}} j^{-1-\zeta} \alpha_{nj}(x) - f_n(x).$$

By Lemma 5.4 there is a constant c independent of α, v and k such that

$$\lambda_{d_k}^N (A_\alpha) \leq c (d_k^{1/2} k^{1+\zeta} \epsilon \|P_k x\|^{-1})^N .$$

Let $P = \mu \{ L \in Q : \ |(L-v)(x)| < \epsilon \}$. Then

$$P \leq \mu \{ L \in Q : \ |(l_n - f_n)(x)| < \epsilon \text{ for all } n \} .$$

Let

$$\Phi_N = \left\{ \big((\phi_{nk})_{k=1}^\infty \big)_{n=1}^N : \ C_\zeta \Big| \sum_{k=1}^\infty k^{-1-\zeta} (\phi_{nk} - f_n)(x) \Big| < \epsilon, \forall n = 1, \ldots, N \right\}$$

Then by Fubini's theorem

$$\begin{aligned} P &\leq \Big(\bigotimes_{j=1}^{\infty} \lambda_{d_j}\Big)^N \Phi_N \\ &= \int_{\alpha \in B} \int_{\phi \in A_\alpha} \mathrm{d}\lambda_{d_k}^N(\phi)\, \mathrm{d}\Big(\bigotimes_{j \in \mathcal{J}} \lambda_{d_j}\Big)^N(\alpha) \\ &\leq \int_{\alpha \in B} c(d_k^{1/2} k^{1+\zeta} \epsilon \|P_k x\|^{-1})^N\, \mathrm{d}\Big(\bigotimes_{j \in \mathcal{J}} \lambda_{d_j}\Big)^N(\alpha) \\ &= c(d_k^{1/2} k^{1+\zeta} \epsilon \|P_k x\|^{-1})^N. \end{aligned}$$

This finishes the proof. □

5.2. **Almost bi-Lipschitz embeddings.** We are now in a position to state and prove our main theorem, that a compact subset X of a Hilbert space with $X - X$ almost homogeneous admits almost bi-Lipschitz linear embeddings into finite-dimensional spaces. Unfortunately homogeneity of X is not automatically inherited by $X - X$: Olson (2002) exhibits an example of a set X with $d_{\mathrm{A}}(X) = 0$ but $d_{\mathrm{A}}(X - X) = +\infty$ (for more see Section 8).

Theorem 5.6. *Let X be a compact subset of a Hilbert space H such that $X - X$ is (α, β)-almost homogeneous with $d_{\mathrm{A}}^{\alpha,\beta}(X - X) < s < N$. If*

$$\gamma > \frac{2 + N(3 + \alpha + \beta) + 2(\alpha + \beta)}{2(N - s)}$$

then a prevalent set of linear maps $f : H \to \mathbb{R}^N$ are injective on X and, in particular, γ-almost bi-Lipschitz.

Proof. First choose $\zeta > 0$ in the definition of Q small enough such that

$$\gamma > \frac{2 + N(3 + 2\zeta + \alpha + \beta) + 2(\alpha + \beta)}{2(N - s)}. \tag{5.3}$$

Since $\tau(X) \leq d_{\mathrm{F}}(X) \leq d_{\mathrm{F}}(X - X) \leq d_{\mathrm{A}}^{\alpha,\beta}(X - X)$ we can apply Hunt & Kaloshin's result (Theorem 5.2, above) with θ chosen so that

$$\theta > \frac{N(1 + s/2)}{N - s}.$$

to obtain a prevalent set S_0 of linear functions $f : H \to \mathbb{R}^N$ such that $f \in S_0$ implies there exists a $\theta < 1$ and $c_1 > 0$ such that

$$|f(x) - f(y)| \geq c_1 \|x - y\|^\theta \qquad \text{for all} \qquad x, y \in X. \tag{5.4}$$

(We note here that the compactly supported probability measure used in the definition of prevalence for S_0 differs from the measure μ constructed in Section 5.1, but is defined on the same normed linear space V of linear maps from H to $\mathbb{R}^N$). We use this result to bootstrap a refined argument that makes use of the stronger hypothesis that $d_{\mathrm{A}}^{\alpha,\beta}(X - X) < \infty$.

Let S_1 be the subset of V consisting of those linear functions $f : H \to \mathbb{R}^N$ such that $f \in S_1$ implies there exists $\delta > 0$ such that

$$|f(x) - f(y)| \geq \frac{\|x - y\|}{\operatorname{slog}(\|x - y\|)^\gamma} \qquad \text{for all} \qquad \|x - y\| < \delta. \tag{5.5}$$

We now show that the set S_1 is also prevalent. Given $f \in V$, let K be the Lipschitz constant of f. We wish to show that $\mu(f + S_1) = 1$. This is equivalent to showing that $\mu(Q \setminus (f + S_1)) = 0$.

Define the layers of $X - X$ by

$$Z_j = \left\{ z \in X - X : \ 2^{-(j+1)} \le \|z\| \le 2^{-j} \right\} \tag{5.6}$$

and the set Q_j of linear maps that fail to satisfy the required continuity property[3] for some $z \in Z_j$ by

$$Q_j = \left\{ L \in Q : \ |(L - f)(z)| \le \Psi_{-\gamma}(2^{-j}) \ \text{ for some } z \in Z_j \right\},$$

where

$$\Psi_{-\gamma}(2^{-j}) := \frac{2^{-j}}{\sigma^\gamma \operatorname{slog}(2^{-j})^\gamma}$$

and σ is the constant occurring in (p4) in Lemma 2.1. We now bound $\mu(Q_j)$.

By assumption $d_{\mathrm{A}}^{\alpha,\beta}(X - X) < s$, and so Z_j can be covered by

$$M_j \le M \operatorname{slog}(2^{-j})^{\gamma s} \operatorname{slog}(2^{-j})^\beta \operatorname{slog}(\Psi_{-\gamma}(2^{-j}))^\alpha \le c_2 (1 + j)^{\alpha+\beta+\gamma s} \tag{5.7}$$

balls of radius $\Psi_{-\gamma}(2^{-j})$. Let the centres of these balls be $z_i^{(j)} \in Z_j$ where $i = 1, \ldots, M_j$. Given any $z \in Z_j$ there is $z_i^{(j)}$ such that $\|z - z_i^{(j)}\| \le \Psi_{-\gamma}(2^{-j})$. Thus

$$\begin{aligned} |(L - f)(z)| &\ge |(L - f)(z_i^{(j)})| - |(L - f)(z - z_i^{(j)})| \\ &\ge |(L - f)(z_i^{(j)})| - (K + \sqrt{N})\Psi_{-\gamma}(2^{-j}) \end{aligned}$$

implies

$$Q_j \subseteq \bigcup_{i=1}^{M_j} \left\{ L \in Q : \ |(L - f)(z_i^{(j)})| \le (K + 2\sqrt{N})\Psi_{-\gamma}(2^{-j}) \right\}.$$

