Finite Fractal Dimension and Hölder-Lipshitz Parametrization

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In this paper we first extend the Hölder-Mañé Theorem [1] from a finite dimensional space to the general Hilbert space setting. Namely, if H is a real Hilbert space and $X \subseteq H$ has fractal dimension less than $m/2$, then for any orthogonal projection P of rank m and $\delta \in (0,1)$ there is an orthogonal projection $\tilde{P}$ such that $\|P - \tilde{P}\| < \delta$ and $\tilde{P}|_X$ has Hölder inverse. Moreover, for any metric space M of finite fractal dimension less than $m/2$ there exists a Lipshitz function $g\colon H \to \mathbf{R}^m$ with Hölder inverse on its image.

1. Introduction

In [4] first and then in [7] and [2], Appendix A, it was proven among others that if X is a compact subset of a real Hilbert space H with fractal dimension less than $m/2$ and if P is a projection in H of rank m then for every $\delta \in (0,1)$ there exists a projection $\tilde{P}$ of rank m such that $\|P - \tilde{P}\| < \delta$, and $\tilde{P}|_X$ is one to one. In [2], Chapter 10 it was almost explicitly shown that if the inverse $f = (\tilde{P}|_X)^{-1}\colon \tilde{P}X \to H$ is not merely continuous but a Hölder function, then the following important theoretical fact is valid: *The exponential attractors of infinite dimensional dissipative evolution equations are also the exponential attractors of some finite dimensional ordinary differential equations.* The existence of such projections $\tilde{P}$ was established in [1] and [2] but only in the case when the dimension of H is finite. Our aim is to show that this last assumption is unnecessary. Namely, we have the following:

THEOREM 1.1. *Let H be a real Hilbert space and $X \subseteq H$ be such that $d_F(X) < m/2$. Then for any orthogonal projection P of rank m and $\delta > 0$ there is an orthogonal projection $\tilde{P}$ such that $\|\tilde{P} - P\| < \delta$ and $\tilde{P}|_X$ has Hölder inverse.*

In fact, the statement of Theorem 1.1 remains valid also if all references to orthogonality are removed (Theorem 8.2). Finally, as a general application of this result we obtain

THEOREM 1.2. *Given a metric space M of fractal dimension less than $m/2$, there is a function $g\colon M \to \mathbf{R}^m$ and constants $L > 0$ and $\gamma \geq 1$ such that*

$$Ld(a,b)^\gamma \leq \big\|g(a) - g(b)\big\| \leq d(a,b) \quad \textit{for all } a, b \in M.$$

In this way any metric space of finite fractal dimension is the range of a Hölder expanding map defined on a bounded set in a finite dimensional euclidean space. Clearly one cannot replace fractal dimension by Hausdorff dimension in these theorems (see also [1]).

The proof of our main result, namely Theorem 1.1, is outlined in the next section. The details of the proof of Theorem 1.1 are presented in the rest of the paper, with the exception of the last paragraph devoted to the proof of Theorem 1.2.

2. Preliminary

Let M be a totally bounded metric space. Recall that the fractal dimension of M is defined by

$$d_F(M) = \limsup_{\rho \to 0} \frac{\log \mathcal{N}_M(\rho)}{\log(1/\rho)}$$

where $\mathcal{N}_M(\rho) = \min\big\{\, n : a_1, \ldots, a_n \in M \text{ and } M \subseteq \bigcup_{i=1}^n B_\rho(a_i) \,\big\}$ is the minimum number of open balls of radius ρ needed to cover M. If M is not totally bounded, its fractal dimension $d_F(M)$ is, by definition, infinite.

Remark 2.1. The following are simple consequences of the definition:

a. $d_F(M_1 \times M_2) \leq d_F(M_1) + d_F(M_2)$,

b. If $f: M_1 \to M_2$ is Lipshitz continuous then $d_F(M_2) \leq d_F(M_1)$.

c. If $d_F(M) < d$ then there is a constant C_M such that $\mathcal{N}_X(\rho) \leq C_M(1/\rho)^d$ holds for all $X \subseteq M$ and $\rho \in (0,1)$.

We now outline the proof of Theorem 1.1 that will be presented in the rest of this paper. Let H be a real Hilbert space, X a compact subset of H and P an orthogonal projection. For $P|_X$ to have Hölder continuous inverse $f: P(X) \to H$ means there are constants K and $\theta > 0$ so that $\|f(x_1) - f(x_2)\| \leq K\|x_1 - x_2\|^\theta$ for $x_1, x_2 \in X$. If $Y = X - X$, then this is equivalent to saying that $\|Py\| \geq C\|y\|^\eta$ for $y \in Y$ where $\eta = 1/\theta$ and $C = (1/K)^\eta$. Furthermore, since $d_F(Y) \leq 2d_F(X)$, we will focus our attention on obtaining a projection P such that $\|Py\| \geq C\|y\|^\eta$ for $y \in Y$ where the rank of P is greater than the fractal dimension of Y.

