ON THE SOLUTIONS TO THE NORMAL FORM OF THE NAVIER–STOKES EQUATIONS

CIPRIAN FOIAS[1,2], LUAN HOANG[2], ERIC OLSON[3] & MOHAMMED ZIANE[4]

Abstract. We introduce a construction of regular solutions to the Navier–Stokes equations which is specifically designed for the study of their asymptotic expansions. Using this construction, we give sufficient conditions for the convergence of those expansions. We also construct suitable normed spaces in which they converge. Moreover, in these spaces, the normal form of the Navier–Stokes equations associated with the terms of the asymptotic expansions [9] is a well-behaved infinite system of differential equations.

1. Introduction

The Navier–Stokes equations describe the dynamics of the incompressible, viscous fluid flows. It is a long standing challenge to use the mathematical theory of the Navier–Stokes equations in order to explain phenomena in fluid mechanics. However that mathematical theory is still incomplete. For example, the basic question about the existence of regular solutions for large times in the three-dimensional case is not yet answered. Even in cases when the regular solutions exist for all time, their dynamics is not well understood. One of the main difficulties in studying the three dimensional Navier–Stokes equations is the analysis of the role of the nonlinear terms in the equations. It is therefore appropriate to consider the simplest case when that role is minimal. One such case occurs when the solutions are periodic in the space variables and the body forces are potential. In this case the velocity filed $\mathbf{u}(\mathbf{x}, t), \mathbf{x} \in \mathbb{R}^3, t \in \mathbb{R}$, satisfies both the periodicity condition

$$\mathbf{u}(\mathbf{x} + L\mathbf{e}_j, t) = \mathbf{u}(\mathbf{x}, t), \quad j = 1, 2, 3,$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$, $L > 0$ is the spatial period; and the Navier–Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} = -\nabla p - \nabla P,$$

$$\nabla \cdot \mathbf{u} = 0,$$

where ν is the kinematic viscosity, p is the pressure and $(-\nabla P)$ is a potential body force. Thanks to the Galileian invariance of the Navier–Stokes equations we can consider only the solutions satisfying the following zero space average condition

$$\int_\Omega \mathbf{u}(\mathbf{x}, t)d\mathbf{x} = 0,$$

where $\Omega = (-L/2, L/2)^3$ and $d\mathbf{x} = dx_1dx_2dx_3$ is the usual volume element in $\mathbb{R}^3$.

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As shown in [8, 9], under these circumstances, the regular solutions $u(t) = \mathbf{u}(\cdot, t)$ possess an asymptotic expansion

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-4\pi^2 \nu n t/L^2}, \text{ for } t \to \infty,$$

where $q_n(t)$ are polynomials in t valued in the space of divergence-free vector valued trigonometric polynomials. This expansion means that for each $N = 1, 2, 3, \ldots$,

$$\frac{1}{\nu L^{1/2}} \| u(t) - \sum_{n=1}^{N} q_n(t) e^{-4\pi^2 \nu n t/L^2} \|_{L^2(\Omega)^3} = O(e^{-4\pi^2 \nu (N+\varepsilon) t/L^2}), \quad t \to \infty,$$

for some $\varepsilon > 0$ depending on N. A normal form of the Navier–Stokes equations is obtained in an explicit way from the map $u(0) \to (q_n(0))_{n=1}^{\infty}$. The associated normalization map W is defined by $W(u(0)) = (R_n q_n(0))_{n=1}^{\infty}$, where R_n, $n = 1, 2, 3, \ldots$, are specified projections (see the subsection 2.2).

The following three related questions are still open:

1. When does the asymptotic expansion actually converge?
2. In what natural normed spaces does our normal form of the Navier–Stokes equations constitute a well-behaved infinite-dimensional system of ordinary differential equations?
3. What is the range of the normalization map?

This paper is devoted to the study of those questions to which we give some partial answers. For instance, we show that if the asymptotic expansion of a regular solution $u(t)$ is absolutely convergent in the Sobolev space H^1 at time $t = 0$ then it converges in H^1 to $u(t)$ for all time t large enough (see Theorem 5.3 and Proposition 5.4; see also Corollaries 5.11 and 5.12 relevant to Question 3); also we give examples of normed spaces in which the answer to Question 2 is positive (see Section 7). Although the questions 1, 2 and 3 are not yet settled, the asymptotic expansions of regular solutions of the Navier–Stokes equations were recently used as an essential tool in the study of the asymptotic behavior of the helicity [4] as well as in the statistical analysis of a decaying flow [5]. The analytic method developed in [4] also turns out to be useful in the study of the asymptotic expansions. Moreover, the method can be applied to a new construction of the regular solutions to the Navier–Stokes equations that gives additional insight into the algebraic role played by the nonlinear terms in those equations. This construction is motivated by the system of differential equations satisfied by the terms $u_n(t) = q_n(t) e^{-4\pi^2 \nu n t/L^2}$ in the above asymptotic expansion and is first shown to yield the classical existence results for the regular solutions to the Navier–Stokes equations. Furthermore, the construction is well adapted to the study of the asymptotic expansions. With some new *a priori* estimates we show that the solutions obtained in this way are global solutions in suitable normed spaces to an extended system of the Navier–Stokes equations. These spaces turn out to provide a suitable setting for the study of the normalization map and the normal form and the extended Navier–Stokes equations. In this case the formal inverse of the normalization map is Lipschitz continuous in some of those normed spaces.

At present, our estimates for the terms in the asymptotic expansions are not sharp. Thus, there is still room for improvements in our method. With more refined estimates it may be possible to establish classical convergence of the asymptotic expansions for small values of the normalization map. Moreover, we still do not know for which general initial data the corresponding normalization map will belong to those large normed spaces mentioned above. Also, the relation between the global solutions to the extended Navier–Stokes equations (see Section 4) and Leray's classical weak solutions needs to be clarified.

This article is organized as follows.

In Section 2 we recall the functional setting of the Navier–Stokes equations and some known results about their regular solutions, the asymptotic expansions, the normalization map and the resulting normal form.

The construction of the regular solutions is described in Section 3. We also establish basic estimates for the summands $u_n(t)$ in the formal sum $u(t) = \sum_{n=1}^{\infty} u_n(t)$. We then apply those estimates to prove the existence theorems for the Navier–Stokes equations using this scheme.

In Section 4 we define the extended Navier–Stokes equations with a suitable normed space in which the system has global solutions. This suggests the natural normed spaces for our study of the normalization map and normal form in subsequent sections.

The connection between the extended Navier–Stokes equations and the asymptotic expansions of the regular solutions is studied in Section 5 using the key estimates obtained in Section 3 and some complex analysis techniques. The convergence of the expansions to the regular solutions is established for large times, under some specific assumptions. Moreover, under some small initial value condition, this convergence holds for all time $t \geq 0$.

In Section 6 the asymptotic expansions are estimated in terms of the normalization map's components in some of the normed spaces introduced in Section 4.

In Section 7 we construct a normed space, based on the estimates in Section 6, with respect to which the extended Navier–Stokes equations have unique global solutions, the normal form of the Navier–Stokes equations are well-posed infinite dimensional differential equations and the formal inverse of the normalization map is continuous.

In Appendices A and B some technical inequalities used in the other sections are proved in detail.

2. Preliminaries

2.1. Mathematical setting. The initial value problem for the Navier–Stokes equations in the three-dimensional space $\mathbb{R}^3$ with a potential body force is

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} = -\nabla p - \nabla P, \\ \text{div } \mathbf{u} = 0, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}), \end{cases} \tag{2.1}$$

where $\nu > 0$ is the kinematic viscosity, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is the unknown velocity field, p is the unknown pressure, $(-\nabla P)$ is the body force specified by a given function P and $\mathbf{u}^0(\mathbf{x})$ is the known initial velocity field. We consider only solutions $\mathbf{u}(\mathbf{x}, t)$ such that for any $t \geq 0$, $\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x}, t)$ satisfies

$$\mathbf{u}(\mathbf{x} + L\mathbf{e}_j) = \mathbf{u}(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^3, \quad j = 1, 2, 3, \tag{2.2}$$

and

$$\int_\Omega \mathbf{u}(\mathbf{x}) d\mathbf{x} = 0. \tag{2.3}$$

where $L > 0$ is fixed and $\Omega = (-L/2, L/2)^3$. We call the functions satisfying (2.2) L-periodic functions. Throughout this paper we take $L = 2\pi$ and $\nu = 1$. The general case is easily recovered by a change of scale.

Let $\mathcal{V}$ be the set of all L-periodic trigonometric polynomials on Ω with values in $\mathbb{R}^3$ which are divergence-free as well as satisfy the condition (2.3). We define

$$\begin{cases} H = \text{ closure of } \mathcal{V} \text{ in } L^2(\Omega)^3 = H^0(\Omega)^3, \\ V = \text{ closure of } \mathcal{V} \text{ in } H^1(\Omega)^3, \end{cases}$$

where $H^l(\Omega)$ with $l = 0, 1, 2, \dots$ denotes the Sobolev space of functions $\varphi \in L^2(\Omega)$ such that for every multi-index α with $|\alpha| \le l$ the distributional derivative $D^\alpha \varphi \in L^2(\Omega)$.

For $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$, define $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ and $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. Let $\langle \cdot, \cdot \rangle$ and $| \cdot |$ denote the scalar product and norm in $L^2(\Omega)^3$ given by

$$\langle u, v \rangle = \int_\Omega \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) dx, \quad |u| = \langle u, u \rangle^{1/2}, \quad u = \mathbf{u}(\cdot), v = \mathbf{v}(\cdot) \in L^2(\Omega)^3.$$

Note that we use $| \cdot |$ for the length of vectors in $\mathbb{R}^3$ as well as the L^2-norm of vector fields in $L^2(\Omega)^3$. In each case the context clarifies the precise meaning of this notation.

Let P_L denote the orthogonal projection in $L^2(\Omega)^3$ onto H. On V we consider the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and the norm $\|\cdot\|$ defined by

$$\langle\!\langle u, v \rangle\!\rangle = \sum_{j,k=1}^{3} \int_\Omega \frac{\partial u_j(\mathbf{x})}{\partial x_k} \frac{\partial v_j(\mathbf{x})}{\partial x_k} dx \quad \text{and} \quad \|u\| = \langle\!\langle u, u \rangle\!\rangle^{1/2},$$

for $u = \mathbf{u}(\cdot) = (u_1, u_2, u_3)$ and $v = \mathbf{v}(\cdot) = (v_1, v_2, v_3)$ in V.

Define the Stokes operator A with domain $\mathcal{D}_A = V \cap H^2(\Omega)^3$ by

$$Au = -\Delta u \quad \text{for all} \quad u \in \mathcal{D}_A. \tag{2.4}$$

The inner product of $u, v \in \mathcal{D}_A$ and the norm of $w \in \mathcal{D}_A$ are defined by $\langle Au, Av \rangle$ and $|Aw|$, respectively. Note for $w \in \mathcal{D}_A$ that (2.3) implies the norm $|Aw|$ is equivalent to the usual Sobolev norm of $H^2(\Omega)^3$. We also define the bilinear mapping associated with the nonlinear term in the Navier–Stokes equations by

$$B(u, v) = P_L(u \cdot \nabla v) \quad \text{for all} \quad u, v \in \mathcal{D}_A. \tag{2.5}$$

A classical result tracing back to Leray's pioneering works on the Navier–Stokes equations in the 1930's (see, e.g., [13, 14, 15]) is that for any initial data $\mathbf{u}^0(\mathbf{x})$ in H there exists a weak solution $\mathbf{u}(\mathbf{x}, t)$ defined for all $\mathbf{x} \in \mathbb{R}^3$ and $t > 0$ which eventually becomes analytic in space and time and $\|\mathbf{u}(\cdot, t)\|_{H^1(\Omega)^3}$ converges exponentially to zero as $t \to \infty$ (see also [3, 11, 6]). Thus there is $t_0 \ge 0$ such that the solution $u(t) = \mathbf{u}(\cdot, t)$ is continuous from $[t_0, \infty)$ into V and satisfies the following functional form of the Navier–Stokes equations

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = 0, \quad t > t_0,$$

where the equation holds in H. We say that $u(t), t \ge t_0$, is a regular solution to the Navier–Stokes equations on $[t_0, \infty)$. We denote $\mathcal{R}$ the set of all initial value $u^0 \in V$

such that there is a (unique) solution $u(t), t > 0$, satisfying

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = 0, \quad t > 0, \\ u(0) = u^0 \in V, \end{cases} \tag{2.6}$$

where the equation holds in H, and $u(t)$ is continuous from $[0, \infty)$ into V. In other words, $\mathcal{R}$ is the set of all initial data $u^0 \in V$ such that the solution $u(t)$ of the Navier–Stokes equations (2.6) is regular (hence also unique) on $[0, \infty)$.

We recall that the spectrum $\sigma(A)$ of the Stokes operator A consists of the eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ of the form $\lambda_j = |\mathbf{k}|^2$ for some $\mathbf{k} \in \mathbb{Z}^3 \setminus \{0\}, j = 1, 2, 3, \ldots$ Note that $\lambda_1 = 1 = |\mathbf{e}_1|^2$ and hence the additive semigroup generated by $\sigma(A)$ coincides with the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of all natural numbers. For $n \in \mathbb{N}$ we denote by R_n the orthogonal projection of H onto the eigenspace of A associated to n. Thus,

$$R_n H = \{ u \in H : Au = nu \}.$$

If n is an eigenvalue of A, then $R_n H$ is generated by functions of the form

$$(a^1_{\mathbf{k}} + i a^2_{\mathbf{k}}) e^{i(\mathbf{k}\cdot\mathbf{x})} + (a^1_{\mathbf{k}} - i a^2_{\mathbf{k}}) e^{-i(\mathbf{k}\cdot\mathbf{x})}, \quad \mathbf{k} \in \mathbb{Z}^3, \quad |\mathbf{k}|^2 = n,$$

where

$$a^1_{\mathbf{k}}, a^2_{\mathbf{k}} \in \mathbb{R}^3 \quad \text{and} \quad a^1_{\mathbf{k}} \cdot \mathbf{k} = a^2_{\mathbf{k}} \cdot \mathbf{k} = 0.$$

Otherwise $R_n = 0$. For example, $R_7 = 0, R_{15} = 0, R_{23} = 0, \ldots$. Define

$$P_n = R_1 + R_2 + \cdots + R_n \qquad \text{and} \qquad Q_n = I - P_n. \tag{2.7}$$

2.2. **The asymptotic behavior of solutions.** Let us recall some known results on the asymptotic expansions and the normal form of the regular solutions to the Navier–Stokes equations (see [7, 8, 9, 12] for more details). First, for any $u^0 \in \mathcal{R}$ there is an eigenvalue n_0 of A such that

$$\lim_{t\to\infty} \frac{\|u(t)\|^2}{|u(t)|^2} = n_0 \quad \text{and} \quad \lim_{t\to\infty} u(t) e^{n_0 t} = w_{n_0}(u^0) \in R_{n_0} H \setminus \{0\}. \tag{2.8}$$

Furthermore, $u(t)$ has the asymptotic expansion

$$u(t) \sim q_1(t) e^{-t} + q_2(t) e^{-2t} + q_3(t) e^{-3t} + \cdots, \tag{2.9}$$

where $q_j(t)$, also denoted by $W_j(t, u^0)$, is a polynomial in t of degree at most $j - 1$ with values trigonometric polynomials in H. This means that for any $N \in \mathbb{N}$ the correction term $v_N(t) = u(t) - \sum_{j=1}^N q_j(t) e^{-jt}$ satisfies

$$|v_N(t)| = O\big(e^{-(N+\varepsilon)t}\big) \quad \text{as} \quad t \to \infty \quad \text{for some} \quad \varepsilon = \varepsilon_N > 0.$$

In fact, $v_N(t)$ belongs to $C^1\big([0, \infty), V\big) \cap C^\infty\big((0, \infty), C^\infty(\mathbb{R}^3)\big)$, and for each $m \in \mathbb{N}$

$$\|v_N(t)\|_{H^m(\Omega)} = O\big(e^{-(N+\varepsilon)t}\big) \quad \text{as} \quad t \to \infty \quad \text{for some} \quad \varepsilon = \varepsilon_{N,m} > 0.$$

Define the normalization map W by $W(u^0) = W_1(u^0) \oplus W_2(u^0) \oplus \cdots$, where $W_j(u^0) = R_j q_j(0)$ for $j \in \mathbb{N}$. Then W is an one-to-one analytic mapping from $\mathcal{R}$ to the Frechet space $S_A = R_1 H \oplus R_2 H \oplus \cdots$ endowed with the component-wise topology. Also, $W'(0) = Id$, meaning

$$W'(0) u^0 = R_1 u^0 \oplus R_2 u^0 \oplus R_3 u^0 \oplus \cdots. \tag{2.10}$$

Therefore, if Π_N denotes the canonical projection of S_A onto $R_1 H \oplus \cdots \oplus R_N H$ then

$$\{ \Pi_N W(u^0) : \|u^0\| < \rho \} \text{ is a neighborhood of 0 in } \Pi_N S_A$$

for all $\rho > 0$ and $N \in \mathbb{N}$.