It follows, setting $k = j$ in Lemma 5.5, that

$$\begin{aligned} \mu(Q_j) &\le \sum_{i=1}^{M_j} \mu \left\{ L \in Q : \ |(L - f)(z_i^{(j)})| \le (K + 2\sqrt{N})\Psi_{-\gamma}(2^{-j}) \right\} \\ &\le M_j \big(d_j^{1/2} j^{1+\zeta} (K + 2\sqrt{N}) \Psi_{-\gamma}(2^{-j}) \|P_j(z_i^{(j)})\|^{-1} \big)^N. \end{aligned}$$

Now (5.7) and Lemma 5.3 imply that

$$\mu(Q_j) \le c_2 (1 + j)^{\alpha+\beta+\gamma s} \big(d_j^{1/2} j^{1+\zeta} (K + 2\sqrt{N}) 2^{j+3} \Psi_{-\gamma}(2^{-j}) \big)^N.$$

In particular (recall that $d_j \le C(1+j)^{\alpha+\beta+1}$) there is a constant $c_3 > 0$ independent of j such that

$$\mu(Q_j) \sim c_3 j^{\alpha+\beta+\gamma s + N(\alpha+\beta+3+2\zeta-2\gamma)/2} \quad \text{as} \quad j \to \infty.$$

Since (5.3) implies $N(2\gamma - 3 - 2\zeta - (\alpha + \beta))/2 > 1 + \alpha + \beta + \gamma s$, we have

$$\sum_{j=1}^{\infty} \mu(Q_j) < c_4.$$

It follows from the Borel-Cantelli Lemma that μ-almost every L is contained in only a finite number of the Q_j; i.e. there exists a J such that for all $j \ge J$,

[3] Strictly speaking the union of the Q_j form a set strictly larger than the complement of S_1.

$2^{-(j+1)} \leq \|z\| \leq 2^{-j}$ implies that $|(L-f)(z)| \geq \Psi_{-\gamma}(2^{-j})$. It follows from (p4) in Lemma 2.1 that

$$|(L-f)(z)| \geq \sigma^{\gamma}\Psi_{-\gamma}(\|z\|) = \frac{\|z\|}{\mathrm{slog}(\|z\|)^{\gamma}} \quad \text{for every} \quad \|z\| \leq 2^{-J}.$$

Thus $L-f \in S_1$ and so $L \in S_1 + f$ for μ-almost every L.

Define $S = S_0 \cap S_1$. Since the intersection of prevalent sets is prevalent (Fact 3′ in Hunt et al. (1992)) S is prevalent. Let $f \in S$. Then there is c_1 and δ such that both (5.4) and (5.5) hold. Thus

$$|f(x)-f(y)| \geq c_5 \frac{\|x-y\|}{\mathrm{slog}(\|x-y\|)^{\gamma}} \qquad \text{for all} \qquad x, y \in X$$

where $c_5 = \min\{1, c_1\delta/\Psi_{-\gamma}(R)\}$ and $R > 0$ is such that $X - X \subseteq B(0,R)$. □

Note that for a space X with $X - X$ homogeneous, i.e. $\alpha = \beta = 0$ in the above theorem, for any $\gamma > 3/2$ we can choose N large enough to obtain a γ-almost bi-Lipschitz embedding into $\mathbb{R}^N$.

We will prove a Banach space version of Theorem 5.6 in Section 7. However, we delay this while, in the next section, we consider in more detail almost homogeneity in a Hilbert space.

6. Lipschitz approximating dimension of Hilbert subsets and Hölder-Lipschitz embeddings

The strong result of the previous section requires that $X - X$ is almost homogeneous, while for a general almost homogeneous metric space (X, d) the embedding result of Theorem 2.3 only provides a subset $f(X)$ of a Hilbert space that is itself almost homogeneous.

Here we investigate further some of the properties of $f(X)$, and are lead to define the 'Lipschitz approximating dimension' and the 'Lipschitz deviation'. In particular we show that it is possible to replace Hunt & Kaloshin's thickness exponent with the Lipschitz deviation.

6.1. **Further properties of the image** $f(X)$**.** First we consider the almost bi-Lipschitz image $f(X)$ of a compact almost homogeneous metric space (X, d) in a Hilbert space, as provided by Theorem 2.3. We show that $f(X)$ can be very well approximated by linear subspaces: it has 'better than zero' thickness.

As remarked after the proof of Theorem 2.3, when (X, d) is compact the function f defined by the simplified series

$$f(x) = \sum_{j=-k}^{\infty} \frac{2^{-j}}{(1+|j|)^{\delta} M_j} \phi_j(x) \otimes e_j$$

still provides a γ-almost bi-Lipschitz embedding of X into a Hilbert space (choosing a k such that $d(x_1, x_2) \leq 2^k$ for all $x_1, x_2 \in X$). Now, for $n \in \mathbb{N}$ any element of $f(X)$ can be approximated to within

$$B \sum_{j=n+1}^{\infty} \frac{2^{-j}}{(1+|j|)^{\delta}} \;\leq\; B \sum_{j=n+1}^{\infty} 2^{-j} \;\leq\; B2^{-n}$$

by an element of the subspace

$$U = \bigoplus_{j=-k}^{n} \mathbb{R}^{M_j} \otimes e_j,$$

which has dimension

$$\sum_{j=-k}^{n} M_j \leq (n+k+1)C(1+n)^{\alpha+\beta} \leq c_1(1+n)^{\alpha+\beta+1}. \tag{6.1}$$

Here c_1 depends on C, k and the constants in Lemma 2.1 but is independent of n. It follows that

$$d(f(X),\epsilon) \leq c_2\big[\log(\mathrm{e}+1/\epsilon)\big]^{\alpha+\beta+1}. \tag{6.2}$$

One consequence of this inequality is that the thickness exponent of $f(X)$ is zero, but (6.2) is significantly stronger than this.

6.2. **The Lipschitz deviation.** Inspired by the quantity $d(X,\epsilon)$ used to define the thickness we now introduce a more general quantity, the m-Lipschitz deviation: we denote by $\delta_m(X,\epsilon)$ the smallest dimension of a linear subspace U such that

$$\mathrm{dist}(X, G_U[\phi]) < \epsilon$$

for some m-Lipschitz function $\phi : U \to U^\perp$,

$$\|\phi(u)-\phi(v)\| \leq m\|u-v\| \qquad \text{for all} \qquad u,v \in U,$$

where $U^\perp$ is the orthogonal complement of U in H. We will write $G_U[\phi]$ for the graph of ϕ over U:

$$G_U[\phi] = \{\, u+\phi(u) :\ u \in U \,\}.$$

Clearly $\delta_0(X,\epsilon) = d(X,\epsilon)$.