Our result in the case of orthogonal projections will be obtained by taking an increasing sequence of finite dimensional subspaces and applying a finite dimensional result on each one thereby obtaining a sequence of projections that converge to one with the desired properties. In fact, given a projection P of rank m, a set Y of finite fractal dimension less than m, and $\delta > 0$, first project H into a large finite dimensional space of dimension N by a suitably choosen orthogonal projection Q_N. We choose Q_N so that the norms of elements in Y do not get too small. Now we consider $Q_N Y$ in the finite dimensional space $Q_N H$ and use [1] ignoring P and δ to construct an orthogonal projection P_0 from $Q_N H$ onto an m dimensional subspace. Then we focus on approximating the original projection. Let V be a subspace of dimension $2m$ such that $PH \subseteq V$ and $P_0 Q_N H \subseteq V$. Now we use [2] to approximate P in V finding an orthogonal projection $\tilde{P}$ that is δ close to P. Note that, although [1] and [2] each contain complete proofs of the result in the case that H is finite dimensional, the techniques in [2] are better suited for estimating how the result depends on $\dim H$ whereas [1] is better for estimating the dependence on δ. Finally we obtain a sequence of orthogonal projections by letting $N \to \infty$ and $\delta \to 0$ in the above construction. With care this sequence can be made to converge to an orthogonal projection with the desired properties. The passage from orthogonal projections to arbitrary projections is quite standard (see Theorem 8.2).

3. From Hilbert Space to a Large Finite Dimensional Space

The first step is the construction of the projections Q_N referred to above.

LEMMA 3.1. *Given a compact subset Y of a Hilbert space H such that $\mathcal{N}_Y(\rho) \le C_Y(1/\rho)^d$ for $\rho \in (0,1)$. Then there are orthogonal projections Q_n of rank less than or equal n such that $\|y - Q_n y\| \le (C_Y/n)^{1/d}$ for all $y \in Y$.*

Proof. Let $P_0 = I$. We define a sequence of vectors and orthogonal projections inductively as follows: Given P_n let $a_{n+1} \in Y$ be a vector so that $P_n a_{n+1}$ has largest norm in $P_n Y$. Define P_{n+1} to be the orthogonal projection onto $P_n H \ominus \mathbf{R} P_n a_{n+1}$.

Note that if $\|P_n a_{n+1}\| \le \rho \le \|P_{n-1} a_n\|$, then, for $j < k \le n$, one has

$$\|a_k - a_j\| \ge \|P_{k-1} a_k - P_{k-1} a_j\| = \|P_{k-1} a_k\| \ge \|P_{n-1} a_n\| \ge \rho.$$

Thus for $A = \{ a_1, a_2, \ldots \}$ we have that $\mathcal{N}_A(\rho) \ge n$. But $\mathcal{N}_A(\rho) \le C_Y(1/\rho)^d$, since $A \subseteq Y$. It follows that $\rho \le (C_Y/n)^{1/d}$ and thus $\|P_n a_{n+1}\| \le (C_Y/n)^{1/d}$.

Let $Q_n = I - P_n$. Then Q_n is an orthogonal projection of rank less than or equal n and $y - Q_n y \in P_n Y$ implies $\|y - Q_n y\| \le \|P_n a_{n+1}\| \le (C_Y/n)^{1/d}$. $\square$

4. From a Large Finite Dimensional Space to $\mathbf{R}^m$

In this paragraph we present the finite dimension techniques introduced in [1] which we will later apply to $Q_N Y$ in the finite dimensional space $Q_N H$ for N large enough. Only part of the proof is presented here as we don't approximate the original projection in this step. Careful track is kept of how the estimates depend on the dimension N.

Notation. Let $G_{N,k}$ be the Grassmanian manifold of k dimensional subspaces in $\mathbf{R}^N$, and μ be the invariant measure on $G_{N,k}$ normalized as in [5], equation 12.35 so that

$$\mu(G_{N,k}) = \frac{O_{N-1}O_{N-2}\cdots O_{N-k}}{O_{k-1}O_{k-2}\cdots O_1 O_0}$$

where $O_{N-1} = 2\pi^{N/2}/\Gamma(N/2)$ is the surface area of the $N-1$ dimensional unit sphere. Here Γ represents Euler's Gamma function. Also, let $S(B) = \{\, x \in G_{N,k} : x \cap B \neq \emptyset \,\}$. According to [5], equation 17.52, the measure of $S(B_\rho(a))$ where $a \neq 0$ and $\rho < \|a\|$ is calculated as

$$\mu\big(S(B_\rho(a))\big) = \frac{O_{N-2}\cdots O_{N-k-1}}{O_{k-2}\cdots O_0} \int_0^{\arcsin(\rho/\|a\|)} \cos^{k-1}(s)\sin^{N-k-1}(s)\, ds$$

Remark 4.1. We obtain for $r \leq \|a\|$ and $\rho < \|a\|$ that

$$\begin{aligned}
\mu\big(S(B_\rho(a))\big) &\leq \frac{1}{N-k}\frac{O_{N-2}\cdots O_{N-k-1}}{O_{k-2}\cdots O_0}\arcsin^{N-k}(\rho/\|a\|) \\
&\leq \frac{\mu(G_{N,k})}{N-k}\frac{O_{k-1}O_{N-k-1}}{O_{N-1}}\Big(\frac{\pi\rho}{2r}\Big)^{N-k} \\
&= \frac{\mu(G_{N,k})O_{N-k-1}}{N-k}\frac{\Gamma(N/2)}{\Gamma(k/2)}\Big(\frac{\rho}{2r}\Big)^{N-k}\pi^{(N-k)/2} \\
&\leq \frac{\mu(G_{N,k})O_{N-k-1}}{N-k}\Big(\frac{N\pi}{2}\Big)^{(N-k+1)/2}\Big(\frac{\rho}{2r}\Big)^{N-k}.
\end{aligned}$$