The case (2.8) holds if and only if $W_1(u^0) = W_2(u^0) = \cdots = W_{n_0-1}(u^0) = 0$ and $W_{n_0}(u^0) \neq 0$. In this case

$$q_1 = q_2 = \cdots = q_{n_0-1} = 0 \quad \text{and} \quad q_{n_0} = w_{n_0}(u^0) = W_{n_0}(u^0).$$

If $u^0 \in \mathcal{R}$ and $W(u^0) = (\xi_1, \xi_2, \ldots)$, then the polynomials q_j are the unique polynomial solutions to the following equations

$$q_j'(t) + (A - j)q_j(t) + \beta_j(t) = 0, \quad t \in \mathbb{R}, \tag{2.11}$$

with

$$R_j q_j(0) = \xi_j, \tag{2.12}$$

where the terms $\beta_j(t)$ are defined by

$$\beta_1(t) = 0 \quad \text{and} \quad \beta_j(t) = \sum_{k+l=j} B(q_k(t), q_l(t)) \quad \text{for} \quad j > 1. \tag{2.13}$$

Given $\xi = (\xi_n)_{n=1}^\infty \in S_A$, the polynomials $q_j(t) = q_j(t, \xi)$ are explicitly given by the recursive formula

(2.14)

$$q_j(t, \xi) = \xi_j - \int_0^t R_j \beta_j(\tau) d\tau + \sum_{n \geq 0} (-1)^{n+1} [(A - j)(I - R_j)]^{-n-1} \frac{d^n}{dt^n} (I - R_j)\beta_j,$$

for $j \in \mathbb{N}$. Here $[(A - j)(I - R_j)]^{-n-1}$ is defined by

$$[(A - j)(I - R_j)]^{-n-1} u = \sum_{|\mathbf{k}|^2 \neq j} \frac{a_{\mathbf{k}}}{(|\mathbf{k}|^2 - j)^{n+1}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

for $u = \sum_{|\mathbf{k}|^2 \neq j} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \in \mathcal{V}$. Above I denotes the identity map on H.

Finally, the S_A-valued function $\xi(t) = (\xi_n(t))_{n=1}^\infty = \big(W_n(u(t))\big)_{n=1}^\infty = W(u(t))$ satisfies the following system of differential equations

$$\begin{cases} \dfrac{d\xi_1(t)}{dt} + A\xi_1(t) = 0, \\[2ex] \dfrac{d\xi_n(t)}{dt} + A\xi_n(t) + \displaystyle\sum_{k+j=n} R_n B\big(q_k(0, \xi(t)), q_j(0, \xi(t))\big) = 0, \ n > 1. \end{cases} \tag{2.15}$$

This system is the normal form of the Navier–Stokes equations (2.6) associated with the asymptotic expansion (2.9). It is easily to check that the solution of (2.15) with initial data $\xi^0 = (\xi_n^0)_{n=1}^\infty \in S_A$ is precisely $\big(R_n q_n(t, \xi^0) e^{-nt}\big)_{n=1}^\infty$. Thus, formula (2.14) yields the normal form and its solutions.

2.3. **Complexification of the Navier–Stokes equations.** We introduce the Navier–Stokes equations with complex times and their analytic solutions (see [3, 10]). Let X be a real Hilbert space with scalar product $(\cdot, \cdot)_X$. Define the complexification of X as

$$X_{\mathbb{C}} = \{\, u + iv : u, v \in X \,\}$$

with the addition and scalar product defined by

$$(u_1 + iu_2) + (v_1 + iv_2) = (u_1 + v_1) + i(u_2 + v_2)$$

and

$$(\zeta_1 + i\zeta_2)(u_1 + iu_2) = \zeta_1 u_1 - \zeta_2 u_2 + i(\zeta_2 u_1 + \zeta_1 u_2)$$

for $u_1, u_2, v_1, v_2 \in X$ and $\zeta_1, \zeta_2 \in \mathbb{R}$. The complexified space $X_{\mathbb{C}}$ is a Hilbert space with respect to the inner product

$$(u + iv, u' + iv')_{X_{\mathbb{C}}} = (u, u')_X + (v, v')_X + i\big[(v, u')_X - (u, v')_X\big],$$

where $u, v, u', v' \in X$. When $X = H$ or $X = V$, we obtain the complexified spaces $H_{\mathbb{C}}$, $V_{\mathbb{C}}$ and their corresponding inner products and norms. For the sake of simplicity, we use the same notations $|\cdot|$ and $\|\cdot\|$ to denote $|\cdot|_{H_{\mathbb{C}}}$ and $\|\cdot\|_{V_{\mathbb{C}}}$, respectively.

The Stokes operator may be extended to the operator $A_{\mathbb{C}}$ defined on $\mathcal{D}_{A_{\mathbb{C}}} = (\mathcal{D}_A)_{\mathbb{C}}$ as

$$A_{\mathbb{C}}(u + iv) = Au + iAv, \quad u, v \in \mathcal{D}_A.$$

Similarly, $B(\cdot, \cdot)$ can be extended to a bounded bilinear map from $V_{\mathbb{C}} \times \mathcal{D}_{A_{\mathbb{C}}}$ to $H_{\mathbb{C}}$ as

$$B_{\mathbb{C}}(u + iv, u' + iv') = B(u, u') - B(v, v') + i[B(u, v') + B(v, u')]$$

for $u, v \in V,\ u', v' \in \mathcal{D}_A$. Note that unlike the real case we have

$$\langle B_{\mathbb{C}}(u, v), v\rangle_{H_{\mathbb{C}}} \not\equiv 0, \qquad \text{for } u, v \in \mathcal{D}_{A_{\mathbb{C}}}.$$

The Navier–Stokes equations with complex times is defined as

$$\begin{cases} \dfrac{du(\zeta)}{d\zeta} + B_{\mathbb{C}}\big(u(\zeta), u(\zeta)\big) + A_{\mathbb{C}}u(\zeta) = 0, \\ u(\zeta_0) = u^\star, \end{cases} \tag{2.16}$$

where $\zeta_0 \in \mathbb{C}$ and $u^\star \in V_{\mathbb{C}}$ are given. Here $d/d\zeta$ denotes the derivative of $H_{\mathbb{C}}$-valued functions. For simplicity we write A and $B(\cdot, \cdot)$ instead of $A_{\mathbb{C}}$ and $B_{\mathbb{C}}(\cdot, \cdot)$.

2.4. **Fourier formulation.** We recall the formulas for Au and the nonlinear term $B(u, u)$ in terms of Fourier coefficients of u. For $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in C^3, let $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$, $\mathbf{a}^* = (\bar{a_1}, \bar{a_2}, \bar{a_3})$ and $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}^*}$. Again, we use $|\cdot|$ for the length of vectors in $\mathbb{C}^3$ as well as the L^2-norm of vector fields in $L^2(\Omega)^3$, but in each case the context clarifies the precise meaning of this notation.

Recall that every $u \in \mathcal{V}$ has a Fourier series of the form

$$u = \sum_{\mathbf{k} \in \mathbb{Z}^3} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},$$

where $a_{\mathbf{k}} = a^1_{\mathbf{k}} + ia^2_{\mathbf{k}} \in \mathbb{C}^3$, $a_{\mathbf{k}} \cdot \mathbf{k} = 0$, $a_{-\mathbf{k}} = a^*_{\mathbf{k}}$, $a_0 = 0$ and $a_{\mathbf{k}} \neq 0$ for only finitely many $\mathbf{k} \in \mathbb{Z}^3$. If $v = \sum_{\mathbf{k}\in\mathbb{Z}^3} v_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$ is a $\mathbb{R}^3$-valued trigonometric polynomial (that is, if $v_{-\mathbf{k}} = v^*_{\mathbf{k}}$, $v_0 = 0$ and $v_{\mathbf{k}} \neq 0$ for only finitely many $\mathbf{k}$'s) then the Leray projection of v is given by

$$P_L\Big(\sum_{\mathbf{k}\in\mathbb{Z}^3} v_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\Big) = \sum_{\mathbf{k}\in\mathbb{Z}^3} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},$$

where $a_0 = 0$ and $a_{\mathbf{k}} = v_{\mathbf{k}} - \mathbf{k}(v_{\mathbf{k}} \cdot \mathbf{k})/|\mathbf{k}|^2$ for $\mathbf{k} \neq 0$. We apply the Stokes operators to u term by term and obtain

$$Au = \sum_{\mathbf{k}\in\mathbb{Z}^3} |\mathbf{k}|^2 a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

We also define the power of A as

$$A^\alpha u = \sum_{\mathbf{k}\in\mathbb{Z}^3} |\mathbf{k}|^{2\alpha} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \alpha \geq 0.$$

Furthermore, if $v = \sum_{\mathbf{k}\in\mathbb{Z}^3} b_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \in \mathcal{V}$, then the Fourier series of the bilinear mapping $B(u,v)$ is

$$B(u,v) = P_L\Big(\sum_{\mathbf{n}\in\mathbb{Z}^3} Q_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{x}}\Big) = \sum_{\mathbf{n}\in\mathbb{Z}^3} B_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{x}},$$

where $\sum_{\mathbf{n}\in\mathbb{Z}^3} Q_{\mathbf{n}} e^{i\mathbf{n}\cdot\mathbf{x}}$ is the $\mathbb{R}^3$-valued trigonometric polynomial

$$Q_{\mathbf{n}} = i \sum_{\mathbf{k}+\mathbf{j}=\mathbf{n}} (a_{\mathbf{k}}\cdot\mathbf{j}) b_{\mathbf{j}} \quad \text{for} \quad n \in \mathbb{Z}^3.$$

Therefore

$$B_0 = 0 \quad \text{and} \quad B_{\mathbf{n}} = Q_{\mathbf{n}} - \frac{Q_{\mathbf{n}}\cdot\mathbf{n}}{|\mathbf{n}|^2}\mathbf{n} \quad \text{for} \quad \mathbf{n} \neq 0.$$

3. A construction of some families of regular solutions

Motivated by the asymptotic expansion (2.9) of the regular solutions of the Navier–Stokes equations, we consider a system of equations which is satisfied by certain families of the regular solutions as well as by the terms of the asymptotic expansions of those solutions. We introduce and start the study of the system (3.2) and (3.3) by providing estimates for its solutions, which we will exploit throughout. In this section we only show that the classical existence theorems for the regular solutions of the Navier–Stokes equations are easy consequences of those estimates.

3.1. The construction.

Suppose the initial data u^0 in V is of the form

$$u^0 = \sum_{n=1}^{\infty} \rho^n u_n^0, \quad \rho \in \mathbb{R},$$

where $\sum_{n=1}^{\infty} \rho_0^n \|u_n^0\| < \infty$ for some $\rho_0 > 0$ and all $|\rho| \le \rho_0$. We try to find the solutions $u(t,\rho)$ of the form

$$u(t,\rho) = \sum_{n=1}^{\infty} \rho^n u_n(t), \tag{3.1}$$

with the initial condition $u(0,\rho) = \sum_{n=1}^{\infty} \rho^n u_n^0$. By introducing (3.1) in the Navier–Stokes equations (2.6) and assuming that the power series indeed are convergent we see that the $u_n(t)$ must satisfy the following system of equations

$$\begin{cases} \dfrac{du_n(t)}{dt} + Au_n(t) + B_n(t) = 0, \quad t > 0, \\ u_n(0) = u_n^0, \end{cases} \tag{3.2}$$

where

$$B_1(t) = 0 \quad \text{and} \quad B_n(t) = \sum_{j+k=n} B\big(u_j(t), u_k(t)\big) \quad \text{for} \quad n > 1. \tag{3.3}$$

We will show below (see Corollary 3.5) that this construction yields a solution (3.1) of the Navier–Stokes equations whenever

$$\rho_0 = 1/\limsup_{n\to\infty} \|u_n^0\|^{1/n} > 0, \quad 0 \le |\rho| < \rho_0,$$

and $t \ge 0$ is small enough.

For each $n \ge 1$, (3.2) is a nonhomogeneous linear evolutionary equation of the type studied in [17], Chapter 4, Part I. By adapting the method presented in that treatise one can recurrently show that the solutions $(u_n(t))_{n=1}^{\infty}$ of the system of equations (3.2) exist for all $t \geq 0$ and are uniquely determined by their initial data $(u_n^0)_{n=1}^\infty$. However we will need explicit and specific estimates for the norms $\|u_n(\zeta)\|$, $n \geq 1$, of their extensions to the complex-valued time, i.e. for the solutions $(u_n(\zeta))_{n=1}^\infty$ of the system

$$\begin{cases} \dfrac{du_n(\zeta)}{d\zeta} + Au_n(\zeta) + B_n(\zeta) = 0, \quad \zeta \in \mathbb{C},\ \mathbf{Re}\zeta > \mathbf{Re}\zeta_0, \\ u_n(\zeta_0) = u_n^\star, \end{cases} \tag{3.4}$$

where $\zeta_0 \in \mathbb{C}$, $u_n^\star \in V_{\mathbb{C}}$ and

$$B_1(\zeta) = 0 \quad \text{and} \quad B_n(\zeta) = \sum_{j+k=n} B\big(u_j(\zeta), u_k(\zeta)\big) \quad \text{for} \quad n > 1.$$

(Above, $\mathbf{Re}\zeta$ denotes the real part of the complex number ζ.) Therefore we will prove the existence of the $(u_n(\zeta))_{n=1}^\infty$ in a direct elementary way based on the Galerkin approximations of each equation (3.4).

Proposition 3.1. *Given $\zeta_0 \in \mathbb{C}$ and $u_n^\star \in V_{\mathbb{C}}$, for $n = 1, 2, 3, \ldots$, then the solutions $(u_n(\zeta))_{n=1}^\infty$ to (3.4) exist and are analytic in $\mathcal{D}_A$ for $\zeta \in H(\zeta_0) = \{\zeta \in \mathbb{C} : \mathbf{Re}\zeta > \mathbf{Re}\zeta_0\}$. Moreover, for $\theta \in (-\pi/2, \pi/2)$ and $s > 0$ let*

$$\begin{cases} \gamma_1(s,\theta) = \|u_1^\star\|, \\ \gamma_n(s,\theta) = \|u_n^\star\| + \xi(s,\theta) \sum_{k+j=n} \gamma_k(s,\theta)\gamma_j(s,\theta), \quad n > 1, \end{cases} \tag{3.5}$$

where

$$\xi(s,\theta) = \frac{C_1}{\cos\theta}(1 - e^{-s\cos\theta})^{1/4}, \quad s \geq 0, |\theta| < \pi/2; \tag{3.6}$$

here C_1 is the positive constant introduced in Appendix A. Then

$$\|u_n(\zeta_0 + se^{i\theta})\| \leq e^{-s\cos\theta}\gamma_n(s,\theta), \quad s > 0, |\theta| < \pi/2, n \in \mathbb{N}; \tag{3.7}$$

and whenever $u_n(\zeta)$ is not identically zero on $H(\zeta_0)$,

$$\int_0^s \frac{|Au_n(\zeta_0 + \rho e^{i\theta})|^2}{\|u_n(\zeta_0 + \rho e^{i\theta})\|} d\rho \leq \frac{\gamma_n(s,\theta)}{\cos\theta}, \quad s > 0, |\theta| < \pi/2. \tag{3.8}$$

Proof. First, when $n = 1$, $B_1 \equiv 0$, then by the classical theory of analytic semigroups (see, for example, Section 3.6 in [17]), we have

$$u_1(\zeta) = e^{-(\zeta-\zeta_0)A}u_1^\star, \quad \zeta \in H(\zeta_0).$$

If $u_1^\star = 0$, so is $u_1(\zeta)$ for all $\zeta \in H(\zeta_0)$. Otherwise, $u_1(\zeta) \neq 0$ for all ζ, therefore if we denote $v_1(s) = u_1(\zeta_0 + se^{i\theta})$, it follows that

$$\frac{d}{ds}\|v_1(s)\| + \cos\theta \frac{|Av_1(s)|^2}{\|v_1(s)\|} = 0, \quad s > 0.$$