In Section 6.1 we showed that for the almost bi-Lipschitz embedding $f(X)$ of an almost homogeneous metric space into a Hilbert space

$$\delta_0(f(X),\epsilon) \leq c_2\big[\log(\mathrm{e}+1/\epsilon)\big]^{\alpha+\beta+1}.$$

We now show that Lemma 5.3 implies a bound of a similar form on $\delta_8(X,\epsilon)$ for any subset of a Hilbert space with $X-X$ almost homogeneous.

Proposition 6.1. *Let X be a compact subset with the set of differences $X-X$ (α,β)-almost (M,s)-homogeneous. Then there exists a sequence of linear subspaces U_k with $\dim U_k \leq C(1+k)^{\alpha+\beta+1}$ and $U_{k+1} \supseteq U_k$, and 8-Lipschitz functions $\phi_k : U_k \to U_k^\perp$ such that*

$$\mathrm{dist}(X, G_{U_k}[\phi_k]) \leq 2^{-k}.$$

In particular

$$\delta_8(X,\epsilon) \leq K\big[\log(\mathrm{e}+1/\epsilon)\big]^{\alpha+\beta+1}.$$

Proof. Applying Lemma 5.3 to $X-X$ we obtain a nested sequence of linear subspaces for which

$$\frac{1}{8}\|x-y\| \leq \|P_kx-P_ky\| \leq \|x-y\| \qquad \text{for all} \qquad x,y \in X \quad \text{with} \quad \|x-y\| \geq 2^{-k},$$

where P_k is the orthogonal projection onto U_k.

Define $\phi_k : U_k \to U_k^\perp$ as follows. Let N_k be a maximal 2^{-k} net in (X,d) and set $\phi_k(P_kx) = (I-P_k)x$ for $x \in N_k$. Given $P_kx, P_ky \in P_kN_k$ we have

$$\|\phi_k(P_kx)-\phi_k(P_ky)\| \leq \|(I-P_k)(x-y)\| \leq \|x-y\| \leq 8\|P_kx-P_ky\|.$$

Therefore $\phi_k : P_k N_k \to U_k^\perp$ is a 8-Lipschitz function. Now, extend this ϕ_k to a 8-Lipschitz function $U_k \to U_k^\perp$.

Since $N_k \subset G_{U_k}[\phi_k]$ then any point of X lies within 2^{-k} of $G_{U_k}[\phi_k]$. Thus

$$\delta_8(X, 2^{-k}) \le c_2(1+k)^{\alpha+\beta+1}$$

and the result follows. □

We now show that this argument can be reversed, i.e. that the results of Lemma 5.3 and Proposition 6.1 are essentially equivalent.

Proposition 6.2. *Suppose that X is a compact subset of a Hilbert space X. For any $m \ge 0$ let $\{U_k\}_{k=1}^\infty$ be a sequence of linear subspaces such that for each U_k there exists an m-Lipschitz function $\phi_k : U_k \to U_k^\perp$ with*

$$\mathrm{dist}(X, G_{U_k}[\phi_k]) \le 2^{-k}.$$

Then there exists an integer n and a constant $c_m > 0$ (which depends on m but is independent of k) such that for every k

$$\|P_{k+n}(x_1 - x_2)\| \ge c_m \|x_1 - x_2\| \quad \text{for all} \quad x, y \in X \quad \textit{with} \quad \|x_1 - x_2\| \ge 2^{-k}.$$

Proof. First note that for any $x \in H$ we have

$$\mathrm{dist}(x, G_{U_k}[\phi_k])^2 = \inf_{u \in U_k} \left(\|P_k x - u\|^2 + \|(I - P_k)x - \phi_k(u)\|^2 \right)$$

and since for any $u \in U_k$ we have

$$\begin{aligned} \|(I-P_k)x - \phi_k(P_k x)\|^2 &= \|(I-P_k)x - \phi_k(u) + \phi_k(u) - \phi_k(P_k x)\|^2 \\ &\le 2\|(I-P_k)x - \phi_k(u)\|^2 + 2\|\phi_k(u) - \phi_k(P_k x)\|^2 \\ &\le 2\|(I-P_k)x - \phi_k(u)\|^2 + 2m^2\|u - P_k x\|^2 \\ &\le l_m^2 \left(\|P_k x - u\|^2 + \|(I-P_k)x - \phi_k(u)\|^2 \right), \end{aligned}$$

where $l_m^2 = 2\max(1, m^2)$, it follows that for $x \in X$

$$\|(I - P_k)x - \phi_k(P_k x)\| \le l_m \,\mathrm{dist}(x, G_{U_k}[\phi_k]) \le l_m 2^{-k} \tag{6.3}$$

Now suppose that $x_1, x_2 \in X$ with

$$\|x_1 - x_2\| \ge 2^{-k}.$$

Let n be the smallest integer such that $3l_m \le 2^n$ and set

$$\tilde{x}_j = P_{k+n} x_j + \phi_{k+n}(P_{k+n} x_j) \qquad \text{for} \qquad j = 1, 2.$$

Clearly, $P_{k+n}(x_1 - x_2) = P_{k+n}(\tilde{x}_1 - \tilde{x}_2)$. Furthermore, it follows from (6.3) that $|x_j - \tilde{x}_j| \le 2^{-k}/3$ for $j = 1, 2$. Therefore, $|\tilde{x}_1 - \tilde{x}_2| \ge |x_1 - x_2|/3$.

Now, since $\tilde{x}_1, \tilde{x}_2 \in G_{U_{k+n}}[\phi_{k+n}]$,

$$\begin{aligned} \|P_{k+n}\tilde{x}_1 - P_{k+n}\tilde{x}_2\|^2 &= \|\tilde{x}_1 - \tilde{x}_2\|^2 - \|\phi_{k+n}(P_{k+n}\tilde{x}_1) - \phi_{k+n}(P_{k+n}\tilde{x}_2)\|^2 \\ &\ge \|\tilde{x}_1 - \tilde{x}_2\|^2 - m^2 \|P_{k+n}(\tilde{x}_1 - \tilde{x}_2)\|^2, \end{aligned}$$

and so

$$\|P_{k+n}(x_1 - x_2)\| = \|P_{k+n}(\tilde{x}_1 - \tilde{x}_2)\| \ge \frac{\|\tilde{x}_1 - \tilde{x}_2\|}{\sqrt{1+m^2}} \ge \frac{\|x_1 - x_2\|}{3\sqrt{1+m^2}}.$$

□

6.3. **Almost homogeneous subsets of a Hilbert space.** If we assume only the almost homogeneity of X, rather than of $X-X$, we can apply a simplified variant of the argument of Theorem 5.6 to obtain the following minor improvement to the embedding theorem of Hunt & Kaloshin (under our stronger hypothesis). For a zero thickness set X with $d_{\rm F}(X)\le d$ they obtain an upper limit of $N/(N-2d)$ for the Hölder exponent, while under the assumption that $d_{\rm A}^{\alpha,\beta}(X)\le s$ we obtain $(N-s)/(N-2s)$ as the upper limit. Note that we replace any assumption on the thickness by (6.4), which in particular is satisfied by the almost bi-Lipschitz embedding $f(X)$ of an almost homogeneous metric space with $m=0$ (see 6.2).