Lemma 4.2. *Let $Y \subseteq \mathbf{R}^N$ be such that $\mathcal{N}_Y(\rho) < C_Y(1/\rho)^d$ for all $\rho \in (0,1)$. Then for $m > d$ and $r \in (0,1)$ there exists P_0 of rank m such that*

$$\|P_0 y\| \geq C\Big(\frac{r}{\sqrt{N}}\Big)^{(m+1)/(m-d)} \quad \text{for } y \in Y \text{ such that } \|y\| \geq r$$

where $C > 0$ is a constant depending only on m, d and C_Y.

Proof. Define $Z = \{\, y \in Y : \|y\| \geq r \,\}$. Consider a cover of Z by $\mathcal{N}_Z(\rho)$ balls of radius $\rho < r/2$. Let a_j where $j = 1, \ldots, \mathcal{N}_Z(\rho)$ denote the centers of these balls. Estimate the measure α of the projections whose kernels intersect $B_{2\rho}(a_j)$ for some j. Let $k = N - m$ and by Remark 4.1 estimate

$$\begin{aligned}\alpha &= \mu\Big(\bigcup S\big(B_{2\rho}(a_j)\big)\Big)\\ &\le \mathcal{N}_Z(\rho)\frac{\mu(G_{N,k})O_{N-k-1}}{N-k}\Big(\frac{N\pi}{2}\Big)^{(N-k+1)/2}\Big(\frac{\rho}{r}\Big)^{N-k}\\ &\le C_Y\frac{\mu(G_{N,k})O_{m-1}}{m}\Big(\frac{N\pi}{2}\Big)^{(m+1)/2}\Big(\frac{1}{r}\Big)^{m}\rho^{m-d}\\ &\le \frac{\mu(G_{N,k})}{2C^{m-d}}\Big(\frac{\sqrt{N}}{r}\Big)^{m+1}\rho^{m-d}\end{aligned}$$

where C is an appropriate constant depending only on m, d, and C_Y. It follows $\alpha < \mu(G_{N,k})$ for $\rho = C(r/\sqrt{N})^{(m+1)/(m-d)}$. Taking $V \in G_{N,k} \setminus \bigcup S\big(B_{2\rho}(a_j)\big)$ and the projection P_0 with kernel V yields

$$\|P_0 y\| \ge \rho \quad \text{for} \quad y \in Z$$

since any $y \in Z$ is at least ρ away from the kernel of P_0. □

5. Sharp Conditions on Close Projections

In this section we present some elementary but sharp results from operator theory on orthogonal projections which we need later. For completeness we provide their proofs.

Lemma 5.1. *Given orthogonal projections P and $\tilde{P}$ of equal and finite rank on a Hilbert space H and $\epsilon \in (0,1)$, if $\tilde{P}x = 0$ implies $\|Px\| \le \epsilon\|x\|$, then $\|P - \tilde{P}\| \le \epsilon$.*

Proof. Define $T = \tilde{P}|_{PH}\colon PH \to \tilde{P}H$. If $Tx = 0$ for some $x \in PH$ then $x = Px$ and $\tilde{P}x = 0$. Therefore $\|x\| = \|Px\| \le \epsilon\|x\|$, which is possible only if $x = 0$. Thus T is one-to-one. But $\dim TPH = \dim PH = \dim \tilde{P}H$ implies $TPH = \tilde{P}H$, and so T is also onto. Define

$$S = (I - \tilde{P})\big|_{(I-P)H}\colon (I-P)H \to (I-\tilde{P})H.$$

By [6], Chapter IX, Lemma 1.1, we have that S is one-to-one and onto and therefore so is its adjoint S^*. Furthermore,

$$\big\|S^*(I-\tilde{P})x\big\|^2 = \big\|(I-\tilde{P})x\big\|^2 - \big\|P(I-\tilde{P})x\big\|^2 \ge (1-\epsilon^2)\big\|(I-\tilde{P})x\big\|^2.$$

We can again apply [6], Chapter IX, Lemma 1.1, to obtain that $T^*: \tilde{P}H \to PH$ is one-to-one and onto with

$$\|T^*\tilde{P}x\|^2 \geq (1-\epsilon^2)\|\tilde{P}x\|^2.$$

It follows that

$$\begin{aligned}\left\|(\tilde{P}-P)x\right\|^2 &= \left\|(I-P)\tilde{P}x - P(I-\tilde{P})x\right\|^2\\ &= \left\|(I-P)\tilde{P}x\right\|^2 + \left\|P(I-\tilde{P})x\right\|^2\\ &= \left\|\tilde{P}x\right\|^2 - \left\|T^*\tilde{P}x\right\|^2 + \left\|P(I-\tilde{P})x\right\|^2\\ &\leq \epsilon^2\left\|\tilde{P}x\right\|^2 + \epsilon^2\left\|(I-\tilde{P})x\right\|^2\end{aligned}$$

thus $\|\tilde{P}-P\| \leq \epsilon$. □

Lemma 5.2. *Let P and $\tilde{P}$ be orthogonal projections of finite rank such that $\|P-\tilde{P}\| < \delta$ for some $\delta \in (0,1)$. Then there is a unitary operator U such that $\tilde{P} = UPU^{-1}$ and $\|U-I\|^2 \leq 2(1-\sqrt{1-\delta^2})$.*