Using Poincaré inequality, we obtain

$$\|v_1(s)\| \leq e^{-s\cos\theta}\|u_1^\star\| \text{ and then } \cos\theta \int_0^s \frac{|Av_1(\rho)|^2}{\|v_1(\rho)\|} d\rho \leq \|u_1^\star\|.$$

For induction, let $N > 1$ and suppose that for $n = 1, 2, \ldots, N-1$, the solutions $u_n(\zeta)$ exist, are analytic in $\mathcal{D}_A$ for $\zeta \in H(\zeta_0)$ and satisfy the inequalities (3.7) and (3.8). For $s > 0$ and $\theta \in (-\pi/2, \pi/2)$, write $\xi(s) = \xi(s,\theta)$, $\gamma_n(s) = \gamma_n(s,\theta)$ and

$v_n(s) = u_n(\zeta_0 + se^{i\theta})$, $n = 1, 2, \dots, N-1$. Note that since $\xi(s)$ is an increasing function of s, so is $\gamma_n(s)$. Define for each $n = 1, 2, \dots, N-1$,

$$\mathcal{G}_n(s) = \begin{cases} \frac{|Av_n(s)|^2}{\|v_n(s)\|}, & \text{when } v_n(s) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have that

$$\|v_n(s)\| \leq e^{-s\cos\theta}\gamma_n(s) \qquad \text{and} \qquad \int_0^s \mathcal{G}_n(\rho)d\rho \leq \frac{\gamma_n(s)}{\cos\theta}, \quad s > 0, \tag{3.9}$$

hold for $n < N$. Now we have to prove that the solution of (3.4) also exists for $n = N$. To this aim we study the Galerkin approximation of $u_n(\zeta)$. For this we recall that for $M \geq 1$, the Galerkin approximation of u_N is the solution $u_N^{(M)}$ of the equation

$$\begin{cases} \dfrac{du_N^{(M)}(\zeta)}{d\zeta} + Au_N^{(M)}(\zeta) + \displaystyle\sum_{k+j=N} P_M B(u_k^{(M)}(\zeta), u_j^{(M)}(\zeta)) = 0, \ \zeta \in \mathbb{C}, \\ u_N^{(M)}(\zeta_0) = P_M u_N^\star, \end{cases} \tag{3.10}$$

where $u_N^{(M)} \in P_M H$. To simplify the notation set $v(s) = u_N^{(M)}(\zeta_0 + se^{i\theta})$, for some $M > 0$ and assume that $v(s)$ is analytic in $V_{\mathbb{C}}$ for $s \in [0, T)$ and some $T \in (0, \infty]$. The equation for dv/ds is given by

$$\frac{dv(s)}{ds} + e^{i\theta}Av(s) = -e^{i\theta}P_M B_N(s), \quad s \in (0, T),$$

where $B_N(s) = \sum_{j+k=n} B(v_j(s), v_k(s))$. Since

$$\frac{d}{ds}\|v\|^2 = \Big\langle \frac{dv}{ds}, Av \Big\rangle + \Big\langle Av, \frac{dv}{ds} \Big\rangle = 2\mathbf{Re}\big\{\langle \frac{dv}{ds}, Av\rangle\big\},$$

it follows that

$$\frac{1}{2}\frac{d}{ds}\|v(s)\|^2 + \cos\theta|Av(s)|^2 \leq \Big|\Big\langle B_N(s), Av(s)\Big\rangle\Big|. \tag{3.11}$$

Given $\varepsilon \in (0, 1)$, let

$$v_\varepsilon(s) = (\|v(s)\|^2 + \varepsilon)^{1/2} \text{ and } \mathcal{S}_\varepsilon(s) = \cos\theta \int_0^s \frac{\|v(\rho)\|^2}{v_\varepsilon^2(\rho)} d\rho.$$

By (3.11) and inequality (A.4) in Lemma A.1,

$$\frac{1}{2}\frac{d}{ds}\|v\|^2 + \cos\theta|Av|^2 \leq C_1 \sum_{j+k=N} \|v_j\|^{1/2}|Av_j|^{1/2}|Av_k|\|v\| \tag{3.12}$$

We derive

$$\begin{aligned} \frac{d}{ds}v_\varepsilon + \cos\theta\frac{|Av|^2}{v_\varepsilon^2}v_\varepsilon &\leq C_1 \sum_{j+k=N} \|v_j\|^{1/2}|Av_j|^{1/2}|Av_k| \\ &= C_1 \sum_{j+k=N} \mathcal{G}_j^{1/4}\|v_j\|^{3/4}\mathcal{G}_k^{1/2}\|v_k\|^{1/2} \\ &= C_1 \sum_{j+k=N} \Big\{\mathcal{G}_j^{1/4}\|v_k\|^{1/4}\Big\}\Big\{\mathcal{G}_k^{1/2}\|v_j\|^{1/2}\Big\}\Big\{\|v_k\|^{1/4}\|v_j\|^{1/4}\Big\} \\ &\leq C_1\Big\{\sum_{j+k=N} \mathcal{G}_j\|v_k\|\Big\}^{3/4}\Big\{\sum_{j+k=N} \|v_k\|\|v_j\|\Big\}^{1/4}. \end{aligned}$$

Since $k, j < N$, then we may apply the induction hypothesis (3.9) to the sum to obtain

$$\begin{aligned}&\frac{d}{ds}v_\varepsilon + \cos\theta\frac{|Av|^2}{v_\varepsilon^2}v_\varepsilon \\ &\quad\le C_1 e^{-(5/4)s\cos\theta}\Big\{\sum_{j+k=N}\mathcal{G}_j\gamma_k\Big\}^{3/4}\Big\{\sum_{j+k=N}\gamma_j\gamma_k\Big\}^{1/4}.\end{aligned}\tag{3.13}$$

Applying the Poincaré inequality clearly yields

$$\begin{aligned}v_\varepsilon(s) \le e^{-\mathcal{S}_\varepsilon(s)}\Bigg\{&v_\varepsilon(0) + C_1\Big\{\sum_{j+k=N}\gamma_j(s)\gamma_k(s)\Big\}^{1/4} \\ &\times\int_0^s e^{\mathcal{S}_\varepsilon(\rho)-(5/4)\rho\cos\theta}\Big\{\sum_{j+k=N}\mathcal{G}_j(\rho)\gamma_k(s)\Big\}^{3/4}d\rho\Bigg\}.\end{aligned}$$

Now applying Hölder's inequality leads to

$$\begin{aligned}&\int_0^s e^{\mathcal{S}_\varepsilon(\rho)-(5/4)\rho\cos\theta}\Big\{\sum_{j+k=N}\mathcal{G}_j(\rho)\gamma_k(s)\Big\}^{3/4}d\rho \\ &\quad\le\Big\{\int_0^s e^{4\mathcal{S}_\varepsilon(\rho)-5\rho\cos\theta}d\rho\Big\}^{1/4}\Big\{\sum_{j+k=N}\gamma_k(s)\int_0^s\mathcal{G}_j(\rho)d\rho\Big\}^{3/4} \\ &\quad\le\Big\{\int_0^s e^{4\mathcal{S}_\varepsilon(\rho)-5\rho\cos\theta}d\rho\Big\}^{1/4}\Big\{\frac{1}{\cos\theta}\sum_{j+k=N}\gamma_k(s)\gamma_j(s)\Big\}^{3/4}.\end{aligned}$$

It follows that

$$v_\varepsilon(s) \le e^{-\mathcal{S}_\varepsilon(s)}\Bigg\{v_\varepsilon(0) + \frac{C_1}{(\cos\theta)^{3/4}}\Big\{\int_0^s e^{4\mathcal{S}_\varepsilon(\rho)-5\rho\cos\theta}d\rho\Big\}^{1/4}\sum_{j+k=N}\gamma_k(s)\gamma_j(s)\Bigg\}.$$

Since $v(s)$ is assumed to be analytic in $[0,T)$, it has at most countably many zeros. Thus

$$\lim_{\varepsilon\to0}\mathcal{S}_\varepsilon(s) = s\cos\theta, \quad s\in(0,T),$$

and hence by letting $\varepsilon\to0$, we derive

$$\begin{aligned}\|v(s)\| &\le e^{-s\cos\theta}\Bigg\{\|v(0)\| + \frac{C_1}{(\cos\theta)^{3/4}}\Big\{\int_0^s e^{-\rho\cos\theta}d\rho\Big\}^{1/4}\sum_{j+k=N}\gamma_k(s)\gamma_j(s)\Bigg\} \\ &\le e^{-s\cos\theta}\Bigg\{\|u_N^\star\| + \xi(s)\sum_{j+k=N}\gamma_k(s)\gamma_j(s)\Bigg\},\end{aligned}$$

thus

$$\|u_N^{(M)}(\zeta_0+se^{i\theta})\| \le e^{-s\cos\theta}\gamma_N(s,\theta).\tag{3.14}$$

For $\theta\in(-\pi/2,\pi/2)$, let

$$d_1(\theta) = \|u_1^\star\| \text{ and } d_n(\theta) = \|u_n^\star\| + (C_1/\cos\theta)\sum_{k_j=n}d_k(\theta)d_j(\theta),\ n\ge2.$$

Then $\gamma_n(s,\theta)\le d_n(\theta)$ for all $s>0$ and hence

$$\|u_N^{(M)}(\zeta_0+se^{i\theta})\| \le d_N(\theta).\tag{3.15}$$

Since the Galerkin approximation (3.10) is a system of linear nonhomogeneous ordinary differential equations (with analytic nonhomogeneous term), the boundedness of $\|v(s)\|$ implies that $T=\infty$ (for every $\theta\in(-\pi/2,\pi/2)$) and hence $u_N^{(M)}(\zeta)$ is analytic in $\zeta\in H(\zeta_0)$.

We now integrate (3.13) directly. Thus,

$$\cos\theta\int_0^s \frac{|Av(\rho)|^2}{v_\varepsilon(\rho)}d\rho \le v_\varepsilon(0) + C_1\Big\{\sum_{j+k=N}\gamma_j(s)\gamma_k(s)\Big\}^{1/4}\int_0^s e^{-(5/4)\rho\cos\theta}\Big\{\sum_{j+k=N}\mathcal{G}_j(\rho)\gamma_k(s)\Big\}^{3/4}d\rho.$$

Applying Hölder's inequality yields

$$\begin{aligned}\int_0^s & e^{-(5/4)\rho\cos\theta}\Big\{\sum_{j+k=N}\frac{|Av_j(\rho)|^2}{\|v_j(\rho)\|}\gamma_k(s)\Big\}^{3/4}d\rho\\ &\le \Big\{\int_0^s e^{-5\rho\cos\theta}d\rho\Big\}^{1/4}\Big\{\sum_{k+j=N}\gamma_k(s)\int_0^s\frac{|Av_j(\rho)|^2}{\|v_j(\rho)\|}d\rho\Big\}^{3/4}\\ &\le \Big\{\int_0^s e^{-\rho\cos\theta}d\rho\Big\}^{1/4}\Big\{\frac{1}{\cos\theta}\sum_{j+k=N}\gamma_j(s)\gamma_k(s)\Big\}^{3/4}\\ &\le \Big\{\frac{1-e^{-s\cos\theta}}{\cos\theta}\Big\}^{1/4}\Big\{\frac{1}{\cos\theta}\sum_{j+k=N}\gamma_j(s)\gamma_k(s)\Big\}^{3/4}.\end{aligned}$$

It follows that

$$\cos\theta\int_0^s\frac{|Av(\rho)|^2}{v_\varepsilon(\rho)}d\rho\le v_\varepsilon(0)+\frac{C_1(1-e^{-s\cos\theta})^{1/4}}{\cos\theta}\sum_{j+k=N}\gamma_j(s)\gamma_k(s). \tag{3.16}$$

By letting $\varepsilon\to 0$, we have

$$\cos\theta\int_0^s\frac{|Av(\rho)|^2}{\|v(\rho)\|}d\rho\le\|u_N^\star\|+\xi(s,\theta)\sum_{j+k=N}\gamma_j(s)\gamma_k(s)=\gamma_N(s),$$

whenever $v(s)$ is not identically zero for $s>0$; and hence by Poincaré's inequality

$$\cos\theta\int_0^s|Au_N^{(M)}(\zeta_0+\rho e^{i\theta})|d\rho\le\gamma_N(s,\theta)\le d_N(\theta).$$

This inequality and the Cauchy integral formula give the uniform bound of $|Au_N^{(M)}(\zeta)|$ on any compact subset of $H(\zeta_0)$. It is now easy to use a compactness argument (based on Vitali's theorem for vector-valued functions) to obtain a subsequence $(u_N^{(M_j)}(\zeta))_{j=1}^\infty$ converging in $\mathcal{D}_A$ uniformly for ζ in any compact subset of $H(\zeta_0)$ to a solution $u_N(\zeta)$ of (3.4). Therefore, $u_N(\zeta)$ is a $\mathcal{D}_A$-valued analytic function in $\zeta\in H(\zeta_0)$. Moreover, one can establish that for each $\theta\in(-\pi/2,\pi/2)$,

$$\lim_{s\searrow 0}\|u_N(\zeta_0+se^{i\theta})-u_N^\star\|=0. \tag{3.17}$$

Then mapping any sector $\{\zeta_0+se^{i\theta}, s>0, |\theta|<\theta_0\}$, $\theta_0\in(0,\pi/2)$, onto the unit disk and using the Poisson integral, one obtains

$$\lim_{\zeta\to\zeta_0}\|u_N(\zeta)-u_N^\star\|=0, \tag{3.18}$$

where the limit is taken for ζ in any sector $\{\zeta_0 + se^{i\theta}, s > 0, |\theta| < \theta_0\}$, $\theta_0 \in (0, \pi/2)$. Since this solution u_N is obviously unique, the whole sequence $(u_N^{(M)}(\zeta))_{M=1}^{\infty}$ has the above convergence property. By letting $M \to \infty$ in (3.14), we obtain (3.7) for $n = N$. Similarly, by letting $M \to \infty$ in (3.16),

$$\cos\theta \int_0^s \frac{|Av_N(\rho)|^2}{(\|v_N(\rho)\|^2 + \varepsilon)^{1/2}} d\rho \leq (\|u_N^\star\|^2 + \varepsilon)^{1/2} + \xi(s,\theta) \sum_{j+k=N} \gamma_j(s)\gamma_k(s),$$

where $v_N(s) = u_N(\zeta_0 + se^{i\theta})$. Moreover, due to the analyticity of $v_N(s)$, by letting $\varepsilon \to 0$ we obtain (3.8) for $n = N$ and this completes the induction. $\square$

Proposition 3.1 allows us to reestablish some classical results on the regular solutions of the Navier–Stokes equations; namely, the existence of local solutions for arbitrarily large initial data and the existence of global solutions for small initial data.

3.2. **Local solutions.** Since the bounds of $\|u_n(\cdot)\|$ are given by (3.7), the convergence of $\sum_{n=1}^{\infty} \|u_n(\cdot)\|$ is implied by the convergence of $\sum_{n=1}^{\infty} \gamma_n(s,\theta)$, where $\gamma_n(s,\theta)$ are nonnegative numbers defined recursively by (3.5). The convergence of such series is considered in the following lemma.

Lemma 3.2. *Let $\xi \geq 0$ and $a_n \geq 0$, $1 \leq n \leq N$. Define γ_n by*

$$\gamma_1 = a_1 \text{ and } \gamma_n = a_n + \xi \sum_{k+j=n} \gamma_k\gamma_j, \ 1 < n \leq N.$$

If $4\xi \sum_{k=1}^N a_k \leq 1$, then

$$\sum_{k=1}^{N} \gamma_k \leq 2 \sum_{k=1}^{N} a_k \quad \text{for} \quad 1 \leq n \leq N. \tag{3.19}$$

Proof. Let $S_n = \sum_{k=1}^n \gamma_k$ and $X = \sum_{k=1}^N a_k$. Then we have

$$S_n = \sum_{k=1}^{n} a_k + \xi \sum_{k+j \leq n} \gamma_k\gamma_j \leq X + \xi S_{n-1}^2, \quad 1 \leq n \leq N.$$

Denote by $S_\star$ the smaller solution of $\xi S^2 - S + X = 0$, *i.e.*,

$$S_\star = \frac{1 - \sqrt{1 - 4\xi X}}{2\xi} = \frac{2X}{1 + \sqrt{1 - 4\xi X}} \in [X, 2X].$$

Note that $S_1 = a_1 = \gamma_1 \leq X \leq S_\star$. Suppose $S_{n-1} \leq S_\star$ then

$$S_n \leq X + \xi S_{n-1}^2 \leq X + \xi S_\star^2 = S_\star$$

proves (3.19). $\square$

We now prove the existence of local solutions in complexified time and also describe their domains of analyticity.

