Bouligand Dimension and Almost Lipschitz Embeddings

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Abstract. This paper discusses a metric dimension defined by Bouligand in [3] and proves the following new projection theorem: Let $\dim_b(\mathcal{A}-\mathcal{A})$ denote the Bouligand dimension of the set $\mathcal{A}-\mathcal{A}$ of differences between elements of $\mathcal{A}$. Given any set $\mathcal{A} \subseteq \mathbf{R}^n$ such that $\dim_b(\mathcal{A}-\mathcal{A}) < m$, then almost every orthogonal projection P of $\mathcal{A}$ of rank m is injective on $\mathcal{A}$ and $P|_{\mathcal{A}}$ has Lipschitz continuous inverse except for a logarithmic correction term.

1. Introduction

What we shall call Bouligand dimension is the metric dimension defined by Bouligand in [3] and further discussed by Assouad in [1]. In this paper we prove a few results about Bouligand dimension and its relation to the Mañé type projection theorems of [14], [2] and [9]. The use of Bouligand dimension in studying projections was initiated by Movahedi-Lankarani in [16] where he constructs a set $\mathcal{A}$ with finite fractal dimension for which there are no finite rank projections P with $P|_{\mathcal{A}}$ having Lipschitz continuous inverse. Here $\mathcal{A}$ is a subset of a Hilbert space H. To do this, he exhibits a set with finite fractal but infinite Bouligand dimension. He then raises the question: what can happen in the case that the Bouligand dimension of $\mathcal{A}$ is finite? This paper will attempt to shed some light on this question.

When $\mathcal{A}$ is an attractor for a partial differential equation, knowing whether $\mathcal{A}$ may be embedded into a finite dimensional space is of theoretical and computational interest. Work along these lines include Eden, Foias, Nicolaenko and Temam [8], Sauer, Yorke and Casdagli [20], and Robinson [18], [17]. Ideally, given an infinite dimensional dynamical system, we would like to construct a finite dimensional dynamical system that has the same long-term behavior. This would allow computations using the necessarily finite dimensional approximations possible with a computer. Since the long-term behavior of a dynamical system is largely governed by its global attractor, this question may be rephrased as whether it is possible to construct a finite dimensional dynamical system with the same global attractor as the original dynamical system. To construct such a system requires embedding the attractor of the original infinite dimensional dynamical system into a finite dimensional space. This embedding should posses certain continuity properties so as to preserve the original dynamics.

Inertial manifolds, as discussed by Constantin in [4] and Constantin, Foias, Nicolaenko and Temam in [6] and [7], provide a bi-Lipschitz embedding of the global attractor into a finite dimensional space. As the Bouligand dimension of an attractor is preserved under bi-Lipschitz mappings, the following holds:

Fact 1.1. The Bouligand dimension of the global attractor of any dynamical system possessing an inertial manifold must be finite.

Thus, there exists a number of global attractors with finite Bouligand dimension. In light of the main result in this paper, computing the Bouligand dimension of the global attractor directly and more specifically the Bouligand dimension of the set $\mathcal{A}-\mathcal{A}$ of differences between elements of $\mathcal{A}$ is of great interest. Our main result is

Theorem 1.2. Given $\mathcal{A} \subseteq \mathbf{R}^n$ such that $\dim_b(\mathcal{A}-\mathcal{A}) < m$, then almost every orthogonal projection P of $\mathcal{A}$ of rank m is injective on $\mathcal{A}$ and $P|_{\mathcal{A}}$ has Lipschitz continuous inverse except for a logarithmic correction term.

This shows that almost every orthogonal projection of $\mathcal{A}$ has the same Lipschitz properties that the embeddings of an inertial manifold have except for a logarithmic correction. Here, almost every should be understood in terms of the Haar measure invariant with respect to orthogonal transformations on the space of all orthogonal projections of rank m in $\mathbf{R}^n$. This provides a partial converse to Fact 1.1. Note that n may be chosen arbitrarily large. Thus, Theorem 1.2 embeds a fractal subset $\mathcal{A}$ of a large finite dimensional space into a smaller finite dimensional space whose dimension is controlled explicitly by the Bouligand dimension of $\mathcal{A}-\mathcal{A}$. Our theorem is related to the following result essentially proved by Mañé in [14] under the assumption of finite Hausdorff dimension.

Theorem 1.3. [Mañé] If E is a Banach space and $\mathcal{A} \subset E$ is a countable union of compact subsets with $\dim_h(\mathcal{A}) < \infty$ then for every subspace $\mathcal{B}$ of E with $\dim_h(\mathcal{A}-\mathcal{A})+1 < \dim(\mathcal{B}) < \infty$ the set of projections $P\colon \mathcal{A} \to \mathcal{B}$ such that $P|_{\mathcal{A}}$ is injective is a residual subset of the space of projections of $\mathcal{A}$ onto $\mathcal{B}$ endowed with the norm topology.

Hölder continuity for the inverse in Mañé's projection theorem has been proved under the hypothesis of finite fractal dimension by Ben-Artzi, Eden, Foias, and Nicolaenko in [2], and extended by Foias and Olson in [9]. In particular,

Theorem 1.4. [Foias and Olson] Let H be a real Hilbert space and $\mathcal{A} \subseteq H$ be such that $2\dim_f(\mathcal{A}) < m$. Then for any orthogonal projection P of rank m and $\delta > 0$ there is an orthogonal projection $\tilde{P}$ such that $\|\tilde{P}-P\| < \delta$ and $\tilde{P}|_{\mathcal{A}}$ has Hölder inverse.