Proof. First suppose $\|P-\tilde{P}\| < \delta$. Define $T = \tilde{P}|_{PH}$. Since $\delta < 1$ then $T: PH \to \tilde{P}H$ is one-to-one and onto, and thus in the polar decomposition VA of T where $A: PH \to PH$ is a positive symmetric transformation we have that $V: PH \to \tilde{P}H$ is a unitary transformation (see [3], sections 105 and 110). Let N be the rank of P. As A is positive symmetric, there must be N orthogonal eigenvectors e_i of A with corresponding eigenvalues α_i. Thus we have $Ae_i = \alpha_i e_i$ where $i = 1, \ldots, N$. Define $\tilde{e}_n = Ve_n$. Clearly $\tilde{e}_n$ is an orthogonal basis for $\tilde{P}H$. Furthermore,

$$\begin{aligned}(e_i, \tilde{e}_j) = (e_i, Ve_j) &= (\tilde{P}e_i, Ve_j) = (Te_i, Ve_j)\\ &= (VAe_i, Ve_j) = \alpha_i(Ve_i, Ve_j) = \alpha_i(e_i, e_j).\end{aligned}$$

Therefore $\|\tilde{P}e_i\| = \|Te_i\| = \|VAe_i\| = \alpha_i$ implies

$$1 = \|Pe_i\|^2 = \|\tilde{P}e_i + Pe_i - \tilde{P}e_i\|^2 = \|\tilde{P}e_i\|^2 + \|Pe_i - \tilde{P}e_i\|^2 \leq (\tilde{e}_i, e_i)^2 + \delta^2$$

so that $(e_i, \tilde{e}_i) \geq \sqrt{1-\delta^2}$.

For $i = 1, 2, \ldots, N$ define $Z_i = \mathbf{R}e_i + \mathbf{R}\tilde{e}_i$ and write $H = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_N \oplus Z$. Let U_i be the rotation in Z_i taking e_i into $\tilde{e}_i$ and define $U = U_1 \oplus U_2 \oplus \cdots \oplus U_N \oplus I|_Z$. Clearly U is unitary and $\tilde{P} = UPU^{-1}$. Furthermore, denoting by I_i the identity operator on Z_i, we have

$$\begin{aligned} \|U - I\|^2 &= \max\big\{ \|U_i - I_i\|^2 : i = 1, 2, \ldots, N \big\} \\ &= \max\big\{ \|\tilde{e}_i - e_i\|^2 : i = 1, 2, \ldots, N \big\} \\ &= \max\big\{ 2\big(1 - (\tilde{e}_i, e_i)\big) : i = 1, 2, \ldots, N \big\} \\ &\le 2(1 - \sqrt{1 - \delta^2}). \end{aligned}$$

□

Lemma 5.2 has the following strong reciprocal:

Lemma 5.3. *Let P be a finite dimensional orthogonal projection and U a unitary operator. If $\|U - I\|^2 \le 2(1 - \sqrt{1 - \delta^2})$ then $\|P - \tilde{P}\| \le \delta$ where $\tilde{P} = UPU^{-1}$.*

Proof. Let $\tilde{P}x = 0$ and $\|x\| = 1$. Define $y = Ux$. Then $\|y\| = 1$ and

$$\rho^2 = \|x - y\|^2 \le \|I - U\|^2 \le 2(1 - \sqrt{1 - \delta^2})$$

imply $(x, y) = 1 - \rho^2/2 \ge \sqrt{1 - \delta^2}$. Since $\tilde{P}x = 0$ then $\tilde{P} \le I - x^* \otimes x$ and therefore

$$\|Px\| = \|UPx\| = \|\tilde{P}y\| \le \|y - (x, y)x\| = \sqrt{1 - (x, y)^2} \le \delta.$$

Applying Lemma 5.1 gives $\|P - \tilde{P}\| \le \delta$. □

6. Finite Dimensional Hölder Continuity

We now need the one step at a time constructive proof of the finite dimensional case of Theorem 1.1 in [2], Proposition A.1. Using the results from the previous section we give a simplified presentation carefully keeping track of how the constants depend on the closeness δ of the approximating projection.

LEMMA 6.1. *Let P be a projection of rank $N-1$ in $\mathbf{R}^N$ and $A = \{\ker \tilde{P} : \|P - \tilde{P}\| < \delta\}$. Then $\mu(A) \geq C\delta^{N-1}$ where $C > 0$ is a constant that depends only on N.*

Proof. Write $P = I - h^* \otimes h$. It follows from Lemma 5.1 that

$$A = \{\ker \tilde{P} : \|\tilde{P}h\| < \delta\} = \{\mathbf{R}\tilde{h} : \|\tilde{h}\| = 1 \text{ and } \|h - \tilde{h}(\tilde{h}, h)\| < \delta\}$$
$$= \{x \in G_{N,1} : x \cap B_\delta(h) \neq \emptyset\} = S\big(B_\delta(h)\big).$$