Theorem 3.3. *Let $(u_n^\star)_{n=1}^{\infty}$ be a sequence in $V_{\mathbb{C}}$ satisfying $S^\star = \sum_{n=1}^{\infty} \|u_n^\star\| < \infty$ and let $(u_n(\zeta))_{n=1}^{\infty}$ be the solutions of (3.4). Let*

$$\gamma_1 = \|u_1^\star\| \text{ and } \gamma_n = \|u_n^\star\| + \xi \sum_{k+j=n} \gamma_k\gamma_j, \ n > 1, \tag{3.20}$$

where $\xi > 0$ *such that* $4S^\star\xi \le 1$. *Then* $u(\zeta) = \sum_{n=1}^\infty u_n(\zeta)$ *is the unique solution of the complexified Navier–Stokes equations* (2.16) *with* $u^\star \stackrel{\text{def}}{=\!=} \sum_{n=1}^\infty u_n^\star$ *and* ζ *belonging to the following domain*

$$E(\zeta_0,\xi) = \Big\{ \zeta_0 + se^{i\theta} : \xi(s,\theta) < \xi,\ s > 0 \text{ and } |\theta| < \pi/2 \Big\}. \tag{3.21}$$

Note that $\xi(s,\theta)$ *is defined in* (3.6). *More precisely,*

$$\|u_n(\zeta)\| \le \gamma_n e^{-\mathbf{Re}(\zeta-\zeta_0)}, \quad \zeta \in E(\zeta_0,\xi), \tag{3.22}$$

$$|Au_n(\zeta)| \le 2\delta\gamma_n(s+r)(\pi r^2)^{-1}[\min\{\cos(\theta+\delta),\cos(\theta-\delta)\}]^{-1}, \tag{3.23}$$

where $\zeta = \zeta_0 + se^{i\theta} \in E(\zeta_0,\xi)$ *and the closed ball* $\bar B(\zeta,r)$ *is contained in*

$$\Big\{ \zeta_0 + \rho e^{i\omega} : \omega \in (\theta-\delta,\theta+\delta) \text{ and } 0 \le \rho \le s+r \Big\} \subset E(\zeta_0,\xi)$$

with $\delta = \arcsin(r/s) > 0$, *and*

$$\sum_{n=1}^\infty \gamma_n \le 2S^\star. \tag{3.24}$$

Proof. Comparing (3.5) and (3.20), we note that $\gamma_n(s,\theta) \le \gamma_n$ provided $\xi(s,\theta) \le \xi$. By Proposition 3.1 and Lemma 3.2, we obtain (3.22) and (3.24). Consequently, the sum $\sum_{n=1}^\infty u_n(\zeta)$ is convergent in V uniformly for $\zeta \in E(\zeta_0,\xi)$. Combining with (3.18), we derive

$$\lim_{\zeta\to\zeta_0} \sum_{n=1}^\infty u_n(\zeta) = \sum_{n=1}^\infty \lim_{\zeta\to\zeta_0} u_n(\zeta) = \sum_{n=1}^\infty u_n^\star = u^\star,$$

where the limit in ζ is taken for $\zeta \in E(\zeta_0,\xi) \cap \{\zeta_0 + se^{i\theta}, s > 0, |\theta| < \theta_0\}$, with any $\theta_0 \in (0,\pi/2)$. Given $n \ge 1$, for each $w \in H$, define the analytic function $f(\eta) = \langle Au_n(\eta), w\rangle$ for $\eta \in E(\zeta_0,\xi)$. We have

$$f(\zeta) = \frac{1}{\pi r^2}\int_{B(\zeta_0,r)} f(\eta)d\eta. \tag{3.25}$$

Applying the Cauchy–Schwarz inequality followed by (3.7) and (3.8) obtains

$$\begin{aligned}
|f(\zeta)| &\le \frac{1}{\pi r^2}\int_{\theta-\delta}^{\theta+\delta}\int_0^{s+r} |w||Au_n(\zeta_0+\rho e^{i\omega})|\rho d\rho d\omega \\
&\le \frac{|w|}{\pi r^2}\int_{\theta-\delta}^{\theta+\delta}\Big\{\int_0^{s+r}\frac{|Au_n(\zeta_0+\rho e^{i\omega})|^2}{\|u_n(\zeta_0+\rho e^{i\omega})\|}d\rho\int_0^{s+r}\|u_n(\zeta_0+\rho e^{i\omega})\|\rho^2 d\rho\Big\}^{1/2} d\omega \\
&\le \frac{\gamma_n|w|}{\pi r^2}\int_{\theta-\delta}^{\theta+\delta}\Big\{\frac{1}{\cos\omega}\int_0^{s+r} e^{-\rho\cos\omega}\rho^2 d\rho\Big\}^{1/2} d\omega \\
&\le \frac{\gamma_n|w|}{\pi r^2}\int_{\theta-\delta}^{\theta+\delta}\frac{s+r}{\cos\omega}d\omega \\
&\le \frac{2\delta(s+r)\gamma_n|w|}{\pi r^2\min\{\cos(\theta+\delta),\cos(\theta-\delta)\}}.
\end{aligned}$$

Therefore, (3.23) follows.

Given $N \in \mathbb{N}$, define $U_N(\zeta) = \sum_{n=1}^N u_n(\zeta)$ where $\zeta \in E(\zeta_0,\xi)$. We have

$$\frac{dU_N(\zeta)}{d\zeta} = -AU_N(\zeta) - \sum_{k+j\le N} B(u_k(\zeta),u_j(\zeta)). \tag{3.26}$$

We have already shown that $U_N(\zeta)$ converges to $u(\zeta)$ in V uniformly for ζ in $E(\zeta_0,\xi)$ and $AU_N(\zeta)$ converges to $Au(\zeta)$ in H uniformly for ζ in any compact subset of $E(\zeta_0,\xi)$. Now, using inequality (A.4) in Lemma A.1, we obtain $B(U_N(\zeta),U_N(\zeta))$ converges in V to $B(u(\zeta),u(\zeta))$ uniformly for ζ in compact subsets of $E(\zeta_0,\xi)$, and also estimate

$$\begin{aligned}\Big\|B(u(\zeta),u(\zeta)) - \sum_{k+j\le N} B(u_k(\zeta),u_j(\zeta))\Big\| &\le \sum_{k+j>N} \big\|B(u_k(\zeta),u_j(\zeta))\big\|\\ &\le \sum_{k+j>N} C_1\|u_k(\zeta)\|^{1/2}|Au_k(\zeta)|^{1/2}|Au_j(\zeta)|\\ &\le C_1(\zeta)\sum_{k+j>N}\gamma_k\gamma_j,\end{aligned}$$

where the positive constants $C_1(\zeta)$ are bounded on compact subsets of $E(\zeta_0,\xi)$. Let $c_m=\sum_{k+j=m}\gamma_k\gamma_j$, for $m\ge 2$. Then $\sum_{m=2}^\infty c_m$ is the Cauchy product of $\sum_{k=1}^\infty\gamma_k$ multiplied with itself which is convergent by (3.24). Therefore,

$$\lim_{N\to\infty}\sum_{k+j>N}\gamma_k\gamma_j = \lim_{N\to\infty}\sum_{m=N+1}^\infty c_m = 0.$$

Hence $\sum_{k+j\le N} B(u_k(\zeta),u_j(\zeta))$ converges to $B(u(\zeta),u(\zeta))$ in V uniformly for ζ in any compact subset of $E(\zeta_0,\xi)$. It follows that (3.26) can be passed to the limit in H as $N\to\infty$ and hence $u(\zeta)$ is the solution of (2.16). □

We will apply Theorem 3.3 repeatedly in conjunction with the following remark.

Remark 3.4. The system (3.4) has the following invariance property: given $\sigma\in\mathbb{C}$, denote $v_n(\zeta)=\sigma^n u_n(\zeta)$ for $n\in\mathbb{N}$ and $\zeta\in\mathbb{C}$. Then

$$\begin{cases}\dfrac{dv_n(\zeta)}{d\zeta}+Av_n(\zeta)+\tilde B_n(\zeta)=0, \quad \zeta\in\mathbb{C},\\ v_n(\zeta_0)=v_n^\star=\sigma^n u_n^\star,\end{cases}\tag{3.27}$$

where

$$\tilde B_1(\zeta)=0 \quad\text{and}\quad \tilde B_n(\zeta)=\sum_{j+k=n} B\big(v_j(\zeta),v_k(\zeta)\big) \quad\text{for}\quad n>1.\tag{3.28}$$

Corollary 3.5. *Let $(u_n^0)_{n=1}^\infty$ be a sequence in V and $\rho_0\in\mathbb{R}$ such that*

$$\rho_0 \stackrel{\text{def}}{=\!=} 1/\limsup_{n\to\infty}\|u_n^0\|^{1/n}>0.$$

Let $(u_n(t))_{n=1}^\infty$ be the solutions to (3.2), then for $\rho\in\mathbb{R}$, $|\rho|<\rho_0$, $u(t)=\sum_{n=1}^\infty\rho^n u_n(t)$ is the regular solution to the Navier–Stokes equations (2.6)with initial data $u^0=\sum_{n=1}^\infty\rho^n u_n^0$, for $t\in[0,T)$, some $T>0$.

Proof. For such ρ in the statement, let $v_n^0=\rho^n u_n^0$ and $v_n(t)=\rho^n u_n(t)$, $n=1,2,3,\ldots$ Then the sum $\sum_{n=1}^\infty\|v_n^0\|$ is finite. By Remark 3.4 and Theorem 3.3, there is $T>0$ such that $v(t)=\sum_{n=1}^\infty v_n(t)$ is the regular solution of the Navier–Stokes equations on $(0,T)$ with the initial data $v^0=\sum_{n=1}^\infty v_n^0=\sum_{n=1}^\infty\rho^n u_n^0$. □

Remark 3.6. When $u^0=\sum_{n=1}^\infty u_n^0\in V\setminus\{0\}$ such that $\sum_{n=1}^\infty\|u_n^0\|<\infty$, take $\zeta_0=0$, $\theta=0$, $u_n^\star=u_n^0$ and $\xi>0$ satisfying $4\xi\sum_{n=1}^\infty\|u_n^0\|=1$, Theorem 3.3 implies that $u(t)=\sum_{n=1}^{\infty}u_n(t)$ is the regular solution to the Navier–Stokes equations (2.6) on the set

$$\Big\{t>0 : C_1(1-e^{-t})^{1/4} < \big(4\sum_{n=1}^{\infty}\|u_n^0\|\big)^{-1}\Big\}. \tag{3.29}$$

In the case $4C_1\sum_{n=1}^{\infty}\|u_n^0\|>1$, the regular solution exists on $[0,T)$ where

$$T=-\log\Big[1-\big\{4C_1\sum_{n=1}^{\infty}\|u_n^0\|\big\}^{-4}\Big]$$

Note that setting $u_n^0=0$ for $n>1$ and $u_1^0=u^0$ gives an estimate

$$T=-\log\Big[1-\big\{4C_1\|u^0\|\big\}^{-4}\Big] \tag{3.30}$$

of the time interval of existence for regular solutions to the Navier–Stokes equations with large initial data.

3.3. **Global solutions with small initial data.** If $4C_1\sum_{n=1}^{\infty}\|u_n^0\|\le 1$, the regular solution $u(t)=\sum_{n=1}^{\infty}u_n(t)$ exists for all time $t>0$ by (3.29). However, under this condition the domain of analyticity is not large enough for our purpose of estimating the terms in the asymptotic expansion of a regular solution. Therefore, we will derive a larger domain of analyticity for $u_n(\zeta)$ and hence for $u(\zeta)$ when $\sum_{n=1}^{\infty}\|u_n^0\|$ satisfies a slightly more stringent condition.

Theorem 3.7. *Given $\epsilon\in(0,(8C_1)^{-1})$ and $u_n^0\in V, n=1,2,3,\dots$ such that $S^0=\sum_{n=1}^{\infty}\|u_n^0\|\le\epsilon$. Let $(u_n(\zeta))_{n=1}^{\infty}$ be the solutions to (3.4) with $\zeta_0=0$, $u_n^\star=u_n^0$. Then $u(\zeta)=\sum_{n=1}^{\infty}u_n(\zeta)$ is the unique solution of the complexified Navier–Stokes equations (2.16) with initial condition $u(0)=u^0\overset{\text{def}}{=\!=}\sum_{n=1}^{\infty}u_n^0\in V$ and ζ belonging to the domain*

$$E(\epsilon)=\Big\{t+se^{i\theta} : \frac{8C_1\epsilon e^{-t}}{\cos\theta}<1,\ s>0,\ t>0 \text{ and } |\theta|<\pi/2\Big\}. \tag{3.31}$$

Moreover, define

$$\gamma_1=\|u_1^0\|,\quad \gamma_n=\|u_n^0\|+\frac{1}{4\epsilon}\sum_{k+j=n}\gamma_k\gamma_j,\ n>1,$$

and

$$\tilde\gamma_1=\gamma_1,\quad \tilde\gamma_n=\gamma_n+\frac{1}{8\epsilon}\sum_{k+j=n}\tilde\gamma_k\tilde\gamma_j,\ n>1.$$

Then, for each $n\in\mathbb{N}$,

$$\|u_n(t)\|\le\gamma_n e^{-t},\quad t>0, \tag{3.32}$$

$$\|u_n(\zeta)\|\le\tilde\gamma_n e^{-\mathbf{Re}\zeta},\quad \zeta\in E(\epsilon), \tag{3.33}$$

and

$$\sum_{n=1}^{\infty}\tilde\gamma_n\le 2\sum_{n=1}^{\infty}\gamma_n\le 4S^0. \tag{3.34}$$

Consequently,

$$\sum_{n=1}^{\infty}\|u_n(t)\|\le 2S^0e^{-t},\quad t>0, \tag{3.35}$$

and

$$\sum_{n=1}^{\infty} \|u_n(\zeta)\| \le 4S^0 e^{-\mathbf{Re}\zeta}, \quad \zeta \in E(\epsilon). \tag{3.36}$$

Proof. First, apply Theorem 3.3 for $\zeta_0 = 0$, $\theta = 0$ and $\zeta = t_0 > 0$ noting that

$$4S^0 \frac{1}{4\epsilon} \le 1 \quad \text{and} \quad \xi(t_0, 0) = C_1(1 - e^{-t_0})^{1/4} \le C_1 < \frac{1}{4\epsilon}.$$

We obtain

$$\|u_n(t_0)\| \le \gamma_n e^{-t_0} \tag{3.37}$$

and

$$S_n(t_0) \overset{\text{def}}{=\!=} \sum_{j=1}^{n} \|u_j(t_0)\| \le 2S^0 e^{-t_0}. \tag{3.38}$$

Hence

$$S_\infty(t_0) \overset{\text{def}}{=\!=} \sum_{n=1}^{\infty} \|u_n(t_0)\| \le 2S^0 e^{-t_0}. \tag{3.39}$$

For $t_0 > 0$, set

$$\begin{cases} \tilde{\gamma}_1(t_0) = \|u_1(\tau_0)\|, \\ \tilde{\gamma}_n(t_0) = \|u_n(\tau_0)\| + \dfrac{e^{t_0}}{8\epsilon} \displaystyle\sum_{j+k=n} \tilde{\gamma}_j(\tau_0)\tilde{\gamma}_k(\tau_0), \quad n > 1. \end{cases}$$

Next, we apply Theorem 3.3 to $\zeta_0 = t_0$, noting that

$$4S_\infty(t_0)\frac{e^{t_0}}{8\epsilon} \le 4(2S^0 e^{-t_0})\frac{e^{t_0}}{8\epsilon} \le 1$$

and for $\zeta = t_0 + se^{i\theta} \in E(\epsilon)$ that

$$\xi(s,\theta) \le \frac{C_1}{\cos\theta} = \frac{8C_1\epsilon e^{-t_0}}{\cos\theta}\frac{e^{t_0}}{8\epsilon} < \frac{e^{t_0}}{8\epsilon},$$

hence $\zeta \in E(t_0, e^{t_0}(8\epsilon)^{-1})$. It follows that

$$\|u(t_0 + se^{i\theta})\| \le \tilde{\gamma}_n(t_0)e^{-s\cos\theta}, \quad t_0 + se^{i\theta} \in E(\epsilon). \tag{3.40}$$

We can show by induction and the use of (3.37) that $\tilde{\gamma}_n(t_0) \le \tilde{\gamma}_n e^{-t_0}$. Therefore

$$\|u(\zeta)\| = \|u(t_0 + se^{i\theta})\| \le \tilde{\gamma}_n e^{-t_0 - s\cos\theta} = \tilde{\gamma}_n e^{-\mathbf{Re}\zeta}, \quad \zeta \in E(\epsilon).$$

Moreover, Lemma 3.2 implies (3.34), thus yields (3.35) and (3.36). □

When S^0 is uniformly small, the analytic domains of regular solutions contain a common subregion which is significant to our subsequent study.