Hunt and Kaloshin in [12] have recently shown that such projections $\tilde{P}$ are in fact prevalent according to the sense of prevalence given by Hunt, Sauer and Yorke in [13].

Note that Theorem 1.4 is stated with a hypothesis only on the fractal dimension of $\mathcal{A}$. This simplification may be made because the fractal dimension of $\mathcal{A}-\mathcal{A}$ is controlled by the inequality $\dim_f(\mathcal{A}-\mathcal{A}) < 2\dim_f(\mathcal{A})$. Such inequalities are not true for the Bouligand and Hausdorff dimensions. Therefore, Theorem 1.2 and Theorem 1.3 are stated under hypotheses about the set $\mathcal{A}-\mathcal{A}$ directly. In this paper we give an example for Bouligand dimension for which $\dim_b(\mathcal{A}) = 0$ and $\dim_b(\mathcal{A}-\mathcal{A}) = \infty$ to show how things are different.

We also show the Bouligand dimension of a set may increase under an orthogonal projection. Note that for fractal and Hausdorff dimensions, the dimension of the projected image is always less than or equal the dimension of the original set. Thus, Bouligand dimension gives us further insight over what can happen to the geometry of a set under orthogonal projection. In particular, there may be bad projections that make a global attractor appear more complicated than it really is.

This paper is organized as follows: First, we define Bouligand dimension and give an alternate characterization of it. In Section 3 we state some properties and show that Bouligand dimension agrees with other notions of dimension for self-similar fractals. In Section 4 we prove Theorem 4.8 and in Section 5 we prove Theorem 1.2.

2. The Bouligand Dimension

In this section we define fractal and Bouligand dimensions giving alternative easier to work with characterizations for each. By fractal dimension we will refer to the metric dimension that is sometimes also called the box-counting dimension, the capacity, or the Minkowski dimension. Denote the fractal dimension of a compact metric space $\mathcal{A}$ by $\dim_f(\mathcal{A})$.

Definition 2.1. Let $\mathcal{N}_{\mathcal{A}}(\rho)$ be the minimum number of ρ-radius balls required to cover all of $\mathcal{A}$. Then, the fractal dimension

$$\dim_f(\mathcal{A}) = \limsup_{\rho\to 0} \frac{\log \mathcal{N}_{\mathcal{A}}(\rho)}{\log \rho}.$$

An equivalent definition of fractal dimension in terms of an infimum follows.

Fact 2.2. The fractal dimension $\dim_f(\mathcal{A})$ is the infimum over all d for which there exists K such that

$$\mathcal{N}_{\mathcal{A}}(\rho) \le K(1/\rho)^d \qquad \text{for } 0 < \rho < 1. \tag{2.1}$$

The definition of Bouligand dimension given by Movahedi-Lankarani in [16] is

Definition 2.3. The Bouligand dimension $\dim_b(\mathcal{A})$ is defined as

$$\dim_b(\mathcal{A}) = \lim_{\epsilon\to 0} \lim_{t\to\infty} \Delta_{\epsilon,t}(\mathcal{A})$$

where

$$\Delta_{\epsilon,t}(\mathcal{A}) = \sup\left\{ \frac{\log \mathcal{N}(r,\rho)}{\log(r/\rho)} : 0 < \rho < r < \epsilon \text{ and } r > t\rho \right\}$$

and $\mathcal{N}(r,\rho)$ is the number of ρ-balls required to cover any r-ball in $\mathcal{A}$.

Bouligand dimension may also be characterized as an infimum in a way similar to that of fractal dimension. In particular,

Theorem 2.4. The Bouligand dimension $\dim_b(\mathcal{A})$ is the infimum over all d for which there exists K such that

$$\mathcal{N}_{\mathcal{A}}(r,\rho) \le K(r/\rho)^d \qquad \text{for } 0 < \rho < r < 1. \tag{2.2}$$

First note that the exact form of the upper bound for r in condition (2.2) is not critical. Equivalently we may require for some $\epsilon > 0$ that there exists K_ϵ such that

$$\mathcal{N}_{\mathcal{A}}(r,\rho) \le K_\epsilon (r/\rho)^d \qquad \text{for } 0 < \rho < r < \epsilon. \tag{2.3}$$

It is obvious if $\epsilon < 1$ that the condition (2.2) implies (2.3) with $K_\epsilon = K$. Conversely suppose (2.3) holds for some $0 < \epsilon < 1$. Since $\mathcal{A}$ is compact, then a finite number N of $\epsilon/4$ balls will cover it. Let r and ρ be such that $0 < \rho < r < 1$. Consider the following cases:

Case $\rho \ge \epsilon/2$. Then

$$\mathcal{N}_{\mathcal{A}}(r,\rho) \le \mathcal{N}_{\mathcal{A}}(r,\epsilon/2) \le \mathcal{N}_{\mathcal{A}}(\epsilon/4) \le N.$$

Case $\rho < \epsilon/2 \le r$. Then

$$\mathcal{N}_{\mathcal{A}}(r,\rho) \le \mathcal{N}_{\mathcal{A}}(r,\epsilon/2)\mathcal{N}_{\mathcal{A}}(\epsilon/2,\rho) \le NK_\epsilon \Big(\frac{\epsilon/2}{\rho}\Big)^d \le NK_\epsilon \Big(\frac{r}{\rho}\Big)^d.$$

Thus taking $K = NK_\epsilon$ we obtain that

$$\mathcal{N}_{\mathcal{A}}(r,\rho) \le K\Big(\frac{r}{\rho}\Big)^d \qquad \text{for } 0 < \rho < r < 1.$$

In the situation where $\epsilon > 1$ the argument is similar. Moreover, if there exists one ϵ and K_ϵ for which condition (2.3) holds, then for each ϵ there is a K_ϵ such that it holds.