Theorem 6.3. *Suppose that X is a compact subset of a Hilbert space H with $d_{\rm A}^{\alpha,\beta}(X)<s$ and that for some $m>0$, $\sigma\ge 0$,*

$$\delta_m(X,\epsilon)\le K[\log({\rm e}+1/\epsilon)]^\sigma. \tag{6.4}$$

Then for any integer $N>2s$, if $\theta>(N-s)/(N-2s)$ there is a prevalent set S of linear maps $f:H\to\mathbb{R}^N$ such that for every $f\in S$ there exists $c>0$ such that

$$|f(x)-f(y)|\ge c\|x-y\|^\theta \quad \text{for all} \quad x,y\in X. \tag{6.5}$$

Proof. Set

$$d_j=\delta_m(X,2^{-j})\le K\big[\log({\rm e}+2^j)\big]^\sigma$$

and define Q as in (5.2) with $\zeta=1$. Define the layers Z_j as in (5.6) and

$$Q_j=\big\{\,L\in Q:\ |(L-v)(z)|\le 2^{-j\theta}\text{ for some }z\in Z_j\,\big\}.$$

Let $R>0$ be chosen so large that $X\subset B(0,R)$. Cover X by

$$\begin{aligned}\mathcal{N}_X(R,2^{-(j+1)\theta}) &\le M\left(\frac{R}{2^{-(j+1)\theta}}\right)^s {\rm slog}(R)^\beta {\rm slog}(2^{-(j+1)\theta})^\alpha\\ &\le c_1 2^{j\theta s}(1+j\theta)^\alpha\end{aligned}$$

balls of radius $2^{-(j+1)\theta}$ centred at points $x_i\in X$. Denote these as

$$X_i=\Big\{\,x\in X:\|x-x_i\|<2^{-(j+1)\theta}\,\Big\}.$$

Now consider the larger balls

$$B_i=\Big\{\,y\in X:\ \|x_i-y\|\le 2^{-(j+1)\theta}+2^{-j}\,\Big\}.$$

Cover each of these balls by at most

$$\begin{aligned}&\mathcal{N}_X(2^{-(j+1)\theta}+2^{-j},2^{-(j+1)\theta})\\ &\quad\le M\left(1+2^{(j+1)\theta-j}\right)^s {\rm slog}(2^{-(j+1)\theta}+2^{-j})^\beta\,{\rm slog}(2^{-(j+1)\theta})^\alpha\\ &\quad\le c_2 2^{j(\theta-1)s}(1+j)^\beta(1+j\theta)^\alpha\end{aligned}$$

balls of radius $2^{-(j+1)\theta}$. Since

$$Z_j=\bigcup_i\bigcup_{x\in X_i}\Big\{\,x-y:2^{-(j+1)}<\|x-y\|<2^{-j}\,\Big\}\subseteq\bigcup_i(X_i-B_i)$$

it follows that Z_j can be covered by

$$M_j=c_1c_2 2^{js(2\theta-1)}(1+j\theta)^{2\alpha}(1+j)^\beta$$

balls of radius $2^{-j\theta}$. Let $z_i^{(j)}$ denote the centres of these balls.

Applying similar estimates as in the proof of Theorem 5.6 (these rely on Proposition 6.2 to ensure that $\|P_k z_i^{(j)}\| \geq c\|z_i^{(j)}\|$ for some $c > 0$) shows that

$$\mu(Q_j) \sim 2^{js(2\theta-1)} j^{2\alpha+\beta} [j^{2+\sigma} 2^{j(1-\theta)}]^N \quad \text{as} \quad j \to \infty.$$

Thus $\sum \mu(Q_j)$ converges provided that $\theta > (N-s)/(N-2s)$. The argument is now concluded as in Theorem 5.6. □

By combining this with Proposition 2.3 we obtain the following Hölder-Lipschitz embedding result for homogeneous metric spaces (cf. Lemma 9.1 in Foias and Olson (1996) which has a similar result for spaces with finite upper box-counting dimension).

Corollary 6.4. *Let (X, d) be an almost homogeneous metric space with $d_{\mathrm{A}}^{\alpha,\beta} < s$. If $N > 2s$ and $\theta > (N-s)/(N-2s)$ there exists a map $\phi : (X, d) \to \mathbb{R}^N$ such that*

$$c^{-1}\, d(x, y)^\theta \leq |\phi(x) - \phi(y)| \leq c\, d(x, y) \qquad \textit{for all} \qquad x, y \in X.$$

Of course one can prove finite-dimensional versions of Theorems 5.6 and 6.3 using very similar techniques.

6.4. **The Lipschitz deviation.** It is interesting that our argument shows that for any fixed $m > 0$ the thickness exponent in the statement of Theorem 5.2 can be replaced by the *m-Lipschitz deviation*, $\mathrm{dev}_m(X)$, which we define by analogy with the thickness exponent (cf. (5.1))

$$\mathrm{dev}_m(X) = \limsup_{\epsilon \to 0} \frac{\log \delta_m(X, \epsilon)}{-\log \epsilon}.$$

We note that $\mathrm{dev}_m(X) \leq \tau(X)$ and that this gives an indication of why the thickness exponent can be expected to play a rôle in determining the Hölder exponent in (6.5). We state without proof:

Theorem 6.5. *Let X be a compact subset of a Hilbert space H, D an integer with $D > d_{\mathrm{F}}(X - X)$, and let $\mathrm{dev}_m(X)$ be the m-Lipschitz deviation of X. If θ is chosen with*

$$\theta > \frac{D(1 + \mathrm{dev}_m(X)/2)}{D - d_{\mathrm{F}}(X - X)}$$

then for a prevalent set of linear maps $L : B \to \mathbb{R}^D$ there exists a $c > 0$ such that

$$c\|x - y\|^\theta \leq |Lx - Ly| \leq \|L\|\|x - y\| \quad \textit{for all} \quad x, y \in X;$$

in particular these maps are injective on X.