In the following proposition and the remainder of this article, we denote the positive constant

$$\varepsilon_0 = \frac{1}{24C_1}, \tag{3.41}$$

where the constant $C_1 > 0$ is introduced in Lemma A.1.

Proposition 3.8. *The domain $E(\varepsilon_0)$ defined by* (3.31) *contains the following subregion*

$$D = \{\tau + i\sigma : \tau > 0, |\sigma| < \sqrt{2}\tau e^{\tau/2}\}, \tag{3.42}$$

where the constant ε_0 is defined by (3.41).

Proof. Let $\varepsilon_1 = (8C_1)^{-1}$ and $\zeta = \tau + i\sigma = \tau/2 + se^{i\theta} \in E(\epsilon)$, $\epsilon \in (0, \varepsilon_1)$. Then, $\cos\theta > 8C_1\epsilon e^{-\tau/2} = e^{-\tau/2}\epsilon/\varepsilon_1$ and

$$|\sigma| = \frac{\tau}{2}\tan\theta < \frac{\tau}{2}\sqrt{\frac{\varepsilon_1^2 e^{\tau}}{\epsilon^2} - 1}.$$

This holds if

$$|\sigma| < \frac{\tau e^{\tau/2}}{2}\sqrt{\frac{\varepsilon_1^2}{\epsilon^2} - 1}.$$

We consider those ϵ that satisfy

$$\frac{1}{2}\sqrt{\frac{\varepsilon_1^2}{\epsilon^2} - 1} \geq \sqrt{2}.$$

Hence $\epsilon \leq \varepsilon_1/3 = (24C_1)^{-1}$. This is our definition of ε_0. $\square$

When only finitely many u_j^0 are given, we consider the finite sum $\sum_{j=1}^n \|u_j(\zeta)\|$ for some $n > 1$ rather than the infinite sum. The following proposition is an obvious consequence of Theorems 3.7 and Proposition 3.8 when we take $u_j^0 = 0$ for all $j > n$.

Proposition 3.9. *Given $n \geq 1$ and suppose that $S_n = \sum_{k=1}^n \|u_k^0\| < \varepsilon_0$. Then*

$$S_n(t) = \sum_{k=1}^n \|u_k^0(t)\| \leq 2S_n e^{-t}, \quad t \geq 0, \tag{3.43}$$

and

$$S_n(\zeta) = \sum_{k=1}^n \|u_k^0(\zeta)\| \leq 4S_n e^{-\mathbf{Re}\zeta}, \quad \zeta \in E(\varepsilon_0), \tag{3.44}$$

where the analytic domain $E(\varepsilon_0)$ defined in (3.31) *contains the subdomain D defined in* (3.42). *More specifically, let*

$$\gamma_1 = \|u_1^0\|, \quad \gamma_m = \|u_m^0\| + \frac{1}{4\varepsilon_0}\sum_{k+j=m}\gamma_k\gamma_j, \ m > 1,$$

and

$$\tilde{\gamma}_1 = \gamma_1, \quad \tilde{\gamma}_m = \gamma_m + \frac{1}{8\varepsilon_0}\sum_{k+j=n}\tilde{\gamma}_k\tilde{\gamma}_j, \ m > 1.$$

Then

$$\|u_j(t)\| \leq \gamma_j e^{-t}, \quad t > 0, 1 \leq j \leq n, \tag{3.45}$$

$$\|u_j(\zeta)\| \leq \tilde{\gamma}_j e^{-\mathbf{Re}\zeta}, \quad \zeta \in D, 1 \leq j \leq n, \tag{3.46}$$

and

$$\sum_{k=1}^n \tilde{\gamma}_j \leq 2\sum_{k=1}^n \gamma_j \leq 4S_n. \tag{3.47}$$

4. Extended Navier–Stokes Equations

We consider now the system of equations (3.2) in the space V^∞ of all sequences $\bar{u} = (u_n)_{n=1}^\infty$ where $u_n \in V$ for all $n = 1, 2, 3, \ldots$ We will refer to this system of linear nonhomogeneous differential equations (i.e., nonhomogeneous Stokes equations) as the *extended Navier–Stokes equations.* Note that in Section 3 we have implicitly proved that for each initial data $\bar{u}^0 \stackrel{\text{def}}{=\!=} (u_n^0)_{n=1}^\infty \in V^\infty$ there exists a unique solution $\bar{u}(t) = (u_n(t))_{n=1}^\infty$ for $t \geq 0$ of the extended Navier–Stokes equations. Moreover, we have also shown that if $\sum_{n=1}^\infty \|u_n^0\|$ is small enough then $\sum_{n=1}^\infty \|u_n(t)\|$ is absolutely convergent and its sum is a regular solution of the Navier–Stokes equations (2.6) for all $t \geq 0$. We do not know if solutions of the extended Navier–Stokes equations leave invariant the space

$$V_{*1} = \Big\{ u = (u_n)_{n=1}^\infty \in V^\infty : \|u\|_{*1} \stackrel{\text{def}}{=\!=} \sum_{n=1}^\infty \|u_n\| < \infty \Big\}.$$

However we will show that there exist norms of the form

$$\|(u_n)_{n=1}^\infty\|_* = \sum_{n=1}^\infty \rho_n \|u_n\|, \tag{4.1}$$

with positive $\rho_n \to 0$ such that those solutions leave invariant each space

$$V_* = \Big\{ u = (u_n)_{n=1}^\infty \in V^\infty : \|u\|_* < \infty \Big\}. \tag{4.2}$$

It will turn out that those spaces play a similar role for the normal form of the Navier–Stokes equations (see Section 7). We now specify the weights $(\rho_n)_{n=1}^\infty$ in (4.1).

Definition 4.1. Let $(\kappa_n)_{n=2}^\infty$ be a sequence of positive numbers satisfying

$$\limsup_{n\to\infty} \kappa_n^{1/n} = 0. \tag{4.3}$$

Let $(\rho_n)_{n=1}^\infty$ be a sequence of positive numbers satisfying

$$\rho_n = \kappa_n C_1^{-1} \min\{\rho_k \rho_j \ : \ k, j \geq 1 \ \text{ and } \ k + j = n\}, \quad n > 1. \tag{4.4}$$

Let $\|\cdot\|_*$ and V_* be the norm and space defined by (4.1) and (4.2), respectively. Then $(V_*, \|\cdot\|_*)$ is a Banach space.

The following lemma will be used to estimate $\|\bar{u}(t)\|_*$ later.

Lemma 4.2. *Let $(\kappa_n)_{n=2}^\infty$ be as in Definition 4.1. Let $a_n \geq 0$. Define*

$$d_1 = a_1 \text{ and } d_n = a_n + \kappa_n \sum_{k+j=n} d_k d_j, \ n \geq 2.$$

If $\sum_{n=1}^\infty a_n$ is finite, then so is $\sum_{n=1}^\infty d_n$.

Proof. Let $K = \max\big\{1/2, \sum_{n=2}^\infty (n-1)\kappa_n\big\} < \infty$ and $M = \max\big\{1/2, \sum_{n=1}^\infty a_n\big\}$. Define $S_n = \sum_{m=1}^n d_m$. We will first prove by induction that

$$S_n \leq (2M)^n (2K)^{n-1} \quad \text{for} \quad n \in \mathbb{N}. \tag{4.5}$$

Since $S_1 = a_1 \leq M$, then (4.5) holds for $n = 1$. Let $N > 1$ and suppose (4.5) holds for all $n < N$. Then

$$S_N = \sum_{n=1}^N a_n + \sum_{n=2}^N \kappa_n \sum_{k+j=n} d_k d_j \leq M + \sum_{n=2}^N \Big(\kappa_n \sum_{k+j=n} S_k S_j \Big).$$

Using the induction hypothesis, we have

$$\begin{aligned}
S_N &\le M + \sum_{n=2}^{N} \kappa_n \sum_{k+j=n} (2M)^k (2K)^{k-1} (2M)^j (2K)^{j-1} \\
&\le \frac{(2M)^N}{2} + \sum_{n=2}^{N} \kappa_n (2M)^n (2K)^{n-2} (n-1) \\
&\le \frac{(2M)^N (2K)^{N-1}}{2} + (2M)^N \frac{(2K)^{N-1}}{2K} \sum_{n=2}^{N} (n-1)\kappa_n \\
&\le (2M)^N (2K)^{N-1}.
\end{aligned}$$

Hence (4.5) holds for all $n \ge 1$. This, by virtue of (4.3), implies that

$$\begin{aligned}
S_n &\le \sum_{m=1}^{n} a_m + \sum_{m=2}^{n} (m-1)\kappa_m (2M)^m (2K)^{m-1} \\
&\le M\Big\{1 + 2\sum_{m=2}^{\infty} (m-1)\kappa_m (4MK)^{m-1}\Big\} < \infty
\end{aligned}$$

for each $n \ge 2$. $\square$

It turns out that the extended Navier–Stokes equations has the global solution in the normed space $(V_*, \|\cdot\|_*)$.

Theorem 4.3. *Let $\bar u^0 = (u_n^0)_{n=1}^\infty \in V_*$ and $u_n(t)$, $n \in \mathbb{N}$, be the solutions of (3.2) for $n \in \mathbb{N}$. Then $\bar u(t) = (u_n(t))_{n=1}^\infty, t \ge 0$ is the solution to the extended Navier–Stokes equations in the space V_*. Moreover,*

$$\|\bar u(t)\|_* \le e^{-t}\Big\{\|\bar u^0\|_* + \sum_{n=2}^{\infty} \kappa_n (n-1) M_0^n\Big\} \quad \textit{for all} \quad t > 0, \tag{4.6}$$

where

$$M_0 = 4\max\big(1/2, \|\bar u^0\|_*\big) \max\Big(1/2, \sum_{n=2}^{\infty} (n-1)\kappa_n\Big).$$

Proof. From the *a priori* estimates given in Proposition 3.1 with $\zeta_0 = 0$, $\theta = 0$ and $s = t > 0$, we have $\|u_n(t)\| \le e^{-t}\gamma_n$ for $n \in \mathbb{N}$ where

$$\gamma_1 = \|u_1^0\| \text{ and } \gamma_n = \|u_n^0\| + C_1 \sum_{k+j=n} \gamma_k \gamma_j,\ n > 1.$$

Therefore, $\rho_1\gamma_1 = \rho_1\|u_1^0\|$ and

$$\rho_n \gamma_n \le \rho_n \|u_n^0\| + \kappa_n \sum_{k+j=n} (\rho_k\gamma_k)(\rho_j\gamma_j), \quad n > 1.$$

Apply Lemma 4.2 with $a_n = \rho_n\|u_n^0\|$ and $d_n = \rho_n\gamma_n$. We have

$$\|\bar u(t)\|_* \le e^{-t} \sum_{n=1}^{\infty} \rho_n \gamma_n \le e^{-t}\Big\{\|\bar u^0\|_* + \sum_{n=2}^{\infty} \kappa_n (n-1) M_0^n\Big\}, \quad t > 0.$$

$\square$

5. Connection to the asymptotic expansion

In this section, we apply the study of the system (3.2) in Section 3 to the asymptotic expansion (2.9), the normalization map W and the normal form (2.15). Note that for $u^0 \in \mathcal{R}$ and the regular solution $u(t)$ of (2.6) with the asymptotic expansion (2.9), the functions $u_n(t) = q_n(t)e^{-nt}, n \in \mathbb{N}$, satisfy the extended Navier–Stokes equations (3.2). We know that $q_n(t)$ is $q_n(t,\xi)$, given by (2.14), where $\xi = W(u^0) \in S_A$. More generally, let $\xi = (\xi_1, \xi_2, \ldots) \in S_A$ and the polynomials $q_n(t) = q_n(t,\xi)$ be recursively computed by (2.14). Then $u_n(t) = q_n(t)e^{-nt}, n \in \mathbb{N}$, satisfy (3.2) with initial value $u_n^0 \overset{\text{def}}{=\!=} q_n(0)$. Moreover, $\xi(t) = (\xi_n(t))_{n=1}^\infty = (R_n u_n(t))_{n=1}^\infty$ is the solution to the normal form (2.15) with the initial value ξ. The results in Section 3 infer that when $\sum_{n=1}^\infty \|q_n(0,\xi)\|$ is finite, $\sum_{n=1}^\infty u_n(t)$ converges in V to a regular solution $u(t)$ for small time t. We do not know whether the convergence holds for large t, and even when this is the case we still do not know whether the sum $\sum_{n=1}^\infty q_n(t,\xi)e^{-nt}$ is the asymptotic expansion of that solution $u(t)$. Note that in the case when $u(t)$ is a regular solution on $[0,\infty)$ and $\sum_{n=1}^\infty q_n(t,\xi)e^{-nt}$ is indeed its asymptotic expansion, the above element ξ of S_A is equal to $W(u(0))$, i.e., ξ belongs to the range of the normalization map.

For our study in this section, let $\xi = (\xi_1, \xi_2, \xi_3, \ldots)$ be in S_A and $q_n(t) = q_n(t,\xi)$, for $n \in \mathbb{N}, t \in \mathbb{R}$. We extend the polynomials $q_n(t), t \in \mathbb{R}$, to polynomials $q_n(\zeta), \zeta \in \mathbb{C}$, and let $u_n(\zeta) = e^{-n\zeta}q_n(\zeta)$ for $\zeta \in \mathbb{C}$. Then $u_n(\zeta)$ satisfies (3.4) with initial condition ($\zeta_0 = 0$)

$$u_n(0) = u_n^0 \overset{\text{def}}{=\!=} q_n(0,\xi), \quad n \in \mathbb{N}. \tag{5.1}$$

We first apply Corollary B.3 to estimate $\|q_n(\zeta)\|$ for $\zeta \in \mathbb{C}$.

Lemma 5.1. *Suppose $\sum_{n=1}^N \|u_n^0\| < \varepsilon_0$ for some $N > 0$, where ε_0 is defined in* (3.41)*. Let $\tilde{\gamma}_n$ be defined as in Proposition 3.9 for $n = 1, 2, \ldots, N$. Then for each $n = 1, 2, \ldots, N$ we have*

$$\|q_n(\zeta)\| \le \tilde{\gamma}_n |1 + \alpha_0\zeta|^{(n-1)/\alpha_0} \quad \textit{for} \quad \zeta \in \phi_{\alpha_0}^{-1}(H_0), \tag{5.2}$$

and

$$\|q_n(\zeta)\| \le n\tilde{\gamma}_n C_3^{n-1}(|\zeta| + a_0)^{n-1} \quad \textit{for} \quad \zeta \in \mathbb{C}, \tag{5.3}$$

where $\alpha_0 = 1/2, \phi_{\alpha_0}$ is defined by (B.4)*, H_0 is the right half plane, the positive constant*

$$C_3 = C(\alpha_0, a_0, r_0) \textit{ is defined by } (\text{B.20}), \tag{5.4}$$

with $a_0 > r_0 > 0$ are fixed numbers such that the disk $B(a_0, r_0)$ in the complex plane is contained in $\phi_{\alpha_0}^{-1}(H_0)$ (see Lemma B.4).

Proof. For $n \ge 2, w \in V$, define $q(\zeta) = \langle\!\langle q_n(\zeta), w\rangle\!\rangle$, for $\zeta \in \mathbb{C}$. Note that the degree of $q(\zeta)$ is less than or equal to $(n-1)$. According to Proposition 3.9, we have

$$|e^{-(n-1)\zeta}q(\zeta)| \le \|e^{\zeta}u_n(\zeta)\|\|w\| \le \tilde{\gamma}_n\|w\|,$$

for ζ in D, the domain defined by (3.42). This domain D is in fact the domain $D(\sqrt{2}, 1/2)$ in (B.3). Hence applying Corollary B.6 to $q(\zeta)$ with $p = n-1$ we have

$$|q(\zeta)| \le \tilde{\gamma}_n|1 + \alpha_0\zeta|^{(n-1)/\alpha_0}\|w\|, \quad \zeta \in \phi_{\alpha_0}^{-1}(H_0)$$

and

$$|q(\zeta)| \le n\tilde{\gamma}_n C_3^{n-1}(|\zeta| + a_0)^{n-1}\|w\|, \quad \zeta \in \mathbb{C},$$

thus obtaining (5.2) and (5.3). □

When the initial data in (5.1) are small, the above estimates are used to establish the convergence of the series $\sum_{n=1}^{\infty} u_n(t)$.