Proof of Theorem 2.4: Suppose there is d and K such that (2.2) holds. Then

$$\frac{\log \mathcal{N}_{\mathcal{A}}(r,\rho)}{\log(r/\rho)} \le d + \frac{\log K}{\log(r/\rho)} \qquad \text{for } 0 < \rho < r < 1.$$

It follows that

$$\Delta_{1,t}(\mathcal{A}) \le d + \frac{\log K}{\log(t)}$$

and therefore

$$\dim_b(\mathcal{A}) = \lim_{\epsilon\to 0}\lim_{t\to\infty} \Delta_{1,t}(\mathcal{A}) \le d.$$

Conversely, suppose d is chosen so that $\dim_b(\mathcal{A}) < d$. Then by definition there are values of ϵ and t such that $0 < \epsilon < 1 < t$ and $\Delta_{\epsilon,t}(\mathcal{A}) < d$. It follows that

$$\frac{\log \mathcal{N}_{\mathcal{A}}(r,\rho)}{\log(r/\rho)} < d \qquad \text{for } 0 < t\rho < r < \epsilon$$

and therefore

$$\mathcal{N}_{\mathcal{A}}(r,\rho) < (r/\rho)^d \qquad \text{for } 0 < t\rho < r < \epsilon.$$

Now, suppose $0 < \rho < r < \epsilon$. Then $0 < t(\rho/t) < r < \epsilon$ and

$$\mathcal{N}_{\mathcal{A}}(r,\rho) \le \mathcal{N}_{\mathcal{A}}(r,\rho/t) \le \Big(\frac{r}{\rho/t}\Big)^d = K\Big(\frac{r}{\rho}\Big)^d \qquad \text{for } 0 < \rho < r < \epsilon$$

where $K = t^d$. It follows that $\dim_b(\mathcal{A})$ is the infimum over all d for which (2.2) holds. $\square$

Note that the Bouligand dimension covers only a portion of the set $\mathcal{A}$ but requires that the number of balls needed to cover that portion scale with its size. In particular, (2.2) may be obtained from (2.1) by the addition of this scaling condition.

It is often of interest to know whether a set has finite Bouligand dimension or not. The following lemma provides a simple test for determining this.

Lemma 2.5. If there is some K such that $\mathcal{N}_{\mathcal{A}}(r, r/2) < K$ holds for all $r < 1$ then $\mathcal{A}$ has finite Bouligand dimension. Moreover, $\dim_b(\mathcal{A}) \le \log_2 K$.

Proof: Given r and ρ such that $0 < \rho < r < 1$ choose n so that $r/2^n \le \rho < r/2^{n-1}$. Then

$$\mathcal{N}_{\mathcal{A}}(r, \rho) \le \mathcal{N}_{\mathcal{A}}(r, r/2)\mathcal{N}_{\mathcal{A}}(r/2, r/2^2) \cdots \mathcal{N}_{\mathcal{A}}(r/2^{n-1}, r/2^n) \le K^n$$

and since $n - 1 \le \log_2(r/\rho)$ it follows that

$$\mathcal{N}_{\mathcal{A}}(r, \rho) \le K(K^{n-1}) \le K(r/\rho)^{\log_2 K}.$$

Therefore $\dim_b(\mathcal{A}) \le \log_2 K < \infty$. $\square$

3. Properties of Bouligand Dimension

The Bouligand dimension satisfies many of the usual properties that a reasonable dimension should satisfy. In this section we state and prove a few of these properties. In particular, we prove Bouligand dimension is well behaved with respect to Cartesian products and that Bouligand dimension agrees with the similarity dimension for self-similar fractals.

First, we will state for reference as Theorem 3.1 a few properties of Bouligand dimension as stated by Movahedi-Lankarani in [16].

Theorem 3.1. The Bouligand dimension has the following properties:

(i) If $\mathcal{A} \subseteq \mathcal{B}$ then $\dim_b(\mathcal{A}) \le \dim_b(\mathcal{B})$.
(ii) If $\mathcal{A}$ is an open subset of $\mathbf{R}_n$ then $\dim_b(\mathcal{A}) = n$.
(iii) If $\mathcal{A}$ and $\mathcal{B}$ are bi-Lipschitz isomorphic, then $\dim_b(\mathcal{A}) = \dim_b(\mathcal{B})$.