7. Embedding subsets X of Banach spaces with $X - X$ homogeneous

In this section we extend the Hilbert space result to cover subsets of Banach spaces. In particular this enables us to prove a new almost bi-Lipschitz embedding result for a class of metric spaces.

The key point is, of course, that enough of Lemma 5.3 can be salvaged to follow a very similar proof:

Lemma 7.1. *Let X be an (α, β)-almost (M, s)-homogeneous subset of a Banach space B. Then there exists a nested sequence of subsets $U_{n+1} \supseteq U_n$ such that*

$$\dim U_n \leq C(1 + n)^{\alpha+\beta+1}$$

and

$$\operatorname{dist}(x, U_n) \leq \frac{1}{4}\|x\| \qquad \textit{for all} \qquad \|x\| \geq 2^{-n}$$

In particular, if we apply this lemma to $Z = X - X$, there exists a nested sequence of linear subspaces of B, $U_k \subseteq U_{k+1}$ such that given $z \in X - X$ with $\|z\| \geq 2^{-n}$ there exists a point $\tilde{z} \in U_n$ such that

$$\|z - \tilde{z}\| \leq \frac{1}{4}\|z\| \qquad \text{and} \qquad \|\tilde{z}\| \geq \frac{3}{4}\|z\|.$$

We now let S_k denote the closed unit ball in the dual of U_k, and denote by S_k^E an isometric embedding of S_k into B^*, whose existence is guaranteed by the Hahn-Banach theorem. We then define our probe set Q as

$$Q = \left\{ (l_1, \ldots, l_N) : \ l_n = C_\zeta \sum_{k=1}^{\infty} k^{-1-\zeta} \phi_{nk} \text{ with } \phi_{nk} \in S_k^E \right\}.$$

Choosing a basis for S_k we identify S_k with a convex set $U_k \subset \mathbb{R}^{d_k}$, and induce a probability measure on S_k (and hence on S_k^E) via the uniform probability measure on U_k.

We now outline the proof of the following result:

Theorem 7.2. *Let X be a compact subset of a Hilbert space B such that $X - X$ is (α, β)-almost homogeneous with $d_{\mathrm{A}}^{\alpha,\beta}(X - X) < s < N$. If*

$$\gamma > \frac{1 + N(2 + \alpha + \beta) + (\alpha + \beta)}{N - s}$$

then a prevalent set of linear maps $f : B \to \mathbb{R}^N$ are injective on X and, in particular, γ-almost bi-Lipschitz.

Proof. The proof proceeds identically to that of Theorem 5.6 until we have to estimate

$$\mu\left(L : \ |(L - f)z_i^{(j)}| \leq c\Psi_{-\gamma}(2^{-j})\right).$$

We can now follow the argument from Hunt & Kaloshin (1999), with some small changes—we only highlight these here. In our case we know that there exists a point $\tilde{z}_i^{(j)} \in U_j$ such that

$$\|\tilde{z}_i^{(j)} - z_i^{(j)}\| \leq \frac{1}{4}\|z_i^{(j)}\|.$$

It follows that there exists a $\psi \in S_j$ such that

$$\psi(z_i^{(j)}) \geq \|z_i^{(j)}\| - \|z_i^{(j)} - \tilde{z}_i^{(j)}\| \geq \frac{3}{4}\|z_i^{(j)}\| \geq 3 \cdot 2^{-(j+3)}.$$

We can then follow Hunt & Kaloshin's argument to show that

$$\mu\left(L : \ |(L - f)z_i^{(j)}| \leq c\Psi_{-\gamma}(2^{-j})\right) \leq \left(j^{1+\zeta} d_j 2^{j+3} \Psi_{-\gamma}(2^{-j})\right)^N,$$

and the proof is completed exactly as in the Hilbert space case, noting that we now have a factor of d_j rather than only $d_j^{1/2}$. ☐

One significant consequence of extending the result to Banach spaces is it allows for a new result for metric spaces via the Kuratowski isometric embedding of (X, d) into $L^\infty(X)$: choosing an arbitrary point $x_0 \in X$, this is given by

$$x \mapsto \rho_x, \qquad \text{where} \qquad \rho_x(y) = d(x, y) - d(x_0, y) \tag{7.1}$$

(see Heinonen, 2003, for example). In this way we can attach meaning to $X - X$ for an arbitrary metric space (X, d), i.e.

$$X - X = \{f \in L^\infty(X) : \ f = d(x, \cdot) - d(y, \cdot), \ x, y \in X\}. \tag{7.2}$$

We then have the following result:

Theorem 7.3. *Let (X, d) be a compact metric space such that $X - X$ is an almost homogeneous subset of $L^\infty(X)$. Then there exists an injective almost bi-Lipschitz map $f : (X, d) \to \mathbb{R}^N$.*

Proof. Denote by $\mathscr{F} : (X, d) \to L^\infty(X)$ the isometric embedding in (7.1). Then $\mathscr{F}(X)$ is isometric to (X, d), while the set of differences $\mathscr{F}(X) - \mathscr{F}(X)$ is almost homogeneous by assumption. The existence of an injective almost bi-Lipschitz embedding of $\mathscr{F}(X)$ into $\mathbb{R}^N$, which follows from the Banach space version of our main theorem, immediately implies the existence of the same type of embedding for (X, d) into $\mathbb{R}^N$. □

8. The relationship between $d_{\mathrm{A}}^{\alpha,\beta}(X)$ and $d_{\mathrm{A}}^{\alpha,\beta}(X - X)$

In this section we give some results relating the homogeneity of X and $X - X$. First, we give an example of a set X for which $d_{\mathrm{A}}(X) = 0$ but $d_{\mathrm{A}}(X - X) = +\infty$. It is easy to show that the set

$$X^* = \left\{ a_n e_n : \ a_n = 4^{-(2^j)}, n = 2^{j-1}, \dots, 2^j - 1 \right\}, \tag{8.1}$$

where e_n is an orthonormal basis of a Hilbert space H, has $d_{\mathrm{A}}(X^*) = +\infty$. Note that $|a_n| \leq 4^{-n}$ for all n. Consider now the subset X of $H \times H$ defined by

$$X = \left\{ (4^{-n} e_n, a_n e_n) \right\}_{n=1}^\infty \cup \left\{ (4^{-n} e_n, 0) \right\}.$$

A simple argument shows that $d_{\mathrm{A}}(X) = 0$, while $X - X$ contains a copy of X^* and so $d_{\mathrm{A}}(X - X) = \infty$.

This negative result appears to be in some ways typical for almost homogeneous sets as well, as we will now show. We begin with a preparatory lemma.

Lemma 8.1. *The orthogonal sequence with algebraic decay*

$$X^* = \left\{ b_n e_n : b_n \sim \epsilon n^{-\gamma} \right\}$$

where $\epsilon, \gamma > 0$ has $d_{\mathrm{A}}^{\alpha,\beta}(X^*) = +\infty$ for any $\alpha, \beta \geq 0$.