Proposition 5.2. *Suppose* $\sum_{n=1}^{\infty} \|u_n^0\| < \varepsilon_0$. *Then* $u(t) = \sum_{n=1}^{\infty} u_n(t)$ *is the regular solution to the Navier–Stokes equations* (2.6) *with initial condition* $u(0) = u^0 \stackrel{\text{def}}{=\!=} \sum_{n=1}^{\infty} u_n^0 \in V$. *Moreover,* $\sum_{n=1}^{\infty} u_n(t)$ *is the asymptotic expansion of* $u(t)$. *More specifically, there exists* $\epsilon > 0$ *such that*

$$\Big\| u(t) - \sum_{n=1}^{N} u_n(t) \Big\| = O\big(e^{-(N+\epsilon)t}\big) \quad \text{as} \quad t \to \infty \tag{5.5}$$

for every $N > 0$.

Proof. The fact that $u(t) = \sum_{n=1}^{\infty} u_n(t)$ is the regular solution to the Navier–Stokes equations (2.6) with $u(0) = \sum_{n=1}^{\infty} u_n^0$ is stated in Theorem 3.7. From Lemma 5.1 and (3.47), we have for each $t > 0$ that

$$\|u_n(t)\| \leq 4\varepsilon_0 n C_3^{n-1}(t + a_0)^{n-1} e^{-nt} = C_0 e^{-t} n \sigma(t)^{n-1},$$

where $C_0 = 4\varepsilon_0$ and $\sigma(t) = C_3(t + a_0)e^{-t}$. Let $t_0 > 0$ such that $\sigma = \sigma(t) \leq 1/2$ for all $t \geq t_0$. Then, for $t > t_0$ we have

$$\sum_{n=N+1}^{\infty} n\sigma^{n-1} = \frac{(N+1)\sigma^N - N\sigma^{N+1}}{(1-\sigma)^2} \leq 4(N+1)\sigma^N.$$

Therefore,

$$\begin{aligned}\Big\| \sum_{n=N+1}^{\infty} u_n(t) \Big\| &\leq C_0 e^{-t} 4(N+1)(C_3(t+a_0)e^{-t})^N \\ &= 4C_0(N+1)C_3^N(t+a_0)^N e^{-(N+1)t} \\ &= o\big(e^{-(N+1/2)t}\big) \quad \text{as} \quad t \to \infty.\end{aligned}$$

This proves that $\sum_{n=1}^{\infty} u_n(t)$ is the asymptotic expansion of $u(t)$. □

When $\xi = W(u^0)$ we have $u_n^0 = W_n(0, u^0)$ and $q_n(t) = W_n(t, u^0)$. It follows, therefore, that $\sum_{n=1}^{\infty} u_n(t) = \sum_{n=1}^{\infty} W_n(t, u^0)e^{-nt}$ is the asymptotic expansion of the regular solution $u(t, u^0)$ of the Navier–Stokes equations (2.6).

Theorem 5.3. *Suppose* $u^0 \in \mathcal{R}$ *with* $\sum_{n=1}^{\infty} \|W_n(0, u^0)\| < \varepsilon_0$. *Then the asymptotic expansion* $\sum_{n=1}^{\infty} W_n(t, u^0)e^{-nt}$ *converges in* V *for all* $t > 0$ *to the regular solution* $u(t, u^0)$ *of the Navier–Stokes equations* (2.6).

Proof. According to Proposition 5.2, both $u(t) = \sum_{n=1}^{\infty} W_n(t, u^0)e^{-nt}$ and $u(t, u^0)$ are regular solutions to the Navier–Stokes equations with the same asymptotic expansion. Since the normalization map W is one-to-one (see Subsection 2.2), the two solutions $u(t)$ and $u(t, u^0)$ coincide. □

When the initial data (5.1) are not quite so small, we prove the convergence of the asymptotic expansion only for large time.

Proposition 5.4. *Let* $\xi \in S_A$, $u_n(t) = q_n(t, \xi)e^{-nt}$ *and* $u_n^0 = q_n(0, \xi)$, $n \in \mathbb{N}$. *If* $\sum_{n=1}^{\infty} \|u_n^0\| = S^0 \geq \varepsilon_0$, *then there is* $T > 0$ *such that* $u(t) = \sum_{n=1}^{\infty} u_n(t)$ *is absolutely convergent in* V *uniformly for* $t \in [T, \infty)$, $u(t)$ *is a regular solution of* *the Navier–Stokes equations* (2.6) *for* $t \geq T$, *and* $\sum_{n=1}^{\infty} u_n(t)$ *is the asymptotic expansion of* $u(t)$. *In addition, if* $\xi = W(u^0)$ *for some* $u^0 \in \mathcal{R}$, *then* $u(t)$ *is equal to the regular solution* $u(t, u^0)$ *of* (2.6) *for all* $t \in [T, \infty)$.

Proof. Let $\epsilon > 0$ be small enough such that $\rho = \varepsilon_0/S^0 - \epsilon \in (0,1)$. Define $\tilde{u}_n(\zeta) = \rho^n u_n(\zeta)$ for $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} \|\tilde{u}_n(0)\| = \sum_{n=1}^{\infty} \rho^n \|u_n^0\| \leq \rho \sum_{n=1}^{\infty} \|u_n^0\| < \varepsilon_0.$$

As indicated in Remark 3.4 we can apply Theorem 3.7 to $\tilde{u}_n(\zeta)$. Hence,

$$\sum_{n=1}^{\infty} \|\tilde{u}_n(\zeta)\| = \sum_{n-1}^{\infty} \rho^n \|u_n(\zeta)\| \leq 4\varepsilon_0 e^{-\mathbf{Re}\zeta} \quad \text{for} \quad \zeta \in D.$$

Letting $\epsilon \to 0$ we obtain

$$\|u_n(\zeta)\| \leq 4\varepsilon_0 \Big(\frac{S^0}{\varepsilon_0}\Big)^n e^{-\mathbf{Re}\zeta} \quad \text{for} \quad \zeta \in D.$$

Similar to Lemma 5.1, we derive for $t > 0$ that

$$\|u_n(t)\| \leq 4\varepsilon_0 \Big(\frac{S^0}{\varepsilon_0}\Big)^n n C_3^{n-1} (t + a_0)^{n-1} e^{-nt} = n\tilde{C}_0 \tilde{C}_3^{n-1} (t + a_0)^{n-1} e^{-nt}, \tag{5.6}$$

where $\tilde{C}_0 = 4S^0$ and $\tilde{C}_3 = S^0 C_3 \varepsilon_0^{-1}$.

Let $t_0 > 0$ such that $\tilde{C}_3(t + a_0)e^{-t}$ is small, say, less that $1/2$. Then $u(t) = \sum_{n=1}^{\infty} u_n(t)$ is absolutely convergent in V uniformly for $t \in [t_0, \infty)$. As shown in Proposition 5.2, $\sum_{n=1}^{\infty} u_n(t)$ is the asymptotic expansion of $u(t)$ in V. Moreover, by (5.6), we can choose $T > t_0$ and further large enough that

$$\sum_{n=1}^{\infty} \|u_n(T)\| < \varepsilon_0. \tag{5.7}$$

Note that $v_n(\tau) \overset{\text{def}}{=\!=} u_n(\tau + T)$ with $n \in \mathbb{N}$ is the solution of (3.2) for $\tau > 0$ with initial condition $v_n(0) = u_n(T)$. Theorem 5.3 and (5.7) imply $v(\tau) \overset{\text{def}}{=\!=} \sum_{n=1}^{\infty} v_n(\tau) = u(\tau + T)$ is a regular solution of the Navier–Stokes equations for $\tau > 0$.

To finish the proof, suppose there is $u^0 \in \mathcal{R}$ such that $\xi = W(u^0)$. Then $u_n^0 = W_n(0, u^0)$ for all $n \in \mathbb{N}$. We next show that the regular solution $u(t, u^0)$ is equal to the asymptotic expansion $u(t)$ for t large enough. Define $v^0 = u(T, u^0)$. Then $\tilde{v}(\tau) \overset{\text{def}}{=\!=} u(\tau + T, u^0)$ is the regular solution with initial condition v^0. Moreover, since the normalizing map is one-to-one we have that the asymptotic expansion of $\tilde{v}(\tau)$ is the same as the asymptotic expansion of $u(\tau + T, u^0)$. Thus,

$$\tilde{v}(\tau) \sim \sum_{n=1}^{\infty} v_n(\tau).$$

Therefore, Theorem 5.3 implies that $\tilde{v}(\tau) = v(\tau) = \sum_{n=1}^{\infty} v_n(\tau) = u(\tau + T)$ for $\tau \geq 0$. It follows that $u(\tau + T, u^0) = \tilde{v}(\tau) = u(\tau + T)$ for $\tau \geq 0$. $\square$

Our next goal is to weaken the condition on the initial data in Proposition 5.4. To achieve that we need to deal with bounds of each term $\|u_n(\zeta)\|$ rather than of the sum $\sum_{n=1}^{N} \|u_n(\zeta)\|$ for $N > 1$. Let us first define the following reference sequences.

Definition 5.5. We define

$$\begin{cases} b_1 = 1/4, \\ b_n = \sum_{k+j=n} b_k b_j, \ n > 1, \end{cases} \quad \text{and} \quad \begin{cases} \mu_1 = 1, \\ \mu_n = \sum_{k+j=n} \mu_k \mu_j, \ n > 1. \end{cases}$$

One can verify that

$$b_n = \frac{\mu_n}{4^n} \geq \frac{1}{4^n} \quad \text{and} \quad \sum_{n=1}^{\infty} b_n \leq \frac{1}{2}$$

by Lemma 3.2.

Before working with the norms $\|u_n(\zeta)\|$, we first establish some properties of series of non-negative real numbers.

Lemma 5.6. *Let* $\chi > 0, r > 0$ *and* $a_n \geq 0$ *for* $n \in \mathbb{N}$. *Assume that*

$$a_n \leq \frac{1}{2}(2\chi)^{n-1} r^n b_n \quad \textit{for} \quad n \in \mathbb{N}. \tag{5.8}$$

Let

$$\gamma_1 = a_1, \quad \gamma_n = a_n + \chi \sum_{k+j=n} \gamma_k \gamma_j, \ n > 1,$$

and

$$\tilde{\gamma}_1 = \gamma_1, \quad \tilde{\gamma}_n = \gamma_n + \frac{\chi}{2} \sum_{k+j=n} \tilde{\gamma}_k \tilde{\gamma}_j, \ n > 1.$$

Then

$$\gamma_n \leq (2\chi)^{n-1} r^n b_n \tag{5.9}$$

and

$$\tilde{\gamma}_n \leq 2(2\chi)^{n-1} r^n b_n. \tag{5.10}$$

Proof. We prove (5.9) by induction. Clearly (5.9) holds for $n = 1$, by (5.8). Given $n > 1$, assume (5.9) holds for all $k < n$. Then

$$\begin{aligned} \gamma_n &\leq a_n + \chi \sum_{k+j=n} (2\chi)^{k-1} r^k b_k (2\chi)^{j-1} r^j b_j \\ &\leq \frac{1}{2}(2\chi)^{n-1} r^n b_n + \frac{(2\chi)^{n-1} r^n \chi}{2\chi} \sum_{k+j=n} b_k b_j \\ &= (2\chi)^{n-1} r^n b_n. \end{aligned}$$

Hence (5.9) holds for all $n \geq 1$.

To prove (5.10), apply the first part to $\chi' = \chi/2$, $r' = 2r$, $a'_n = \gamma_n$ and $\gamma'_n = \tilde{\gamma}_n$. Since

$$a'_n = \gamma_n \leq (2\chi)^{n-1} r^n b_n = \frac{1}{2}(2\chi')^{n-1}(r')^n b_n,$$

we obtain

$$\tilde{\gamma}_n \leq (2\chi')^{n-1}(r')^n b_n = 2(2\chi)^{n-1} r^n b_n.$$

□

We now relax the convergence condition on the initial sum in Proposition 5.4.

Lemma 5.7. *Suppose* $\sup\{\|u_n^0\| : n \in \mathbb{N}\} = M_0 < \infty$. *Then*

$$\|u_n(\zeta)\| \leq 2\varepsilon_0 \big\{4 \max(1, \varepsilon_0^{-1} M_0)\big\}^n e^{-\mathbf{Re}\zeta} \quad \textit{for} \quad \zeta \in D, \tag{5.11}$$

where D *is the domain defined by* (3.42).

Proof. Given $N \geq 1$, let

$$M_N \stackrel{\text{def}}{=\!=} \max\left\{ \left(\frac{\|u_k^0\|}{\varepsilon_0}\right)^{1/k} : k = 1, 2, \ldots, N \right\}.$$

If $M_N = 0$, then $u_n(\zeta) = 0$ for $\zeta \in D$ and $n = 1, 2, \ldots, N$, hence (5.11) trivially holds for $n = N$. In the remaining case when $M_N \neq 0$, let $\chi = (4\varepsilon_0)^{-1}$ and choose $r > 0$ small enough such that

$$\sum_{n=1}^{\infty} \frac{1}{2}(2\chi)^{n-1} r^n b_n < \varepsilon_0. \tag{5.12}$$

Let $\rho = r\chi/(2M_N)$, $v_n(\zeta) = \rho^n u_n(\zeta), \zeta \in D$, and $v_n^0 = \rho^n u_n^0$, for $n = 1, 2, \ldots N$. Define

$$\gamma_1 = \|v_1^0\|, \quad \gamma_n = \|v_n^0\| + \chi \sum_{k+j=n} \gamma_k \gamma_j, \ n > 1,$$

and

$$\tilde{\gamma}_1 = \gamma_1, \quad \tilde{\gamma}_n = \gamma_n + \frac{\chi}{2} \sum_{k+j=n} \tilde{\gamma}_k \tilde{\gamma}_j, \ n > 1.$$

Estimate

$$\|v_n^0\| = \rho^n \|u_n^0\| \leq \varepsilon_0 \left(\frac{r\chi}{2}\right)^n = \frac{1}{2}(2\chi)^{n-1} r^n \frac{1}{4^n} \leq \frac{1}{2}(2\chi)^{n-1} r^n b_n.$$

Setting $a_n = \|v_n^0\|$ for $n = 1, 2, \ldots, N$ and $a_n = 0$ for $n > N$ yields a sequence of non-negative numbers that satisfy the conditions of Lemma 5.6. Thus,

$$\tilde{\gamma}_n \leq 2(2\chi)^{n-1} r^n b_n \leq (2\chi)^{n-1} r^n \quad \text{for} \quad n = 1, 2, \ldots, N.$$

Also from (5.12) we have $\sum_{n=1}^{N} \|v_n^0\| < \varepsilon_0$. By virtue of Remark 3.4 and Proposition 3.9, $\|v_N(\zeta)\| \leq \tilde{\gamma}_N e^{-\mathbf{Re}\zeta}$ for $\zeta \in D$. We obtain

$$\begin{aligned}
\|u_N(\zeta)\| &\leq \rho^{-N} (2\chi)^{N-1} r^N e^{-\mathbf{Re}\zeta} \\
&\leq (2\chi)^{N-1} r^N \left(\frac{2}{r\chi}\right)^N \left\{ \max_{1 \leq k \leq N} \left(\frac{\|u_k^0\|}{\varepsilon_0}\right)^{1/k} \right\}^N e^{-\mathbf{Re}\zeta} \\
&\leq \frac{4^N}{2\chi} \left\{ \max_{1 \leq k \leq N} \left(\frac{M_0}{\varepsilon_0}\right)^{1/k} \right\}^N e^{-\mathbf{Re}\zeta} \\
&\leq 2\varepsilon_0 \left\{ 4 \max(1, \varepsilon_0^{-1} M_0) \right\}^N e^{-\mathbf{Re}\zeta} \quad \text{for} \quad \zeta \in D.
\end{aligned}$$

□

A direct consequence of Lemma 5.7 allows $\|u_n^0\|$ to grow as a power function with exponent n.

Lemma 5.8. *Suppose* $\limsup_{n\to\infty} \|u_n^0\|^{1/n} < \infty$. *Then*

$$\|u_n(\zeta)\| \leq 2\varepsilon_0 M^n e^{-\mathbf{Re}\zeta}, \quad \zeta \in D, \tag{5.13}$$

for some $M > 0$, *where* D *is the domain defined by* (3.42).