We now discuss how Bouligand dimension behaves with respect to Cartesian products. We would like

$$\dim_b(\mathcal{A} \times \mathcal{B}) = \dim_b(\mathcal{A}) + \dim_b(\mathcal{B}).$$

However, as with any metric dimension, the Bouligand dimension depends on what metric is used. Therefore, care must be taken in choosing a metric. Let $\mathcal{A}$ and $\mathcal{B}$ be compact metric spaces with metrics $d_{\mathcal{A}}$ and $d_{\mathcal{B}}$. Define the metric spaces $X_p = \mathcal{A} \times \mathcal{B}$ for $1 \le p \le \infty$ by the metrics d_p given by

$$d_p\big((a_1, b_1), (a_2, b_2)\big) = \begin{cases} \big(d_{\mathcal{A}}(a_1, a_2)^p + d_{\mathcal{B}}(b_1, b_2)^p\big)^{1/p} & \text{for } 1 \le p < \infty \\ \max\big(d_{\mathcal{A}}(a_1, a_2), d_{\mathcal{B}}(b_1, b_2)\big) & \text{for } p = \infty. \end{cases}$$

Theorem 3.2. $\dim_b(X_p) = \dim_b(\mathcal{A}) + \dim_b(\mathcal{B})$.

Proof: Since d_∞ is Lipschitz equivalent to d_p for $1 \le p < \infty$, it follows from property (iii) of Theorem 3.1 that

$$\dim_b(X_\infty) = \dim_b(X_p) \qquad \text{for} \qquad 1 \le p < \infty.$$

Hence, we may prove the theorem for $X = X_\infty$ without loss of generality.

For every α and β such that $\dim_b(\mathcal{A}) < \alpha$ and $\dim_b(\mathcal{B}) < \beta$ there exists $K_\mathcal{A}$ and $K_\mathcal{B}$ such that

$$\mathcal{N}_\mathcal{A}(r,\rho) < K_\mathcal{A}\Big(\frac{r}{\rho}\Big)^\alpha \qquad \text{and} \qquad \mathcal{N}_\mathcal{B}(r,\rho) < K_\mathcal{B}\Big(\frac{r}{\rho}\Big)^\beta$$

for $0 < \rho < r < 1$. Let B be a ball of radius r in X. Then by definition $B = U \times V$ where U and V are balls of radius r. Cover U by balls U_i of radius ρ in $\mathcal{A}$ and V by balls V_j of radius ρ in $\mathcal{B}$. Since $U_i \times V_j$ form a cover of B, it follows that

$$\mathcal{N}_X(r,\rho) \le \mathcal{N}_\mathcal{A}(r,\rho)\mathcal{N}_\mathcal{B}(r,\rho) \le K_\mathcal{A}K_\mathcal{B}\Big(\frac{r}{\rho}\Big)^{\alpha+\beta}.$$

Hence $\dim_b(X) \le \alpha + \beta$ and so $\dim_b(X) \le \dim_b(\mathcal{A}) + \dim_b(\mathcal{B})$.

Now, let n be the maximum number of disjoint $\rho/2$-radius balls with centers u_i in U. Let U_i be balls of radius ρ with the same centers. Thus the points u_i are at least a distance $\rho/2$ apart from each other and U_i covers U. Do the same for V to obtain m balls V_j of radius ρ whose centers v_j are at least a distance $\rho/2$ apart from each other which cover V. Let $z_k \in B$ be an enumeration of the points (u_i, v_j) and define $Z = \{\, z_k : k = 1 \ldots nm \,\}$. It follows that

$$\mathcal{N}_X(r,\rho/4) \ge \mathcal{N}_Z(\rho/2) \ge nm \ge \mathcal{N}_U(\rho)\mathcal{N}_V(\rho).$$

Choose U and V such that $\mathcal{N}_U(\rho)$ and $\mathcal{N}_V(\rho)$ attain their maximum values. Given d such that $\dim_b(X) < d$, there is K such that

$$\mathcal{N}_X(r,\rho) \le K\Big(\frac{r}{\rho}\Big)^d \qquad \text{for } 0 < \rho < r < 1.$$

Thus

$$\mathcal{N}_\mathcal{A}(r,\rho)\mathcal{N}_\mathcal{B}(r,\rho) \le \mathcal{N}_X(r,\rho/4) \le 4^d K\Big(\frac{r}{\rho}\Big)^d,$$

and therefore $\dim_b(\mathcal{A}) + \dim_b(\mathcal{B}) = \dim_b(X)$. $\square$

It is interesting to note that the Hausdorff dimension $\dim_h$ does not behave well under Cartesian products. For example, Falconer exhibits sets $\mathcal{A}$ and $\mathcal{B}$ in [5] such that $\dim_h(\mathcal{A}) + \dim_h(\mathcal{B}) < \dim_h(\mathcal{A} \times \mathcal{B})$.

We end this section with a calculation of the Bouligand dimension for a self-similar fractal. In particular, we show the Bouligand dimension agrees with the similarity dimension for such sets, and therefore the fractal and Hausdorff dimensions as well. First note that Hausdorff dimension is bounded by the fractal dimension which is in turn bounded by the Bouligand dimension. Thus given a compact metric space $\mathcal{A}$ we have that

$$\dim_h(\mathcal{A}) \le \dim_f(\mathcal{A}) \le \dim_b(\mathcal{A}). \tag{3.1}$$

Let $f_i\colon \mathbf{R}^n \to \mathbf{R}^n$ where $i = 1, \ldots, N$ be similarities with ratios $c_i \in (0,1)$ and define $F(M) = \cup f_i(M)$ for $M \subseteq \mathbf{R}^n$. Let $\mathcal{A}$ be the compact invariant set such that $F(\mathcal{A}) = \mathcal{A}$.

Definition 3.3. The similarity dimension of $\mathcal{A}$ is defined as $\dim_s(\mathcal{A}) = s$ where $\Sigma c_i^s = 1$.