Therefore

$$\mu(A) = O_{N-2} \int_0^{\arcsin(\delta)} \sin^{N-2}(s)\, ds \geq C\delta^{N-1}$$

where C is a constant that depends only on N. □

LEMMA 6.2. *Let $Y \subseteq B_1(0) \subseteq \mathbf{R}^N$ be such that $\mathcal{N}_Y(\rho) \leq C_Y(1/\rho)^d$ for $\rho \in (0,1)$ where $d < N-1$. There are constants $C > 0$ and $\eta \geq 1$ depending only on N, d, and C_Y such that for any projection P of rank $N-1$ and $\delta \in (0,1)$ there is a projection $\tilde{P}$ such that*

$$\|P - \tilde{P}\| < \delta \quad \textit{and} \quad \|\tilde{P}y\| \geq C\big(\delta\|y\|\big)^\eta \quad \textit{for all } y \in Y.$$

Proof. Let $\eta = 2(N-1)/(N-1-d)$. Since P is of rank $N-1$ there is h with $\|h\| = 1$ such that $P = I - h^* \otimes h$. Define $Z_n = \{y \in Y : r_n \leq \|y\| \leq r_{n-1}\}$ where $r_n = 1/2^n$. Consider a cover of Z_n by $\mathcal{N}_{Z_n}(\rho_n)$ balls of radius $\rho_n = \epsilon r_n^\eta$. Let a_j^n where $j = 1, \ldots, \mathcal{N}_{Z_n}(\rho_n)$ denote the centers of these balls.

Estimate the measure α of projections whose kernels intersect the balls $B_{2\rho_n}(a_j^n)$ in $G_{N,1}$:

$$\begin{aligned}
\alpha &= \mu\Big(\bigcup_{n,j} S\big(B_{2\rho_n}(a_j^n)\big)\Big) \\
&\leq \sum_{n=1}^{\infty} \mathcal{N}_{Z_n}(\rho_n) \frac{\mu(G_{N,1}) O_{N-2}}{N-1} \Big(\frac{N\pi}{2}\Big)^{N/2} \Big(\frac{\rho_n}{r_n}\Big)^{N-1} \\
&\leq C_Y \epsilon^{N-1-d} \frac{\mu(G_{N,1}) O_{N-2}}{N-1} \Big(\frac{N\pi}{2}\Big)^{N/2} \sum_{n=1}^{\infty} r_n^{(N-1-d)\eta - N + 1} \\
&\leq C_1 \epsilon^{N-1-d}
\end{aligned}$$

where C_1 is an appropiate constant depending only on N, d and C_Y.

From Lemma 6.1 it follows that $\alpha < \mu\big(\{\ker \tilde P : \|P - \tilde P\| < \delta\}\big)$ by setting $\epsilon = C_2\delta^{(N-1)/(N-1-d)}$ where C_2 is some suitable constant depending only on N, d and C_Y. Take

$$\tilde V \in \{\ker \tilde P : \|P - \tilde P\| < \delta\} \setminus \Big(\bigcup_{j,n} S\big(B_{2\rho_n}(a_j^n)\big)\Big).$$

and the projection $\tilde P$ with kernel $\tilde V$. If $y \in Y$ then $r_n \le \|y\| \le r_{n-1}$ for some n and $y \in B_{\rho_n}(a_j^n)$ for some j. Since $\tilde V \notin S\big(B_{2\rho_n}(a_j^n)\big)$ then y is at least ρ_n away from the kernel of $\tilde P$. Therefore

$$\|\tilde P y\| \ge \rho_n = \epsilon r_n^\eta \ge \epsilon 2^{-\eta}\|y\|^\eta = C_2 2^{-\eta}\delta^{\eta/2}\|y\|^\eta \ge C_2 2^{-\eta}\big(\delta\|y\|\big)^\eta$$

because $\delta \in (0,1)$. The desired conclusion follows with $C = C_2 2^{-\eta}$. ☐

Theorem 6.3 ([1], [2]). *Given $Y \subseteq B_1(0) \subseteq \mathbf{R}^N$ such that $\mathcal{N}_Y(\rho) \le C_Y(1/\rho)^d$ for $\rho \in (0,1)$ where $d < m \le N-1$. There are constants $C > 0$ and $\eta \ge 1$ depending only on N, m, d and C_Y such that for any projection P on $\mathbf{R}^N$ of rank m and $\delta \in (0,1)$ there is a projection $\tilde P$ of rank m such that*

$$\|P - \tilde P\| < \delta \quad \textit{and} \quad \|\tilde P y\| \ge C\big(\delta\|y\|\big)^\eta \quad \textit{for every } y \in Y.$$

Proof. If $N - 1 = m$ then this is simply Lemma 6.2, otherwise assume for induction on N that the result holds in $\mathbf{R}^{N-1}$.

Let $Q : \mathbf{R}^N \to \mathbf{R}^{N-1}$ be chosen so $P \le Q$. By Lemma 6.2 there is a $\tilde Q$ such that $\|Q - \tilde Q\| < \delta/2$ and $\|\tilde Q y\| \ge C_1\big(\delta\|y\|\big)^{\eta_1}$ for $y \in Y$. Let U be as in Lemma 5.2 so that $\tilde Q = UQU^{-1}$ and $\|U - I\|^2 \le 2(1 - \sqrt{1 - \delta^2/4})$. From Lemma 5.3 it follows that $\|P - UPU^{-1}\| < \delta/2$.