Proof. Take $R > \limsup_{n\to\infty} \|u_n^0\|^{1/n}$. Then there is $M_0 > 0$ such that $\|u_n^0\| \leq M_0 R^n$ for all $n \in \mathbb{N}$. According to Remark 3.4 we can apply Theorem 5.7 to $v_n^0 = u_n^0/R^n$ and $v_n(\zeta) = u_n(\zeta)/R^n$. We obtain

$$\|v_n(\zeta)\| \leq 2\varepsilon_0 \left\{ 4 \max(1, \varepsilon_0^{-1} M_0) \right\}^n e^{-\mathbf{Re}\zeta} \quad \text{for} \quad \zeta \in D.$$

Taking $M = 4R \max(1, \varepsilon_0^{-1} M_0)$ finishes the proof. □

Combining Lemma 5.8 and the proof of Proposition 5.4 now gives the following results easily.

Proposition 5.9. *Suppose* $\limsup_{n\to\infty} \|u_n^0\|^{1/n} < \infty$. *Then there is* $T > 0$ *such that* $u(t) = \sum_{n=1}^\infty u_n(t)$ *is absolutely convergent in* V *uniformly for* $t \in [T, \infty)$, $u(t)$ *is a regular solution of the Navier–Stokes equations for* $t \ge T$ *and* $\sum_{n=1}^\infty u_n(t)$ *is the asymptotic expansion of* $u(t)$.

Theorem 5.10. *Suppose* $u^0 \in \mathcal{R}$ *and*

$$\limsup_{n\to\infty} \|W_n(0, u^0)\|^{1/n} < \infty.$$

Then there is $T > 0$ *such that*

$$u(t) = \sum_{n=1}^{\infty} W_n(t, u^0) e^{-nt}$$

is absolutely convergent in V *uniformly for* $t \in [T, \infty)$, $\sum_{n=1}^\infty W_n(t, u^0)e^{-nt}$ *is the asymptotic expansion of* $u(t)$, *and* $u(t)$ *is equal to the regular solution* $u(t, u^0)$ *of the Navier–Stokes equations* (2.6) *for all* $t \in [T, \infty)$.

Proposition 5.2 can be applied to the study of the normalization map. Indeed, we have

Corollary 5.11. *If* $\xi \in S_A$ *and* $\sum_{n=1}^\infty \|q_n(0, \xi)\| < \varepsilon_0$, *then* ξ *is in the range of the normalization map, i.e.,* $\xi = W(u^0)$ *where* $u^0 = \sum_{n=1}^\infty q_n(0, \xi) \in \mathcal{R}$.

Proof. From Proposition 5.2 we know that $u(t) = \sum_{n=1}^\infty q_n(t, \xi)e^{-nt}$ is the regular solution of the Navier–Stokes equations with initial condition u^0 defined above. In addition, the series $\sum_{n=1}^\infty q_n(t, \xi)e^{-nt}$ is the asymptotic expansion of the solution $u(t)$. Therefore, by the definition of the normalization map in Subsection 2.2 and by (2.12) we have

$$W(u^0) = (R_n q_n(0, \xi))_{n=1}^\infty = \xi.$$

□

Similarly, Proposition 5.9 has the following consequence concerning the solutions of the normal form.

Corollary 5.12. *If* $\xi \in S_A$ *and* $\limsup_{n\to\infty} \|q_n(0, \xi)\|^{1/n} < \infty$, *then the solution* $\xi(t) = (R_n q_n(t, \xi))_{n=1}^\infty$ *of the normal form* (2.15) *with initial condition* ξ *is in the range of the normalization map for* $t \in [T, \infty)$ *for some* $T > 0$. *More precisely, there is* $u^\star \in \mathcal{R}$ *and* $T > 0$ *such that* $\xi(t) = W(u(t - T, u^\star))$ *for* $t \in [T, \infty)$, *where* $u(\cdot, u^\star)$ *denotes the regular solution of* (2.6) *with initial condition* $u^\star$.

Proof. From Proposition 5.9, $u(t) = \sum_{n=1}^\infty q_n(t, \xi)e^{-nt}$ is a regular solution of the Navier–Stokes equations. Let $u^\star = u(T)$. Hence $u^\star \in \mathcal{R}$. For each $t \ge T$ fixed, let $v^\star = u(t) = u(t - T, u^\star)$. Then the regular solution $u(\tau, v^\star)$ with initial condition $v^\star$ has the asymptotic expansion $\sum_{n=1}^\infty q_n(\tau + t, \xi)e^{-n(\tau+t)}$ as $\tau \to \infty$. Hence

$$W(u(t)) = W(v^\star) = (R_n q_n(t, \xi)e^{-nt})_{n=1}^\infty = \xi(t).$$

□

6. A norms connection between the normalization map and the asymptotic expansion

We know from Proposition 5.9 of the previous section that if $\xi \in S_A$ satisfies

$$\frac{1}{\rho_0} = \limsup_{n\to\infty} \|q_n(0,\xi)\|^{1/n} > 0, \tag{6.1}$$

then $\sum_{n=1}^{\infty} q_n(t,\xi)e^{-nt}$ is absolutely convergent in V to a regular solution $u(t)$ at least for large time t. Moreover, according to Corollary 3.5, for any $\rho \in (0,\rho_0)$, $\sum_{n=1}^{\infty} \rho^n q_n(t,\xi)e^{-nt}$ is absolutely convergent in V to a regular solution $v_\rho(t)$ on $[0,T_\rho)$, for some positive $T_\rho \le \infty$. However, up to now, the condition (6.1) is not known to hold even for a simple value of ξ such as $(\xi_1,0,0,\ldots)$. This is due to the presence of the nonlinear term in the recursive formula (2.14). From another point of view, $\bar{u}(t) = (u_n(t))_{n=1}^{\infty} = (q_n(t,\xi)e^{-nt})_{n=1}^{\infty}$ is a solution to the extended Navier–Stokes equations (3.2). The two results recalled above are concerned with $\bar{u}(t)$ having finite norms $\|\bar{u}(t)\|_* = \sum_{n=1}^{\infty} \rho_n\|u_n(t)\|$, where $\rho_n = 1$ for all n or $\rho_n = \rho^n$ for all n. However, the study in Section 4 shows that with suitably positive weights $(\rho_n)_{n=1}^{\infty}$, $\|\bar{u}(t)\|_*$ is finite for all time $t > 0$ provided $\|\bar{u}(0)\|_*$ is finite initially. This raises a question of whether or not the latter holds when $\|\xi\|_*$ is finite. Overall, we want to study the connection between $\|\xi\|_*$ and $\|(q_n(0,\xi))_{n=1}^{\infty}\|_*$ and, subsequently, $\|(q_n(t,\xi)e^{-nt})_{n=1}^{\infty}\|_*$ for $t > 0$, with appropriate $(\rho_n)_{n=1}^{\infty}$.

This section is devoted to estimating $\|(q_n(t,\xi)e^{-nt})_{n=1}^{\infty}\|_*, t \ge 0$, in terms of $\|\xi\|_*$. It is rather technical and hence we postpone to the next section the definition of our suitable normed spaces and further study of the normalization map and the normal form in those spaces.

Throughout this section, $\xi = (\xi_1,\xi_2,\xi_3,\ldots) \in S_A$ and $q_n(t) = q_n(t,\xi)$ for $n \in \mathbb{N}, t \in \mathbb{R}$. Let $q_n(\zeta), \zeta \in \mathbb{C}$, be the analytic extension of the polynomials $q_n(t), t \in \mathbb{R}$. Let $u_n(\zeta) = q_n(\zeta)e^{-n\zeta}$, for $n \in \mathbb{N}, \zeta \in \mathbb{C}$, $u_n^0 = q_n(0)$, and $\bar{u}(\zeta) = (u_n(\zeta))_{n=1}^{\infty}$.

We consider only positive weights ρ_n, $n \in \mathbb{N}$, satisfying

$$\rho_n \le \min\{\rho_k\rho_j : k + j = n\}, \tag{6.2}$$

for $n = 2,3,4,\ldots$.

This particular condition (6.2) is satisfied by the ρ_n in Definition 4.1 when n is sufficiently large, according to (4.4) and (4.3). Imposing (6.2) for all $n \ge 2$ therefore is natural and it will result in straightforward estimates as shown in Proposition 6.1 below.

We denote

$$S_n(\zeta) = \sum_{j=1}^{n} \rho_j\|u_j(\zeta)\|, \qquad S_n = S_n(0), \qquad n \in \mathbb{N}, \zeta \in \mathbb{C}.$$

First of all, to estimate $\|\bar{u}(t)\|_*$ for $t > 0$ in terms of $\|\bar{u}(0)\|_*$, we use the following version of Proposition 3.9 which is obtained by applying the results in Section 3 to $\rho_n u_n(\zeta)$ and the above $S_n(\zeta)$.

Proposition 6.1. *Suppose $S_N < \varepsilon_0$ for some $N > 0$, where ε_0 is defined by (3.41). Let*

$$\gamma_1 = \rho_1\|u_1^0\|, \quad \gamma_n = \rho_n\|u_n^0\| + \frac{1}{4\varepsilon_0}\sum_{k+j=n} \gamma_k\gamma_j, \quad 1 < n \le N,$$

and

$$\tilde{\gamma}_1 = \rho_1\gamma_1, \quad \tilde{\gamma}_n = \rho_n\gamma_n + \frac{1}{8\varepsilon_0}\sum_{k+j=n}\tilde{\gamma}_k\tilde{\gamma}_j, \quad 1 < n \le N.$$

Then

$$\rho_n\|u_n(t)\| \le \gamma_n e^{-t}, \quad t > 0, 1 \le n \le N, \tag{6.3}$$

$$\rho_n\|u_n(\zeta)\| \le \tilde{\gamma}_n e^{-\mathbf{Re}\zeta}, \quad \zeta \in D, 1 \le n \le N, \tag{6.4}$$

where the domain D is defined in (3.42)*, and*

$$\sum_{k=1}^{n}\tilde{\gamma}_j \le 2\sum_{k=1}^{n}\gamma_j \le 4S_n, \quad 1 \le n \le N. \tag{6.5}$$

In particular,

$$S_n(t) \le 2S_n e^{-t}, \quad t \ge 0, 1 \le n \le N, \tag{6.6}$$

and

$$S_n(\zeta) \le 4S_n e^{-\mathbf{Re}\zeta}, \quad \zeta \in D, 1 \le n \le N. \tag{6.7}$$

We now estimate $\|q_n(0)\|$ provided $\|\xi_n\|$ and $\|q_j(0)\|$ for $j < n$ are given.

Lemma 6.2. *Let $n \ge 2$ and $0 < \sigma_n \le \min\{\rho_k\rho_j : k + j = n$ and $k, j \in \mathbb{N}\}$. Suppose $S_{n-1} < \varepsilon_0$, then*

$$\|q_n(0)\| \le \|\xi_n\| + \frac{L_{3,n}}{\sigma_n}\sum_{k+j=n}\tilde{\gamma}_k\tilde{\gamma}_j, \tag{6.8}$$

where $\tilde{\gamma}_k$ for $k = 1, 2, \ldots, n-1$ are defined in Proposition 6.1 and $L_{3,n} = L_{1,n}+L_{2,n}$, where

$$L_{1,n} = C_4(C_3/2)^{n-2}(n-2)^{3/4}n^2(n-2)!, \tag{6.9}$$

$$L_{2,n} = C_4(C_3/2)^{n-2}n^{3/2}n^2(n-2)!, \tag{6.10}$$

$$C_4 = C_4(a_0) = 8^{-1}C_1e^{2a_0}, \tag{6.11}$$

and the positive numbers a_0 and C_3 are defined in Lemma 5.1.

Proof. Let $\check{q}_n(t) = P_{n-1}q_n(t)$ where P_n is the projection given in (2.7). Note that Lemma A.2 implies that $\check{q}_n(t) = P_{n-2}q_n(t)$. Upon projecting (2.11) we have

$$\frac{d}{dt}\check{q}_n(t) + (A-n)\check{q}_n(t) + P_{n-2}B_n(t) = 0.$$

Then

$$e^{t'(A-n)P_{n-2}}\check{q}_n(t') - e^{t(A-n)P_{n-2}}\check{q}_n(t) = -\int_t^{t'} e^{\tau(A-n)P_{n-2}}P_{n-2}B_n(\tau)d\tau.$$

Letting $t' \to \infty$, we obtain

$$\check{q}_n(t) = \int_t^{\infty} e^{(\tau-t)(A-n)P_{n-2}}P_{n-2}B_n(\tau). \tag{6.12}$$

Now set $t = 0$ and use inequality (A.2) in Lemma A.1 to estimate

$$\begin{aligned}\|\check{q}_n(0)\| &\le \int_0^\infty e^{-2\tau}\|P_{n-2}B_n(\tau)\|d\tau \\ &\le C_1(n-2)^{3/4}\int_0^\infty e^{-2\tau}\sum_{k+j=n}\|q_k(\tau)\|\|q_j(\tau)\|d\tau.\end{aligned}$$

Multiply the above inequality by σ_n and apply the estimates in Lemma 5.1 for $\rho_n q_n$ instead of q_n alone to obtain

$$\begin{aligned}\sigma_n\|\check{q}_n(0)\| &\le C_1(n-2)^{3/4}\int_0^\infty e^{-2\tau}\sum_{k+j=n}\rho_k\|q_k(\tau)\|\rho_j\|q_j(\tau)\|d\tau \\ &\le C_1(n-2)^{3/4}C_3^{n-2}\Big(\sum_{k+j=n}k\tilde{\gamma}_k j\tilde{\gamma}_j\Big)\int_0^\infty e^{-2\tau}(\tau+a_0)^{n-2}d\tau \\ &\le C_1(n-2)^{3/4}\frac{n^2}{4}C_3^{n-2}\Big(\sum_{k+j=n}\tilde{\gamma}_k\tilde{\gamma}_j\Big)\int_0^\infty e^{-2\tau}(\tau+a_0)^{n-2}d\tau.\end{aligned}$$

Take $y = 2(\tau + a_0)$ then

$$\int_0^\infty e^{-2\tau}(\tau+a_0)^{n-2}d\tau \le 2^{-1}e^{2a_0}\int_0^\infty e^{-y}(y/2)^{n-2}dy = \frac{e^{2a_0}}{2^{n-1}}\Gamma(n-1).$$

We obtain

$$\sigma_n\|P_{n-1}q_n(0)\| \le L_{1,n}\sum_{k+j=n}\tilde{\gamma}_k\tilde{\gamma}_j. \tag{6.13}$$

Let $\hat{q}(\zeta) = Q_n q_n(\zeta)$. By Lemmas A.2 and A.3 we write $\hat{q}(\zeta) = P_{n^2}Q_{n+1}q_n(\zeta)$. Similar to the estimate of $\check{q}_n(0)$, we have

$$\hat{q}_n(0) = -\int_{-\infty}^0 e^{\tau(A-n)(P_{n^2}-P_{n+1})}(P_{n^2}-P_{n+1})B_n(\tau)d\tau.$$

Thus,

$$\begin{aligned}\sigma_n\|\hat{q}_n(0)\| &\le C_1 n^{3/2}\int_{-\infty}^0 e^{2\tau}\sum_{k+j=n}\rho_k\|q_k(\tau)\|\rho_j\|q_j(\tau)\|d\tau \\ &\le C_1 n^{3/2}\frac{n^2}{4}C_3^{n-2}\Big(\sum_{k+j=n}\tilde{\gamma}_k\tilde{\gamma}_j\Big)\int_{-\infty}^0 e^{2\tau}(|\tau|+a_0)^{n-2}d\tau.\end{aligned}$$

We obtain

$$\sigma_n\|Q_n q_n(0)\| \le L_{2,n}\sum_{k+j=n}\tilde{\gamma}_k\tilde{\gamma}_j. \tag{6.14}$$

Finally, combine (6.13) and (6.14) along with (2.12) to obtain (6.8). □

We now bootstrap the above estimates to obtain bounds for the asymptotic expansions solely in terms of $\xi = (\xi_1, \xi_2, \ldots)$ using the weighted norm $\|\cdot\|_*$ introduced in Section 4 with the weights $\rho_n, n \in \mathbb{N}$, being specified below. Our choice of these ρ_n is suitable not only for our norm estimates in this section but also for our further study in the next section.

Definition 6.3. Let $(\alpha_n)_{n=1}^{\infty}$ be a sequence of numbers satisfying

$$\alpha_1 \geq 0, \quad \alpha_n > 0 \text{ for } n > 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n \leq 1/2. \tag{6.15}$$

Construct the sequence $(\rho_n)_{n=1}^{\infty}$ as follows: let $\rho_1 = 1$ and for $n > 1$ define

$$\sigma_n = \min\{\, \rho_k \rho_j : k + j = n \text{ and } k, j \in \mathbb{N}\,\}. \tag{6.16}$$

Then let

$$0 < \rho_n = \sigma_n \min\Big\{1, \frac{\alpha_n}{16\varepsilon_0 \max\{1, L_{3,n}, C_1 n^{3/2}\}}\Big\}, \quad n > 1, \tag{6.17}$$

where ε_0 is the constant defined by (3.41) and $L_{3,n}$ is defined in Lemma 6.2.