Theorem 3.4. Let $\mathcal{A}$ be defined as above. Then $\dim_b(\mathcal{A}) = \dim_s(\mathcal{A})$.

Proof: Since $\mathcal{A}$ is compact there is ϵ such that $0 < \epsilon < 1$ and

$$\operatorname{dist}\big(f_i(\mathcal{A}), f_j(\mathcal{A})\big) > \epsilon \quad \text{for} \quad i \neq j.$$

Let $c = \min\{\, c_i : i = 1, \ldots, N\,\}$. Let $R > \epsilon/c$ be chosen so large that $\mathcal{A}$ fits inside an R ball. Let r and ρ be chosen so that $0 < \rho < r < \epsilon/2$. Let I be the set of finite sequences $(i_1, \ldots, i_n)$ such that

$$c_{i_1} \cdots c_{i_n} \epsilon \leq 2r < c_{i_1} \cdots c_{i_{n-1}} \epsilon.$$

Since $2r < \epsilon$, then I is non-empty. Furthermore, $\mathcal{A} = \cup\{ f_{i_1} \circ \cdots \circ f_{i_n}(\mathcal{A}) : (i_1, \ldots, i_n) \in I\,\}$. Let $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_m)$ be distinct elements of I. Suppose for definiteness that $n \leq m$. Consider the sets $f_{i_1} \circ \cdots \circ f_{i_n}(\mathcal{A})$ and $f_{j_1} \circ \cdots \circ f_{j_m}(\mathcal{A})$. Let k be the largest index such that $i_l = j_l$ for all $l \leq k$. Obviously $k < n$ for otherwise $k = n$ would imply $n < m$ leading to the contradiction

$$c_{i_1} \cdots c_{i_n} \epsilon \leq 2r < c_{j_1} \cdots c_{j_{m-1}} \epsilon \leq c_{j_1} \cdots c_{j_n} \epsilon = c_{i_1} \cdots c_{i_n} \epsilon.$$

Define $M = f_{i_k+1} \circ \cdots \circ f_{i_n}(\mathcal{A})$ and $N = f_{j_k+1} \circ \cdots \circ f_{j_m}(\mathcal{A})$. Then $i_{k+1} \neq j_{k+1}$ implies $\operatorname{dist}(M, N) > \epsilon$. It follows that

$$\operatorname{dist}\big(f_{i_1} \circ \cdots \circ f_{i_n}(\mathcal{A}), f_{j_1} \circ \cdots \circ f_{j_m}(\mathcal{A})\big) > c_{i_1} \cdots c_{i_k} \epsilon \geq c_{i_1} \cdots c_{i_{n-1}} \epsilon > 2r.$$

Now let B be a ball of radius r. If follows that $B \cap f_{i_1} \circ \cdots \circ f_{i_n}(\mathcal{A}) \neq \emptyset$ for at most one $(i_1, \ldots, i_n) \in I$. Let $(i_1, \ldots, i_n)$ be that one and fix it.

Hence $B \cap \mathcal{A} = B \cap f_{i_1} \circ \cdots \circ f_{i_n}(\mathcal{A})$. Define $\gamma = c_{i_1} \cdots c_{i_n}$ and let J be the set of $(j_1, \ldots, j_m)$ such that

$$R\gamma c_{j_1} \cdots c_{j_m} \leq \rho < R\gamma c_{j_1} \cdots c_{j_{m-1}}.$$

Since $\rho < r \leq \gamma\epsilon/c \leq R\gamma$, then J is non-empty. Since $\mathcal{A} = \cup\{ f_{j_1} \circ \cdots \circ f_{j_m}(\mathcal{A}) : (j_1, \ldots, j_n) \in J\,\}$ it follows that

$$B \cap \mathcal{A} \subseteq \cup\{\, f_{i_1} \circ \cdots \circ f_{i_n} \circ f_{j_1} \circ \cdots \circ f_{j_m}(\mathcal{A}) : (j_1, \ldots, j_m) \in J\,\}.$$

Furthermore, since $R\gamma c_{j_1} \cdots c_{j_m} \leq \rho$, it follows that

$$f_{i_1} \circ \cdots \circ f_{i_n} \circ f_{j_1} \circ \cdots \circ f_{j_m}(\mathcal{A}) \subseteq B_\rho\big(f_{i_1} \circ \cdots \circ f_{i_n} \circ f_{j_1} \circ \cdots \circ f_{j_m}(\alpha)\big).$$

We now estimate the number of elements in J. Since $\Sigma c_i^s = 1$, it follows by induction that $\sum_J (c_{j_1} \cdots c_{j_m})^s = 1$. Therefore $\sum_J \big(c\rho/(R\gamma)\big)^s \leq 1$ and so J has no more than $\big(R\gamma/(c\rho)\big)^s$ elements. Hence, $\mathcal{N}_{B\cap\mathcal{A}}(\rho) \leq \big(R\gamma/(c\rho)\big)^s$. Since $\gamma = c_{i_1} \cdots c_{i_n} \leq 2r/\epsilon$, it follows that $\mathcal{N}_{\mathcal{A}}(r, \rho) \leq K(r/\rho)^s$ where $K = \big(2R/(c\epsilon)\big)^s$. As this holds for all $0 < \rho < r < \epsilon/2$, it further follows that $\dim_b(\mathcal{A}) \leq s$. By [5], Theorem 9.3, the Hausdorff dimension $\dim_h(\mathcal{A}) = s$; therefore, in light of (3.1) we obtain that $\dim_b(\mathcal{A}) = s$. $\square$