Proof. Let n_0 be chosen so large that

$$\epsilon(2n)^{-\gamma} < |b_n| < \epsilon(n/2)^{-\gamma} \qquad \text{for} \qquad n > n_0.$$

Let $r_n = \epsilon(n/2)^{-\gamma}$ and $\rho_n = \epsilon(4n)^{-\gamma}$. Suppose, for a contradiction, that $d_{\mathrm{A}}(X^*) < s < \infty$. Then there exists an $M \geq 1$ such that

$$\mathcal{N}(r_n, \rho_n) \leq M \left(\frac{r_n}{\rho_n} \right)^s \operatorname{slog}(r_n)^\beta \operatorname{slog}(\rho_n)^\alpha. \tag{8.2}$$

On the other hand,

$$B(0, r_n) \supseteq \left\{ b_k e_k : n < k \leq 2n \right\},$$

where the points $b_k e_k$ with $n < k \leq 2n$ are each a distance greater than $|b_k| > \epsilon(4n)^{-\gamma}$ apart from each other. Therefore,

$$\mathcal{N}(r_n, \rho_n) \geq \operatorname{card}\big(\left\{ b_k e_k : n < k \leq 2n \right\} \big) = n. \tag{8.3}$$

Combining inequality (8.2) with (8.3) and applying (p1) of Lemma 2.1 we obtain

$$n \le M8^{\gamma s}\big(\log 2 + |\log \epsilon (n/2)^{-\gamma}|\big)^{\beta}\big(\log 2 + |\log \epsilon (4n)^{-\gamma}|\big)^{\beta}.$$

Letting $n \to \infty$ yields a contradiction, and so $d_{\rm A}^{\alpha,\beta}(X^*) = \infty$. $\square$

Lemma 8.2. *Given two unit vectors $v, w \in H$ set $e_1 = v$ and choose $\alpha \in \mathbb{R}$ and a unit vector e_2 such that $e_1 \cos\alpha - e_2 \sin\alpha = w$ and $\cos\alpha = (v, w)$. Note that e_2 is orthogonal to e_1. Extend $\{e_1, e_2\}$ to a basis for H, and define the rotation*

$$R_x = \begin{pmatrix} \cos(\alpha\psi(x)) & \sin(\alpha\psi(x)) \\ -\sin(\alpha\psi(x)) & \cos(\alpha\psi(x)) \end{pmatrix} \oplus \mathrm{id},$$

where $\psi : H \to \mathbb{R}$ is a fixed C^∞ function such that

$$\psi(x) = \begin{cases} 0 & \text{if } \|x\| \le 3/4 \text{ or } \|x\| \ge 2, \\ 1 & \text{if } \|x\| = 1. \end{cases}$$

Let $f(x) = R_x x$. Then $f \in C^\infty$ and $f(v) = w$. Moreover, $f_\eta(x) = \eta^{-1} f(\eta x)$ is uniformly bi-Lipschitz continuous for $\eta > 0$ and different from the identity only for $x \in H$ such that $(3/4)\eta^{-1} < \|x\| < 2\eta^{-1}$.

Proof. By construction $f \in C^\infty$, $f(v) = w$ and $f(x) = x$ for $\|x\| \le 3/4$ or $\|x\| \ge 2$. Rescaling shows that $f_\eta(x)$ is different from the identity only for $(3/4)\eta^{-1} < \|x\| < 2\eta^{-1}$. We now show that $f_\eta(x)$ is uniformly bi-Lipschitz continuous for $\eta > 0$.

Let $x, y \in H$ with $\|x\| \le \|y\|$. If $\|x\| \ge 2\eta^{-1}$ then $f_\eta(x) = x$ and $f_\eta(y) = y$, so we consider only the case $\|x\| < 2\eta^{-1}$. Then

$$\begin{aligned} \|f_\eta(x) - f_\eta(y)\| &= \|R_{\eta x}x - R_{\eta y}y\| \\ &\le \|(R_{\eta x} - R_{\eta y})x\| + \|R_{\eta y}(x-y)\| \\ &\le \|R_{\eta x} - R_{\eta y}\|\|x\| + \|R_{\eta y}\|\|x - y\| \\ &\le 2\eta^{-1}\|R_{\eta x} - R_{\eta y}\| + \|x - y\|. \end{aligned}$$

Since

$$\begin{aligned} \|R_{\eta x} - R_{\eta y}\| &= \left\| \begin{pmatrix} \cos(\alpha\psi(\eta x)) - \cos(\alpha\psi(\eta y)) & \sin(\alpha\psi(\eta x)) - \sin(\alpha\psi(\eta y)) \\ -\sin(\alpha\psi(\eta x)) + \sin(\alpha\psi(\eta y)) & \cos(\alpha\psi(\eta x)) - \cos(\alpha\psi(\eta y)) \end{pmatrix} \right\| \\ &\le C_1\alpha\eta\|x - y\| := C_2\eta\|x - y\|, \end{aligned}$$

it follows that

$$\|f_\eta(x) - f_\eta(y)\| \le (2C_2 + 1)\|x - y\|$$

where the Lipschitz constant $2C_2 + 1$ does not depend on η. Since f_η is injective with inverse f_η^{-1} formed by the same construction but with the roles of v and w reversed we obtain the same bound for $\|f_\eta^{-1}(x) - f_\eta^{-1}(y)\|$. $\square$

Proposition 8.3. *Let X be a connected subset of a Hilbert space H that contains more than one point. Then there exists a C^∞ bi-Lipschitz map $\phi : H \to H$ such that*

$$d_{\rm A}^{\alpha,\beta}(\phi(X) - \phi(X)) = +\infty$$

for every $\alpha, \beta \ge 0$. Furthermore ϕ may be chosen such that $\mathrm{dist_H}(\phi(X), X)$ is arbitrarily small.