Now consider the $N-1$ dimensional space $\tilde Q\mathbf{R}^N$. By the induction hypothesis there is $\tilde P \le \tilde Q$ such that $\|\tilde P - UPU^{-1}\| < \delta/2$ and $\|\tilde P\tilde Q y\| \ge C_2\big(\delta\|\tilde Q y\|\big)^{\eta_2}$ for $y \in Y$. Thus

$$\|\tilde P - P\| \le \|\tilde P - UPU^{-1}\| + \|P - UPU^{-1}\| < \delta$$

and

$$\|\tilde{P}y\| \geq C_2\big(\delta\|\tilde{Q}y\|\big)^{\eta_2} \geq C_2\big(\delta C_1(\delta\|y\|)^{\eta_1}\big)^{\eta_2} \geq C\big(\delta\|y\|\big)^{\eta} \quad \text{for } y \in Y$$

where C and η depend only on N, m, d, and C_Y. $\square$

7. Approximation of P in the $2m$-Dimensional Space

In this section the previous results are put together in the way outlined in section 2 to obtain an orthogonal projection $\tilde{P}$ on H that is close to the original projection and behaves reasonably well on Y.

Lemma 7.1. *Let H be a Hilbert space and Y be a compact subset of $B_1(0)$ such that $d_F(Y) < m$. Then there exists constants $\eta \geq 1$, $\epsilon > 0$ and $C > 0$ such that for any orthogonal projection P of rank m and $r \in (0, \epsilon)$ there is an orthogonal projection $\tilde{P}$ with*

$$\|\tilde{P} - P\| < r \quad \text{and} \quad \|\tilde{P}y\| \geq Cr^{\eta} \quad \text{for } y \in Y \text{ such that } \|y\| \geq r.$$

Proof. Choose d so $d_F(Y) < d < m$, let C_Y be as in Remark 2.1c, and take $\epsilon = C_Y^{1/d}$. Thus there is $N \geq 2$ so that

$$\Big(\frac{C_Y}{N}\Big)^{1/d} \leq \frac{r}{2} < \Big(\frac{C_Y}{N-1}\Big)^{1/d}.$$

Defining $Q = Q_N$ from Lemma 3.1 yields

$$\|Qy\| \geq \|y\| - (C_Y/N)^{1/d} \geq r/2 \quad \text{for } y \in Y \text{ such that } \|y\| \geq r.$$

Since $C_{QY} \leq C_Y$ one may use Lemma 4.2 to find an orthogonal projection P_0 of rank m in QH such that

$$\|P_0Qy\| \geq C_1\Big(\frac{r/2}{\sqrt{N}}\Big)^{(m+1)/(m-d)} \quad \text{for } y \in Y \text{ such that } \|Qy\| \geq r/2$$

where C_1 depends only on m, d and C_Y. Since $N \geq 2$ we have

$$\frac{1}{N} > \frac{1}{2C_Y}\Big(\frac{r}{2}\Big)^d$$

and therefore also

$$\|P_0Qy\| \geq C_2 r^{(1+d/2)(m+1)/(m-d)} \quad \text{for } y \in Y \text{ such that } \|y\| \geq r$$

where C_2 depends only on m, d and C_Y.

Choose a subspace $V \subseteq H$ of dimension $2m$ such that $P_0QH \subseteq V$ and $PH \subseteq V$ and let $\tilde{Q}$ be the orthogonal projection from H onto V. By Theorem 6.3 applied to $\tilde{Q}Y$ in V there is an orthogonal projection $\tilde{P}_1$ in V such that

$$\|\tilde{P}_1 - P\| < r \quad \text{and} \quad \|\tilde{P}_1\tilde{Q}y\| \geq C_3\big(r\|\tilde{Q}y\|\big)^{\eta_1} \quad \text{for all } y \in Y$$

where C_3 and $\eta_1 \geq 1$ are constants depending only on m, d and C_Y.

Defining $\tilde{P} = \tilde{P}_1\tilde{Q}$, we obtain

$$\|\tilde{P}y\| \geq C_3\big(r\|P_0Qy\|\big)^{\eta_1} \geq C_3\big(rC_2r^{(1+d/2)(m+1)/(m-d)}\big)^{\eta_1} = Cr^{\eta}$$

for $y \in Y$ such that $\|y\| \geq r$ with $\eta = \big(1 + (1+d/2)(m+1)/(m-d)\big)\eta_1$ and C depending only on $C_2^{\eta_1}$ and C_3, that is, only on m, d, and C_Y. □

8. The Hölder-Mañé Projection in Hilbert Space

In this section we conclude the proof of Theorem 1.1 by creating a Cauchy sequence of orthogonal projections using the previous theorem and letting the distance between successive projections decrease so quickly that a projection with Hölder inverse is obtained in the limit.