Thus, for self-similar sets, the Bouligand dimension agrees with the similarity, fractal, and Hausdorff dimensions. In the case that $c_i = c$ for all i, the similarity dimension has an easy to calculate form and we obtain the following corollary:

Corollary 3.5. Let $f_i\colon \mathbf{R}^n \to \mathbf{R}^n$ where $i = 1, \ldots, N$ be similarities with ratio $c \in (0, 1)$ and define $F(M) = \cup f_i(M)$ for $M \subseteq \mathbf{R}^n$. Take $\mathcal{A}$ to be the compact invariant set such that $F(\mathcal{A}) = \mathcal{A}$. If the images under the f_i are disjoint then $\dim_b(\mathcal{A}) = -\log(N)/\log(c)$.

4. Dimension Increasing Projections

The results in this section are motivated by examples involving orthogonal sequences in a Hilbert space. The fractal dimension of such sequences has been already studied by Ben-Artzi, Eden, Foias and Nicolaenko in [2] and by Ladislav Mišík Jr. and Tibor Žáčik in [15]. In some sense, an orthogonal sequence is the farthest thing possible from the regular self-similar fractals discussed in the previous section. Not surprisingly, it is in the treatment of these sets that the Bouligand dimension differs most dramatically from the fractal dimension. In particular, we exhibit a set $\mathcal{A}$ such that $\dim_b(\mathcal{A}) = 0$ and $\dim_b(\mathcal{A} - \mathcal{A}) = \infty$. This section closes with a proof of Theorem 4.8 concerning the existence orthogonal projections that increase Bouligand dimension.

Recall, Bouligand dimension is characterized by a scaling condition between r and ρ. In particular, the number of ρ balls needed to cover a r sized piece should be no different than the number of $\lambda\rho$ balls needed to cover a λr sized piece. This homogeneous scaling makes the Bouligand dimension sensitive to inhomogeneities in the set $\mathcal{A}$. We will consider orthogonal sequences in a Hilbert space converging to zero. Let H be a Hilbert space and e_i an orthonormal sequence.

Lemma 4.1. Let $\mathcal{A} = \{0\} \cup \{e_n/n^\alpha : n \in \mathbf{N}\}$. Then $\dim_f(\mathcal{A}) = 1/\alpha$ and $\dim_b(\mathcal{A}) = \infty$.

Proof: The first fact appears in [2]. For the second, consider the ball of radius $r = 1/m^\alpha$ centered at the origin

$$B = \{a \in \mathcal{A} : \|a\| < r\} = \{0\} \cup \{e_n/n^\alpha : n > m\}.$$

Cover B by $r/2$ balls. Each point a distance more that $r/2$ from the origin will require a separate ball. Since

$$1/n^\alpha > r/2 \qquad \text{implies} \qquad n < (2/r)^{1/\alpha} = m2^{1/\alpha}$$

then

$$\mathcal{N}_{\mathcal{A}}(r, r/2) \geq \mathcal{N}_B(r/2) > m2^{1/\alpha} - m - 1.$$

This is unbounded; therefore, by Lemma 2.5 it follows that $\dim_b(\mathcal{A}) = \infty$. $\square$

Fact 4.2. If there is K and α such that

$$a_n \leq K/n^\alpha$$

then $\dim_f\left(\{0\} \cup \{a_n e_n : n \in \mathbf{N}\}\right) \leq 1/\alpha$.

The Bouligand dimension of a geometric sequence is finite because geometric sequences have the scaling property needed for $\mathcal{N}_{\mathcal{A}}(r, r/2)$ to be bounded.

Fact 4.3. If there is K and α such that $0 < \alpha < 1$ and

$$(1/K)\alpha^n \leq a_n \leq K\alpha^n \tag{4.1}$$

then $\dim_b\left(\{0\} \cup \{a_n e_n : n \in \mathbf{N}\}\right) = 0$.

The lower bound in the inequality (4.1) is required. This follows immediately from

Theorem 4.4. There exist sequences converging arbitrarily fast to zero that have infinite Bouligand dimension.

Proof: Let b_j be a sequence converging to zero. Consider the sequence $a_n e_n$ where

$$a_n = b_j \quad \text{for } n = 2^{j-1}, \ldots, 2^j - 1.$$

Let B be the ball of radius $r = b_j + \epsilon$ centered at the origin

$$B = \{a_n e_n : n \geq 2^{j-1}\}.$$

Cover B by $r/2$ balls. Then

$$\mathcal{N}_{\mathcal{A}}(r, r/2) \geq \mathcal{N}_B(r/2) \geq 2^j - 2^{j-1} = 2^{j-1}$$

is unbounded; therefore, $\dim_b(\mathcal{A}) = \infty$. Since no conditions on b_j were imposed, then a_n may converge arbitrarily fast to zero. □

Corollary 4.5. There exist a set $\mathcal{A}$ such that $\dim_b(\mathcal{A}) = 0$ and $\dim_b(\mathcal{A} - \mathcal{A}) = \infty$.