Proof. Since X contains more than one point, there exist two disjoint balls $B(x_1, R)$ and $B(x_2, R)$ of radius $R > 0$. Moreover, since X is connected, then there are points $x_{2+i} \in X$ for $i = 1, 2$ such that $\|x_{2+i} - x_i\| = R/4$. Thus, the four balls $B(x_i, R/8)$ with $x_i \in X$ for $i = 1, \ldots, 4$ are disjoint. Moreover

$$\bigcup_{i=1}^{4} B(x_i, R/8) \subseteq \bigcup_{i=1}^{2} B(x_i, 3R/8).$$

Recursively define nested families of disjoint balls such that

$$\bigcup_{i=1}^{2^{j+1}} B(x_i, R8^{-j}) \subseteq \bigcup_{i=1}^{2^j} B(x_i, 3R8^{-j}).$$

For $j = 0, 1, 2, \ldots$ and $i = 1, \ldots, 2^{j+1}$ let $a_j = (1/2)R8^{-j}$ and $e_{ij} = e_{2^{j+1}-2+i}$ where e_i is an orthonormal basis of H. Choose the points $y_{ij} \in B(x_i, R8^{-j})$ such that $\|x_i - y_{ij}\| = a_j$. Further define

$$g_{ij}(x) = x_i + f_\eta(x - x_i),$$

where f_η is the function given in Lemma 8.2 for $v = (y_{ij} - x_i)/a_j$, $w = e_{ij}$ and $\eta = 1/a_j$. If $\|x - x_i\| \geq 2a_j = R8^{-j}$ or $\|x - x_i\| \leq (3/4)a_j = 3R8^{-j-1}$ then $f_\eta(x - x_i) = x - x_i$ and $g_{ij}(x) = x$. Therefore the function g_{ij} is C^∞, bi-Lipschitz and different from the identity only on the annulus $B(x_i, R8^{-j}) \setminus B(x_i, 3R8^{-j-1})$. Moreover, by construction we have

$$g_{ij}(y_{ij}) = x_i + f_\eta(y_{ij} - x_i) = x_i + a_i f(v) = x_i + a_i e_{ij}.$$

Set

$$\phi(x) = \sum_{j=0}^{\infty} \sum_{i=1}^{2^{j+1}} g_{ij}(x).$$

Since the g_{ij} are different from the identity only on disjoint sets and the bi-Lipschitz constant of f_η is independent of η, then the map ϕ is a bi-Lipschitz C^∞ map of H onto H. Since $\phi(X) - \phi(X)$ contains

$$\begin{aligned}&\{\, a_j e_{ij} : j = 0, 1, 2, \ldots \text{ and } i = 1, \ldots, 2^{j+1} \,\}\\ &\quad = \{\, b_n e_n : b_n = (1/2)R8^{-j},\ n = 2^{j+1} - 1, \ldots, 2^{j+2} - 2 \,\}\end{aligned}$$

where $4R/(n+2)^3 \leq b_n \leq 4R/(n+1)^3$, then $b_n \sim 4Rn^{-3}$ and hence Lemma 8.1 implies $d_{\mathrm{A}}^{\alpha,\beta}(\phi(X) - \phi(X)) = \infty$.

Finally, note that $\mathrm{dist}_{\mathrm{H}}(\phi(X), X)$ may be made arbitrarily small by taking $R > 0$ sufficiently small in step one. $\square$

A consequence of this result is that it is not necessary for $X - X$ to be homogeneous in order to obtain a bi-Lipschitz embedding of X into some $\mathbb{R}^k$. Indeed, any set X that can be so embedded has a bi-Lipschitz image that has $d_{\mathrm{A}}^{\alpha,\beta}(X - X) = \infty$. However, it may still be the case that $X - X$ has to be homogeneous in order to obtain a *linear* bi-Lipschitz embedding as in Theorem 5.6.

On a more positive note, if X is an orthogonal sequence then homogeneity of X does imply homogeneity of $X - X$.

Lemma 8.4. *Let $X = \{\, x_j \,\}_{j=1}^{\infty}$ be an orthogonal sequence in H. If $d_{\mathrm{A}}(X) < +\infty$ then $d_{\mathrm{A}}(X - X) \leq 2d_{\mathrm{A}}(X)$.*

Proof. Suppose that X is (M,s)-homogeneous. We write $B_X(r,x)=B(r,x)\cap X$, and consider a ball $B=B_{X-X}(r,x-y)\subseteq X-X$ of radius r centred at $x-y\in X-X$. Since $B\subseteq B_{X-X}(\rho,0)\cup\big(B\setminus\{0\}\big)$, we need only cover $B\setminus\{0\}$.

Suppose that $x=y$, so that $B=B_{X-X}(r,0)$. Let $a-b\in B\setminus\{0\}$. Then $a\neq b$ and therefore a is orthogonal to b. It follows that

$$\big\|(a-b)-(x-y)\big\|^2=\|a\|^2+\|b\|^2<r^2.$$

Hence $a,b\in B_X(r,0)$, and consequently

$$B\setminus\{0\}\subseteq B_X(r,0)-B_X(r,0).$$

Cover $B_X(r,0)$ with $M(2r/\rho)^s$ balls $B_X(\rho/2,a_i)$ of radius $\rho/2$ centred at $a_i\in X$. Then

$$\begin{aligned}\bigcup_{i,j}B_{X-X}(\rho,a_i-a_j)\quad&\supseteq\quad\bigcup_i B_X(\rho/2,a_i)-\bigcup_j B_X(\rho/2,a_j)\\&\supseteq\quad B_X(r,0)-B_X(r,0)\supseteq B_{X-X}(r,0)\setminus\{0\}\,.\end{aligned}$$

It follows that B is covered by $1+M^2(2r/\rho)^{2s}$ balls of radius ρ.

Now suppose that $x\neq y$. Let $a-b\in B\setminus\{0\}$. Again $a\neq b$ and therefore a is orthogonal to b. We have

$$\|(a-b)-(x-y)\|^2=\begin{cases}\|a-x\|^2+\|b-y\|^2 & \\ \|a+y\|^2+\|2x\|^2 & \text{if}\\ \|2y\|^2+\|b+x\|^2 & \end{cases}\quad\begin{array}{l}a\neq y,\ b\neq x\\ a\neq y,\ b=x\\ a=y,\ b\neq x,\end{array}$$

and so

$$\left.\begin{array}{ll}a\in B_X(r,x) & b\in B_X(r,y)\\ a\in B_X(r,-y) & b\in B_X(r,x)\\ a\in B_X(r,y) & b\in B_X(r,-x)\\ a\in B_X(r,y) & b\in B_X(r,x)\end{array}\right\}\quad\text{if}\quad\left\{\begin{array}{l}a\neq y,\ b\neq x\\ a\neq y,\ b=x\\ a=y,\ b\neq x,\\ a=y,\ b=x.\end{array}\right.$$

Therefore

$$\begin{aligned}B\setminus\{0\}\quad\subseteq\quad&\big(B_X(r,x)-B_X(r,y)\big)\cup\big(B_X(r,-y)-B_X(r,x)\big)\\&\cup\big(B_X(r,y)-B_X(r,-x)\big)\cup\big(B_X(r,y)-B_X(r,x)\big).\end{aligned}$$

Cover each of $B_X(r,x)$, $B_X(r,-x)$, $B_X(r,y)$ and $B_X(r,-y)$ by $M(2r/\rho)^s$ balls of radius $\rho/2$. An argument similar to before yields a cover of B by $1+4M^2(2r/\rho)^{2s}$ balls of radius $r/2$.