Lemma 8.1. *Given a set $Y \subseteq B_1(0)$ suppose there are constants $\eta \geq 1$, $\epsilon > 0$ and $C > 0$ such that for every orthogonal projection P of rank m and $r \in (0, \epsilon)$ there is an orthogonal projection $\tilde{P}$ satisfying*

$$\|\tilde{P} - P\| < r \quad \text{and} \quad \|\tilde{P}y\| \geq Cr^{\eta} \quad \text{for} \quad y \in Y \text{ such that } \|y\| > r.$$

Then for every $\gamma > \eta + 1$ and $\delta \in (0,1)$ there is constant K such that for every orthogonal projection P of rank m there is an orthogonal projection $\tilde{P}$ of rank m satisfying

$$\|\tilde{P} - P\| < \delta \quad \text{and} \quad \|\tilde{P}y\| \geq K\|y\|^{\gamma^2} \quad \text{for} \quad y \in Y.$$

Proof. Let $\alpha = \min(\delta, \epsilon)$. Choose $\rho < 1/2$ so that $K = C(\alpha\rho)^{\eta} - 2\alpha\rho^{\gamma} > 0$. Define $r_n = \alpha\rho^{\gamma^n}$ for $n = 0, 1, 2, \ldots$ and choose a sequence of orthogonal projections so that

$$\|P_n - P_{n-1}\| < r_n \quad \text{and} \quad \|P_n y\| \geq C r_n^{\eta} \quad \text{for} \quad \|y\| \geq r_n.$$

for $n = 0, 1, 2, \ldots$ where $P_{-1} = P$. Since $r_{n+1} \leq r_n/2$ then $\sum_{k=n+1}^{\infty} r_k \leq 2r_{n+1}$ implies P_n is a Cauchy sequence of projections. Let $\tilde{P} = \lim P_n$. Clearly $\|\tilde{P} - P\| < \sum_{n=0}^{\infty} r_n \leq 2r_0 = 2\alpha\rho < \delta$. Fix $y \in Y$ with $y \neq 0$. If $r_0 \leq \|y\| \leq 1$ then

$$\begin{aligned} \|\tilde{P}y\| &\geq \|P_0 y\| - \|\tilde{P} - P_0\|\|y\| \geq C r_0^{\eta} - 2r_1 \\ &= C(\alpha\rho)^{\eta} - 2\alpha\rho^{\gamma} = K \geq K\|y\|^{\gamma^2}. \end{aligned}$$

Otherwise there is n so that $r_n \leq \|y\| \leq r_{n-1}$ and

$$\begin{aligned} \|\tilde{P}y\| &\geq \|P_n y\| - \|\tilde{P} - P_n\|\|y\| \geq C r_n^{\eta} - 2r_{n+1} r_{n-1} \\ &= C(\alpha\rho^{\gamma^n})^{\eta} - 2\alpha\rho^{\gamma^{n+1}} \alpha\rho^{\gamma^{n-1}} \geq (C\alpha^{\eta} - 2\alpha^2 \rho^{\gamma^n(\gamma-\eta)})\rho^{\gamma^n \eta} \\ &\geq \rho^{-\eta} K \rho^{\gamma^n \eta} \geq K\rho^{\gamma^n \eta} \geq K r_{n-1}^{\gamma\eta} \geq K\|y\|^{\gamma^2}. \end{aligned}$$

□

We are now able to conclude with the proof of Theorem 1.1.

Proof of Theorem 1.1. Since X must be totally bounded, its closure $\bar{X}$ in H is compact. It is easy to see that $d_F(\bar{X}) = d_F(X)$ and therefore writing $Y = \bar{X} - \bar{X}$ yields $d_F(Y) \leq 2d_F(X) < m$ and Y compact. Clearly Theorem 1.1 is now a consequence of Lemma 7.1 and Lemma 8.1. □

Theorem 1.1 is still true for general projections, that is, operators P on the Hilbert space satisfying only $P^2 = P$.

THEOREM 8.2. *Let H be a Hilbert space and $X \subseteq H$ be such that $d_F(X) < m/2$. Then for any projection P of rank m and $\delta > 0$ there is a projection $\tilde{P}$ such that $\|\tilde{P} - P\| < \delta$ and $\tilde{P}|_X$ has Hölder inverse.*

Proof. Let P_0 be an orthogonal projection such that $P_0H = PH$ and let U be a unitary operator from $(I-P_0)H$ onto $(I-P)H$. Define $V = P_0+U(I-P_0)$. Then

$$VP = \big(P_0 + U(I-P_0)\big)P = P_0P = P$$

and

$$VU^{-1}(I-P) = \big(P_0 + U(I-P_0)\big)U^{-1}(I-P) = I - P$$

imply $V^{-1} = P + U^{-1}(I-P)$ and thus

$$VP_0V^{-1} = VP_0\big(P + U^{-1}(I-P)\big) = VP_0P = VP = P.$$

By Theorem 1.1 applied to $V^{-1}X$ and P_0 there is an orthogonal projection $\tilde{P}_0$ and constants K and η such that

$$\|\tilde{P}_0 - P_0\| \le \delta/\big(\|V\|\|V^{-1}\|\big) \quad \text{and} \quad \|\tilde{P}_0V^{-1}y\| \ge K\|V^{-1}y\|^\eta \quad \text{for} \quad y \in Y$$

where $Y = \bar{X} - \bar{X}$. Define $\tilde{P} = V\tilde{P}_0V^{-1}$. It follows that

$$\|\tilde{P}y\| = \|V\tilde{P}_0V^{-1}y\| \ge \frac{\|\tilde{P}_0V^{-1}y\|}{\|V^{-1}\|} \ge \frac{K\|V^{-1}y\|^\eta}{\|V^{-1}\|} \ge \frac{K\|y\|^\eta}{\|V^{-1}\|\|V\|^\eta}$$

and $\|\tilde{P} - P\| \le \big\|V(\tilde{P}_0 - P_0)V^{-1}\big\| \le \|V\|\|\tilde{P}_0 - P_0\|\|V^{-1}\| \le \delta.$ □