Proof: Let x_j be a sequence of the type given in Theorem 4.4 such that $\|x_j\| \leq 4^{-j}$. We assume the complement of the closed linear span of the x_j to be infinite dimensional and define y_j to be an orthogonal sequence in that complement such that $\|y_j\| = 4^{-j}$. Let $\mathcal{A}$ be the closure of the set of all a_j where $a_{2j} = y_j$ and $a_{2j+1} = x_j + y_j$. Clearly the set of differences $\mathcal{A} - \mathcal{A}$ contains the set x_j and therefore $\dim_b(\mathcal{A} - \mathcal{A}) = \infty$.

Claim $\dim_b(\mathcal{A}) = 0$. If $k = 2j$ then

$$\|a_k\| = \|y_j\| = 4^{-j} = 2^{-k}$$

and so condition (4.1) is satisfied with $\alpha = 1/2$. If $k = 2j + 1$, then

$$\|a_k\| \leq \|x_j\| + \|y_j\| \leq 4^{-j} + 4^{-j} = 2(4^{-(k-1)/2}) \leq 4(2^{-k})$$

and

$$\|a_k\| \geq \|y_j\| = 2(2^{-k}).$$

Therefore, taking $K = 4$ and $\alpha = 1/2$ satisfies (4.1) for all terms of the sequence a_j. Applying Fact 4.3 now yields that $\dim_b(\mathcal{A}) = 0$. □

We will now use the same scaling properties exploited in the above examples to construct orthogonal projections that increase Bouligand dimension. First, we need the following definitions and results.

Definition 4.6. Let $\vee\mathcal{A}$ denote the closed linear span of $\mathcal{A}$.

Lemma 4.7. There exists a projection Q such that $Q\mathcal{A}$ contains an orthogonal sequence if and only if there exists $V \subseteq \mathcal{A}$ such that $\dim(\vee V) = \infty$ and

$$\vee\big(V \setminus \{v\}\big) \neq \vee V \qquad \text{for all } v \in V.$$

Proof: Suppose there is such a projection Q. Choose $v_n \in \mathcal{A}$ such that the Qv_n form an orthogonal sequence. Define $V = \{\, v_n : n \in \mathbf{N}\,\}$. It follows that Qv is orthogonal to $\vee\big(QV \setminus \{Qv\}\big)$ for any $v \in V$. Therefore $\vee\big(QV \setminus \{Qv\}\big) \neq \vee\big(QV\big)$. Since orthogonal projections are continuous and linear, then Q commutes with $\vee$ and we obtain $Q\big(\vee\big(V \setminus \{v\}\big)\big) \neq Q(\vee V)$, which implies $\vee\big(V \setminus \{v\}\big) \neq \vee V$.

Conversely, by omitting some elements of V if necessary, we write $V = \{\, v_n : n \in \mathbf{N}\,\}$ and suppose the orthogonal complement of $\vee V$ to be infinite dimensional. Let f_i be an orthonormal sequence contained in that complement.

Define $V_1 = V$ and $W_1 = V_1 \setminus \{v_1\}$. For induction suppose $\vee W_n \neq \vee V_n$. Then there exists $e_n \in \vee V_n$ such that $\|e_n\| = 1$ and $e_n \perp \vee W_n$. Let $g_n = (e_n + f_n)/\sqrt{2}$. Define $V_{n+1} = V_n \cup \{g_n\}$ and $W_{n+1} = V_{n+1} \setminus \{v_{n+1}\}$. Clearly $\vee W_{n+1} \neq \vee V_{n+1}$.

The above construction yields an orthonormal sequence g_n such that $(g_i, v_i) \neq 0$ and $(g_i, v_j) = 0$ for $i \neq j$. Let Q be the projection onto the space spanned by the g_n. It follows that $Pv_n = \alpha_n g_n$ where $\alpha_n \neq 0$ and so $QV \subseteq Q\mathcal{A}$ contains an orthogonal sequence. □

Theorem 4.8. If $\mathcal{A}$ satisfies Lemma 4.7, then there exists an orthogonal projection P such that the Bouligand dimension of $P\mathcal{A}$ is infinite.

Note that Fact 4.3 provides an example of a set with finite Bouligand dimension that satisfies Lemma 4.7. However, a slight modification of the solution given by Halmos for Problem 11 in [11] yields an infinite dimensional compact set which does not satisfy these hypotheses. It is unknown whether the attractors of naturally occurring physical systems satisfy these hypothesis or not.

Proof of Theorem 4.8: By Lemma 4.7 there exists an orthogonal projection Q such that $Q\mathcal{A}$ contains the orthogonal sequence $\alpha_n g_n$. Let

$$m_j = \min\{\, |\alpha_n| : n = 2^{j-1}, \ldots, 2^j - 1\,\}$$

and choose a monotone sequence β_j strictly decreasing to zero such that $\beta_j \leq m_j$. We may assume that the orthogonal complement of $\vee\{\, \alpha_n g_n : n \in \mathbf{N}\,\}$ in QH is infinite. Let h_n be an orthonormal sequence in that complement and define

$$\tilde{g}_n = \sin(\theta_n) g_n + \cos(\theta_n) h_n$$

where θ_n has been chosen in such a way that

$$\alpha_n \sin(\theta_n) = \beta_j \qquad \text{for} \qquad n = 2^{j-1}, \ldots, 2^j - 1.$$

This can be done since $\beta_j < \alpha_n$. Let P be the projection onto the space spanned by the $\tilde{g}_n$. It follows that $P\mathcal{A} = PQ\mathcal{A}$ contains a sequence similar to the one found in Theorem 4.4 and therefore

$$\dim_b(P\mathcal{A}) \geq \dim_b\{\, \alpha_n \sin(\theta_n) g_n : n \in \mathbf{N} \,\} = \infty. \qquad \square$$