Since we have $N_{X-X}(r,\rho)\leq 1+4M^2(2r/\rho)^{2s}$ it follows that $d_{\mathrm{A}}(X-X)\leq 2s$. $\square$

9. Non-existence of bi-Lipschitz linear embeddings

In this section we give a simple example showing that if we require a linear embedding (as in Theorem 5.6) then we can do no better than almost bi-Lipschitz. First we prove the following simple decomposition lemma for linear maps from H onto $\mathbb{R}^k$ (cf. comments in Hunt & Kaloshin, 1997).

Lemma 9.1. *Suppose $L:H\to\mathbb{R}^k$ is a linear map with $L(H)=\mathbb{R}^k$. Then $U=(\ker L)^\perp$ has dimension k, and L can be decomposed uniquely as MP, where P is the orthogonal projection onto U and $M:U\to\mathbb{R}^k$ is an invertible linear map.*

Note that the result of this lemma shows Theorem 5.6 remains true with linear maps replaced by orthogonal projections. This gives a much more concise proof of the result in Friz & Robinson (1999).

Proof. Let $U = (\ker L)^\perp$ and suppose that there exist $m > k$ linearly independent elements $\{x_j\}_{j=1}^m$ of U for which $Lx_j \neq 0$. Then $\{Lx_j\}$ are elements of $\mathbb{R}^k$; since $m > k$ at least one of the $\{Lx_j\}$ can be written as a linear combination of the others:

$$Lx_i = \sum_{j \neq i} c_j (Lx_j).$$

It follows that

$$\Big(x_i - \sum_{j \neq i} c_j x_j\Big) = 0,$$

which contradicts the definition of U.

Let P denote the orthogonal projection onto U, and M the restriction of L to U. Let $x \in H$, and decompose $x = u + v$, where $u \in U$ and $v \in \ker L$. Note that this decomposition is unique. Clearly $Lx = Lu = Mu = M(Px)$. It remains to show that M is invertible. This is clear since $\dim U = \dim \mathbb{R}^k = k$ and M is linear. $\square$

Following Ben-Artzi et al. (1993) we now prove

Lemma 9.2. *Suppose that $X - X$ contains a set of the form $\{\alpha_n e_n\}_{n=1}^\infty$ with $\{e_n\}_{n=1}^\infty$ an orthonormal set. Then no linear map into any $\mathbb{R}^k$ can be bi-Lipschitz between X and its image.*

Proof. We assume that $L(H) = \mathbb{R}^k$, otherwise it is possible to prune some redundant dimensions from $\mathbb{R}^k$. Suppose that L is bi-Lipschitz from X into $\mathbb{R}^k$. Write $L = MP$ as in Lemma 9.1. Since L is bi-Lipschitz on X then for all $y \in X - X$ we have

$$\|y\| \le c|Ly| \quad = \quad c|MPy| \le C\|Py\|,$$

where $C = c\|M\|$. In particular we have

$$\|\alpha_n e_n\| \le c\|P(\alpha_n e_n)\| \qquad \Rightarrow \qquad c\|Pe_n\| \ge 1.$$

But

$$k = \operatorname{rank} P = \operatorname{Trace} P \ge \sum_{n=1}^\infty (Pe_n, e_n) = \sum_{n=1}^\infty \|Pe_n\|^2 = +\infty$$

a contradiction. $\square$

We note that this result also follows from Lemma 2.4 in Movahedi-Lankarani & Wells (2005) which gives a characterisation of sets X that can be linearly bi-Lipschitz embedded into some $\mathbb{R}^k$: such an embedding is possible if and only if the weak closure of

$$\left\{ \frac{x - y}{\|x - y\|} \,:\, x, y \in X,\ x \neq y \right\}$$

does not contain zero ("weak spherical compactness of X").

Now consider the homogeneous set $X = \{2^{-n} e_n\} \cup \{0\}$, which has $d_{\mathrm{A}}(X) = 0$. Since X is an orthogonal sequence, it follows that $X - X$ (which in particular contains X) is also homogeneous; but Lemma 9.2 shows that no linear map into any finite-dimensional Euclidean space can be bi-Lipschitz on X. This shows that, with the requirement of linearly, our Theorem 5.6 cannot be improved.

However, note that there is a simple *nonlinear* bi-Lipschitz map ϕ from X into $[0,1]$, given by

$$\phi(2^{-n} e_n) = 2^{-n} :$$

for $n < m$ we have

$$\underbrace{\frac{1}{4}(2^{-n}+2^{-m})}_{\frac{1}{4}|2^{-n}e_n-2^{-m}e_m|} \le 2^{-(n+1)} \le \underbrace{|2^{-n}-2^{-m}|}_{|\phi(2^{-n}e_n)-\phi(2^{-m}e_m)|} \le 2^{-n} \le \underbrace{(2^{-n}+2^{-m})}_{|2^{-n}e_n-2^{-m}e_m|}.$$

The relationship between linear embeddings and general bi-Lipschitz embeddings is delicate. Suppose that X is a connected set containing more than one point. The result of Proposition 8.3 shows that even if X can be linearly bi-Lipschitz embedded into some $\mathbb{R}^n$ it is nevertheless bi-Lipschitz equivalent to a space $\phi(X)$ that cannot be bi-Lipschitz embedded into any $\mathbb{R}^n$ using a linear map.

10. Conclusion

We have identified a new class of *almost homogeneous* metric spaces, and shown that such spaces enjoy almost bi-Lipschitz embeddings into Hilbert space. Furthermore we have shown that any compact subset X of a Banach space with $X - X$ almost homogeneous can embedded into a finite-dimensional Euclidean space is an almost bi-Lipschitz way, and used this to deduce the same for any compact metric space (X, d) with $\mathscr{F}(X) - \mathscr{F}(X)$ almost homogeneous, where $\mathscr{F} : X \to L^\infty(X)$ is the isometric Kuratowski embedding of (X, d) into $L^\infty(X)$.

Some outstanding problems remain:

(1) Is there a homogeneous subset of a Hilbert space that cannot be bi-Lipschitz embedded into any $\mathbb{R}^k$?
(2) Can any (almost) homogeneous subset of a Hilbert space be (almost) bi-Lipschitz embedded into some $\mathbb{R}^k$?
(3) Can one construct an almost bi-Lipschitz embedding f of a compact almost homogeneous metric space (X, d) into a Hilbert space in such a way that $X - X$ is almost homogeneous? (This would answer (2) positively.)
(4) Is the exponent γ in Theorem 5.6 (the power of the slog term) in any way optimal?
(5) Can one bound the Assouad dimension of the attractors of dissipative PDEs (or preferably the set of differences of solutions lying on such attractors)?

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