9. Consequences

The Hölder-Lipshitz parametrization of a compact metric space M of finite fractal dimension presented in Theorem 1.2 is a natural application of Theorem 1.1. For the proof of this theorem we need the following Hölder-Lipshitz embedding of M into a Hilbert space:

LEMMA 9.1. *Given a metric space M of finite fractal dimension and a Hilbert space H, there is function $f: M \to H$ and constants $K_1 > 0$, $K_2 > 0$, and $\eta \ge 1$ such that*

$$K_2 d(a,b)^\eta \le \big\|f(a) - f(b)\big\| \le K_1 d(a,b) \quad \textit{for all } a, b \in M.$$

Proof. Fix $d > \max\bigl(1, d_F(M)\bigr)$. By Remark 2.1c there is C_M so that $\mathcal{N}_M(\rho) \le C_M(1/\rho)^d$. For each n cover M with $N_n = \mathcal{N}_M(\rho_n)$ balls of radius $\rho_n = 1/2^n$ and denote the centers of the balls by a_j^n where $j = 1, \ldots, N_n$. Let $I = \bigcup_{n=1}^{\infty} \{\, (n, j) : j = 1, \ldots, N_n \,\}$ and choose a bijection $\sigma\colon I \to \mathbf{N}$. Define $f\colon M \to H$ by

$$f(a) = \sum_{n=1}^{\infty} \sum_{j=1}^{N_n} e_{\sigma(n,j)} \rho_n^d d(a, a_j^n)$$

where $\{\, e_k : k \in \mathbf{N} \,\}$ is an orthogonal basis for H. The existence of the constant K_1 is given by the following computation:

$$\begin{aligned}
\bigl\| f(a) - f(b) \bigr\|^2 &= \sum_{n=1}^{\infty} \sum_{j=1}^{N_n} \rho_n^{2d} \bigl| d(a, a_j^n) - d(b, a_j^n) \bigr|^2 \\
&\le \sum_{n=1}^{\infty} C_M (1/\rho_n)^d \rho_n^{2d} \bigl| d(a, b) \bigr|^2 \\
&\le C_M \bigl| d(a, b) \bigr|^2 \sum_{n=1}^{\infty} \rho_n^d \\
&= K_1^2 d(a, b)^2 \quad \text{for all } a, b \in M
\end{aligned}$$

with $K_1 = \bigl(C_M \sum_{n=1}^{\infty} \rho_n^d\bigr)^{1/2}$. To obtain the constants K_2 and η observe that $\bigl\| f(a) - f(b) \bigr\|^2 \ge \rho_n^{2d} \bigl| d(a, a_j^n) - d(b, a_j^n) \bigr|^2$ and therefore $d(a, a_j^n) \le \epsilon/\rho_n^d + d(b, a_j^n)$ where $\epsilon = \bigl\| f(a) - f(b) \bigr\|$. Choosing j so $d(b, a_j^n) < \rho_n$ yields

$$d(a, b) \le d(a, a_j^n) + d(b, a_j^n) \le \epsilon/\rho_n^d + 2\rho_n.$$

Since M has finite fractal dimension, it is bounded. Let D be the diameter of M. Thus $\epsilon \le DK_1$. Choose n so that $4^{d(n-1)} \le DK_1/\epsilon < 4^{dn}$. Then

$$d(a, b) \le 2^d \bigl(DK_1 \epsilon\bigr)^{1/2} + 2\bigl(\epsilon/(DK_1)\bigr)^{1/2d}$$

and thus

$$K_2 d(a, b)^{\eta} \le \| f(a) - f(b) \| \quad \text{for all } a, b \in M$$

provided that $\eta = 2d$ and $K_2 = DK_1 (2^d DK_1 + 2)^{-2d}$. $\square$

With the above result in hand we are now able to conclude with a proof of Theorem 1.2.

Proof of Theorem 1.2. Let f be the function given by Lemma 9.1 and P be an arbitrary orthogonal projection of rank m. Since f is Lipshitz, Remark 2.1b gives that $d_F(X) < m/2$ where $X = f(M)$. Apply Theorem 1.1 to X and P to obtain an orthogonal projection $\tilde{P}$ and constants C and γ_1 such that $\|\tilde{P}x - \tilde{P}y\| \geq C\|x-y\|^{\gamma_1}$ for $x, y \in X$. Define $g(a) = K_1^{-1}\tilde{P}f(a)$. Thus for all $a, b \in M$ we have

$$\big\|g(a) - g(b)\big\| \leq \frac{1}{K_1}\big\|f(a) - f(b)\big\| \leq d(a,b)$$

and

$$\big\|g(a) - g(b)\big\| \geq \frac{C}{K_1}\big\|f(a) - f(b)\big\|^{\gamma_1} \geq \frac{C}{K_1}K_2^{\gamma_1} d(a,b)^{\gamma_1\eta}.$$

To conclude we let $L = CK_2^{\gamma_1}/K_1$, and $\gamma = \gamma_1\eta$. □

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