5. The Projection Theorem

In this section we prove Theorem 1.2 restated more explicitly in the form of Theorem 5.2 below. We want to find rank m orthogonal projections such that

$$\|Px - Py\| \geq \epsilon f\big(\|x-y\|\big) \qquad \text{for } x, y \in \mathcal{A} \tag{5.1}$$

where f is linear with a logarithmic correction. Moreover, we show that the measure of all the projections which don't satisfy this condition for any constant $\epsilon > 0$ is zero. Writing $Y = \mathcal{A} - \mathcal{A}$ simplifies (5.1) to the requirement that

$$\|Py\| \geq \epsilon f\big(\|y\|\big) \qquad \text{for } y \in Y.$$

Let G be the space of orthogonal projections in $\mathbf{R}^N$ of rank m and μ be the invariant measure on G with respect to orthogonal transformations. Define the shadow of a set B in $\mathbf{R}^N$ to be

$$S(B) = \{\, P \in G : 0 \in PB \,\}.$$

Showing that $\mu\big(S(Y)\big) = 0$ would prove almost every rank m orthogonal projection is injective. To obtain continuity of the inverse we construct a slightly larger set U containing Y and show that $\mu\big(S(U)\big) = 0$. We shall use the following estimate in our computations:

Theorem 5.1. The measure of the shadow of a ρ-ball B centered a distance r from the origin is bounded by $\mu\big(S(B)\big) \leq C(\rho/r)^m$.

This result is given by Santalo in [19]. An explicit computation involving only elementary techniques has recently been given by by Friz and Robinson in [10].

Theorem 5.2. Given $\mathcal{A} \subseteq \mathbf{R}^N$ such that $\mathcal{A} \subseteq B_R(0)$ and $\dim_b(\mathcal{A} - \mathcal{A}) < d < m$. Let $\eta > 1$ and define

$$f(x) = x\Big(\frac{1}{\log_2(2R/x)}\Big)^{\eta/(m-d)}$$

where $R = 2\sup\big\{\, \|a\| : a \in \mathcal{A} \,\big\}$. Then for almost every projection P of rank m there is a constant C such that (5.1) is satisfied.

Proof: Define $r_n = R/2^n$ and $\rho_n = f(r_{n-1})$. Divide Y into shells

$$Z_n = \{\, y \in Y : r_n \leq \|y\| \leq r_{n-1} \,\}$$

and cover Z_n by balls of radius ρ_n.

Let $Y = \mathcal{A} - \mathcal{A}$ and choose K such that $\mathcal{N}_{\mathcal{B}}(r,\rho) \leq K(r/\rho)^d$ for all $\mathcal{B} \subseteq Y$. Further choose $c > 2$ so large that $c^{m-d} > K$. Cover each ρ_n-ball by ρ_n/c-balls and each of those by ρ_n/c^2-balls and so on. Label the centers of the ρ_n/c^i-balls that cover Z_n by a_{nij}. Thus

$$Z_n \subseteq \textstyle\bigcup_j B_{\rho_n/c^i}(a_{nij})$$

where the index j ranges over at most $K^{i+1}c^{id}(r_{n-1}/\rho_n)^d$ balls.

Since $c > 2$, the covers formed by doubling the radius of the balls in each generation will be nested. Let

$$U_i = \bigcup\nolimits_{n,j} B_{2\rho_n/c^i}(a_{nij})$$

be the union of all the radius doubled balls in the i-th generation. For each $y \in Y$ there is a ball $B_\rho(a)$ of radius $\rho = \rho_n/c^i$ centered at a such that $y \in B_\rho(a)$. If $P \notin S(U_i)$ then $0 \notin PB_{2\rho}(a)$ and hence

$$\|Py\| \geq \rho = \rho_n/c^i = f(r_{n-1})/c^i \geq f(\|y\|)/c^i$$

shows that (5.1) holds for the i-th generation with $\epsilon = 1/c^i$.

Let $U = \limsup_i U_i$. It remains to show the measure of $S(U)$ is zero. Estimate

$$\begin{aligned}
\mu\big(S(U)\big) &\leq \mu\big(S(U_i)\big) \leq \sum_{n,j} \mu\Big(S\big(B_{2\rho_n/c^i}(a_{nij})\big)\Big) \leq \sum_{n,j} C\Big(\frac{2\rho_n}{c^i r_n}\Big)^m \\
&\leq \sum_n CK^{i+1}c^{id}\Big(\frac{r_{n-1}}{\rho_n}\Big)^d\Big(\frac{2\rho_n}{c^i r_n}\Big)^m = 4^m CK\Big(\frac{K}{c^{m-d}}\Big)^i \sum_n \Big(\frac{f(r_{n-1})}{r_{n-1}}\Big)^{m-d} \\
&= 4^m CK\Big(\frac{K}{c^{m-d}}\Big)^i \sum_n \frac{1}{n^\eta} = L\Big(\frac{K}{c^{m-d}}\Big)^i
\end{aligned}$$

where L is a constant. Letting i become large, we obtain $\mu\big(S(U)\big) = 0$. Thus, almost every orthogonal projection of rank m satisfies (5.1) for some $\epsilon > 0$. $\square$

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