Proposition 6.4. *Let $(\rho_n)_{n=1}^{\infty}$ be as in Definition 6.3. If $\sum_{j=1}^{\infty} \rho_j \|\xi_j\| = \delta < \varepsilon_0/2$ then*

$$S(0) \stackrel{\text{def}}{=\!=} \sum_{j=1}^{\infty} \rho_j \|q_j(0)\| \leq 2\delta, \tag{6.18}$$

$$S(t) \stackrel{\text{def}}{=\!=} \sum_{j=1}^{\infty} \rho_j \|u_j(t)\| \leq 4\delta e^{-t} \quad \textit{for} \quad t > 0 \tag{6.19}$$

and

$$S(\zeta) \stackrel{\text{def}}{=\!=} \sum_{j=1}^{\infty} \rho_j \|u_j(\zeta)\| \leq 8\delta e^{-\mathbf{Re}\zeta} < 4\varepsilon_0 e^{-\mathbf{Re}\zeta} \quad \textit{for} \quad \zeta \in D, \tag{6.20}$$

where D is defined by (3.42).

Proof. We will prove by induction that

$$\sum_{j=1}^{n} \rho_j \|q_j(0)\| \leq 2\delta \quad \text{and} \quad \rho_n \|q_n(0)\| \leq \rho_n \|\xi_n\| + 2\delta\alpha_n \quad \text{for} \quad n \in \mathbb{N}. \tag{6.21}$$

Since $q_1(t) = \xi_1$ and $\rho_1 = 1$, then (6.21) holds for $n = 1$. Suppose (6.21) holds for all $n < N$. Since $S_{N-1} = \sum_{j=1}^{N-1} \rho_j \|q_j(0)\| \leq 2\delta < \varepsilon_0$, we have $\sum_{n=1}^{N-1} \tilde{\gamma}_n \leq 4S_{N-1} \leq 8\delta$ by Proposition 6.1. By Lemma 6.2 followed by (6.17) it follows that

$$\begin{aligned} \rho_n \|q_n(0)\| &\leq \rho_n \|\xi_n\| + \frac{\rho_n L_{3,n}}{\sigma_n} \Big(\sum_{n=1}^{n-1} \tilde{\gamma}_n \Big)^2 \\ &\leq \rho_n \|\xi_n\| + \frac{\alpha_n}{16\varepsilon_0} (8\delta)^2 \\ &< \rho_n \|\xi_n\| + 2\delta\alpha_n \quad \text{for} \quad 1 \leq n \leq N. \end{aligned} \tag{6.22}$$

In particular,

$$\rho_N \|q_N(0)\| \leq \rho_N \|\xi_N\| + 2\delta\alpha_N.$$

Summing up (6.22) from $n = 1$ to N yields

$$\sum_{n=1}^{N} \rho_n \|q_n(0)\| \leq \sum_{n=1}^{N} \rho_n \|\xi_n\| + 2\delta \sum_{n=1}^{N} \alpha_n \leq \delta + \delta = 2\delta.$$

Therefore, by the induction principle, (6.21) is true for all $n \in \mathbb{N}$. Consequently (6.18) holds. Now (6.19) and (6.20) follow from (6.6) and (6.7). $\square$

7. Application to the normal form

Considering Definition 6.3, we note that $\rho_n = \kappa_n C_1^{-1}\sigma_n, (n > 1)$, where for n large enough $\kappa_n = C_1\alpha_n/(16\varepsilon_0 L_{3,n})$ satisfies (4.3). This property of $(\rho_n)_{n=1}^\infty$ and Proposition 6.4 suggest the natural normed spaces for our study of the normalization map and its associated normal form of the Navier–Stokes equations be those $(V_*, \|\cdot\|_*)$ occurring in the study of the extended Navier–Stokes equations in Section 4. We specifically define some of those suitable normed spaces below. In those spaces, we study the continuity of the formal inverse of the normalization map, the continuous dependence of the formal solutions to the Navier–Stokes equations on their generators ξ of S_A, and the well-posedness of the normal form (2.15) of the Navier–Stokes equations.

Let $V_{\mathbb{C}}$ and $(R_nH)_{\mathbb{C}}$, for $n \in \mathbb{N}$, be the complexifications of V and R_nH, respectively (see the subsection 2.3). Define the complex linear space

$$V_{\mathbb{C}}^\infty = \{\, u = (u_n)_{n=1}^\infty : u_n \in V_{\mathbb{C}} \,\},$$

its subspace

$$(S_A)_{\mathbb{C}} = (R_1H)_{\mathbb{C}} \oplus (R_2H)_{\mathbb{C}} \oplus \cdots,$$

and recall that the real linear space V^∞ is $V \oplus V \oplus V \oplus \cdots$.

For $\bar{u} = (u_n)_{n=1}^\infty \in V_{\mathbb{C}}^\infty$ define

$$\|\bar{u}\|_\star = \sum_{n=1}^\infty \rho_n \|u_n\|,$$

where $(\rho_n)_{n=1}^\infty$ is the particular sequence of positive numbers given in Definition 6.3. Let

$$V_{\mathbb{C}}^\star = \{\, u \in V_{\mathbb{C}}^\infty : \|u\|_\star < \infty \,\}, \quad V^\star = \{\, u \in V^\infty : \|u\|_\star < \infty \,\},$$

and

$$R^\star = \{\, u \in S_A : \|u\|_\star < \infty \,\}.$$

Clearly, the spaces $V_{\mathbb{C}}^\star, V^\star$ and $R^\star$ with the norm $\|\cdot\|_\star$ are complete normed spaces. As noted earlier, these specific $\rho_n, n \in N$, agree with Definition 4.1, and hence the space $V^\star$ is one of the spaces V_* in Section 4, with respect to which the extended Navier–Stokes equations have unique global solutions. The same will be shown to hold for the normal form of the Navier–Stokes equations under a certain small initial data condition.

Define the functions

$$F(\xi,\zeta) = \big(e^{-n\zeta} q_n(\zeta,\xi)\big)_{n=1}^\infty \in V_{\mathbb{C}}^\infty, \quad \zeta \in \mathbb{C},\ \xi \in S_A,$$

and

$$G(\xi,\zeta) = \big(e^{-n\zeta} R_n q_n(\zeta,\xi)\big)_{n=1}^\infty \in (S_A)_{\mathbb{C}}, \quad \zeta \in \mathbb{C},\ \xi \in S_A,$$

where the polynomials $q_n(\zeta,\xi)$ in ζ are generated by ξ as defined in (2.14), first for real ζ and then are extended for complex ζ.

For $\xi \in S_A,\ F(\xi,t), t \geq 0$, is considered as the formal solution to the Navier–Stokes equations generated by ξ, since the sum $\sum_{n=1}^\infty q_n(t,\xi)e^{-nt}$ may be convergent (in V) to and be the asymptotic expansion of a regular solution $u(t)$ for all $t \geq 0$. In particular, when $\xi = W(u^0)$, some $u^0 \in \mathcal{R}$, $F(\xi,0)$ is viewed as the formal inverse of the normalization map W, because $W^{-1}(\xi) = u^0 = \sum_{n=1}^\infty q_n(0,\xi)$ holds in particular cases but is not yet known to be true or false in general. For example, for any given eigenvalue N of the Stokes operator, let $\xi = (\xi_n)_{n=1}^\infty$, where $\xi_n = 0$ if $n \neq N$, and $\xi_N \in R_NH$ such that $B(\xi_N,\xi_N) = 0$. In this case,

$F(\xi, t) = (0, \dots, 0, e^{-Nt}\xi_N, 0, \dots)$, for $t \geq 0$; the sum $\sum_{n=1}^{\infty} q_n(t, \xi)e^{-nt}$ is simply $e^{-Nt}\xi_N = u(t)$, a regular solution; and $W^{-1}(\xi) = u(0) = \xi_N$.

We further note that for $\xi \in S_A$, $G(\xi, t)$ also belongs to S_A for all $t \geq 0$, and is the solution to the normal form (2.15) with the initial value $\xi^0 = \xi$.

Using the above notation we can restate Proposition 6.4 as

Theorem 7.1. *If $\xi \in R^\star$ and $\|\xi\|_\star < \varepsilon_0/2$, then $F(\xi, t) \in V^\star$ for all $t \geq 0$ and $F(\xi, \zeta) \in V^\star_{\mathbb{C}}$ for all $\zeta \in D$. Moreover*

$$\|F(\xi, 0)\|_\star \leq 2\|\xi\|_\star,$$

$$\|F(\xi, t)\|_\star \leq 4\|\xi\|_\star e^{-t}, \quad t > 0,$$

and

$$\|F(\xi, \zeta)\|_\star \leq 8\|\xi\|_\star e^{-\mathbf{Re}\zeta}, \quad \zeta \in D.$$

In particular, $G(\xi, t) \in R^\star$ for all $t \geq 0$ and

$$\|G(\xi, t)\|_\star \leq 4\|\xi\|_\star e^{-t}, \quad t \geq 0.$$

First we show a Lipschitz-like property of $F(\xi, \zeta)$ connected with $F(\xi, 0)$.

Lemma 7.2. *Suppose $\xi, \chi \in R^\star$ and $\|\xi\|_\star, \|\chi\|_\star < \varepsilon_0/2$. Then*

$$\|F(\xi, t) - F(\chi, t)\|_\star \leq e^{1/8}e^{-t}\|F(\xi, 0) - F(\chi, 0)\|_\star, \quad t \geq 0, \tag{7.1}$$

and

$$\|F(\xi, \zeta) - F(\chi, \zeta)\|_\star \leq e^{7/8}e^{-\mathbf{Re}\zeta}\|F(\xi, 0) - F(\chi, 0)\|_\star, \quad \zeta \in D, \tag{7.2}$$

where D is the domain defined by (3.42). *Moreover,*

$$\|F(\xi, t) - F(\chi, t)\|_\star \leq e^{-7t/8}\|F(\xi, 0) - F(\chi, 0)\|_\star, \quad t \geq 0, \tag{7.3}$$

and

$$\|F(\xi, \zeta) - F(\chi, \zeta)\|_\star \leq e^{-\mathbf{Re}\zeta/4}\|F(\xi, 0) - F(\chi, 0)\|_\star, \quad \zeta \in D. \tag{7.4}$$

Proof. Let $q_n(\zeta) = q_n(\zeta, \xi)$ and $p_n(\zeta) = q_n(\zeta, \chi)$ be the polynomials generated by ξ and χ, respectively. Let $u_n(\zeta) = e^{-n\zeta}q_n(\zeta)$, $v_n(\zeta) = e^{-n\zeta}p_n(\zeta)$ and $w_n(\zeta) = u_n(\zeta) - v_n(\zeta)$. Note that $u_n(\zeta), v_n(\zeta)$ and $w_n(\zeta)$ belong to $P_{n^2}H$, by Lemma A.3. Define

$$X_n(\zeta) = \sum_{j=1}^{n} \rho_j \|u_j(\zeta)\|, \qquad X_n = X_n(0),$$

$$Y_n(\zeta) = \sum_{j=1}^{n} \rho_j \|v_j(\zeta)\|, \qquad Y_n = Y_n(0),$$

and

$$Z_n(\zeta) = \sum_{j=1}^{n} \rho_j \|w_j(\zeta)\|, \qquad Z_n = Z_n(0).$$

According to Theorem 7.1 we have for each $n \in \mathbb{N}$ that

$$X_n(t), Y_n(t) \leq 2\varepsilon_0 e^{-t}, \quad t \geq 0, \tag{7.5}$$

and

$$X_n(\zeta), Y_n(\zeta) \leq 4\varepsilon_0 e^{-\mathbf{Re}\zeta}, \quad \zeta \in D. \tag{7.6}$$

Subtracting the evolution equations (3.4) for u_n and v_n we find that w_n satisfies

(7.7)
$$\frac{d}{d\zeta}w_n(\zeta) + Aw_n(\zeta) + \sum_{k+j=n}\Big\{B(w_k(\zeta), u_j(\zeta)) + B(v_j(\zeta), w_k(\zeta))\Big\} = 0, \quad \zeta \in D.$$

Take the inner product of (7.7) with Aw_n and apply Lemma A.1 to obtain

$$\frac{d}{dt}\|w_n(t)\| + \|w_n(t)\| \le C_1 n^{3/2} \sum_{k+j=n} \|w_k(t)\|\big(\|u_j(t)\| + \|v_j(t)\|\big), \quad t > 0.$$

Hence

$$\begin{aligned}\rho_n\frac{d}{dt}\|w_n(t)\| + \rho_n\|w_n(t)\| &\le \frac{\rho_n C_1 n^{3/2}}{\sigma_n}\sum_{k+j=n}\rho_k\|w_k(t)\|\big(\rho_j\|u_j(t)\| + \rho_j\|v_j(t)\|\big)\\ &\le \frac{\alpha_n}{16\varepsilon_0}\sum_{k+j=n}\rho_k\|w_k(t)\|\big(\rho_j\|u_j(t)\| + \rho_j\|v_j(t)\|\big)\\ &\le \frac{1}{32\varepsilon_0}\sum_{k+j=n}\rho_k\|w_k(t)\|\big(\rho_j\|u_j(t)\| + \rho_j\|v_j(t)\|\big).\end{aligned}$$

Sum up from $n = 1$ to N and use (7.5). We have

$$\begin{aligned}\frac{d}{dt}Z_N(t) + Z_N(t) &\le \frac{1}{32\varepsilon_0}Z_{N-1}(t)(X_{N-1}(t) + Y_{N-1}(t))\\ &\le \frac{4\varepsilon_0 e^{-t}}{32\varepsilon_0}Z_{N-1}(t) \le \frac{e^{-t}}{8}Z_N(t),\end{aligned}$$

hence

$$\frac{d}{dt}Z_N(t) \le (\frac{1}{8e^t} - 1)Z_N(t). \tag{7.8}$$

Simple estimating the right hand side by $(-7Z_N(t)/8)$ gives

$$Z_N(t) \le Z_N e^{-7t/8}, \tag{7.9}$$

for all $N \in \mathbb{N}$ and $t \ge 0$. Letting $N \to \infty$ obtains (7.3). We also have from (7.8)

$$Z_N(t) \le Z_N e^{-t+\frac{1}{8}(1-e^{-t})} \le e^{1/8}Z_N e^{-t}, \quad N \in \mathbb{N}. \tag{7.10}$$

Letting $N \to \infty$ we obtain (7.1).

We next deal with $Z_n(\zeta)$ with $\zeta \in D$. First, from (7.7) and Lemma A.1 we have

$$\begin{aligned}&\frac{d}{ds}\|w_n(\tau_0 + se^{i\theta})\| + \cos\theta\|w_n(\tau_0 + se^{i\theta})\|\\ &\qquad\le C_1 n^{3/2}\sum_{k+j=n}\|w_k\|\big(\|u_j\| + \|v_j\|\big)\Big|_{\zeta=\tau_0+se^{i\theta}}.\end{aligned}$$

Therefore

$$\begin{aligned}&\rho_n\frac{d}{ds}\|w_n(\tau_0 + se^{i\theta})\| + \rho_n\cos\theta\|w_n(\tau_0 + se^{i\theta})\|\\ &\qquad\le \frac{C_1 n^{3/2}\rho_n}{\sigma_n}\sum_{k+j=n}\rho_k\|w_k\|\big(\rho_j\|u_j\| + \rho_j\|v_j\|\big)\Big|_{\zeta=\tau_0+se^{i\theta}}\\ &\qquad\le \frac{1}{32\varepsilon_0}\sum_{k+j=n}\rho_k\|w_k\|\big(\rho_j\|u_j\| + \rho_j\|v_j\|\big)\Big|_{\zeta=\tau_0+se^{i\theta}}.\end{aligned}$$

Summing up

$$\begin{aligned}\frac{d}{ds}Z_N(\tau_0+se^{i\theta})&+(\cos\theta)Z_N(\tau_0+se^{i\theta})\\&\le\frac{1}{32\varepsilon_0}Z_{N-1}(\tau_0+se^{i\theta})\big(X_{N-1}(\tau_0+se^{i\theta})+Y_{N-1}(\tau_0+se^{i\theta})\big)\\&\le\frac{8\varepsilon_0e^{-\tau_0-s\cos\theta}}{32\varepsilon_0}Z_{N-1}(\tau_0+se^{i\theta}),\end{aligned}$$

hence

$$\frac{d}{ds}Z_N(\tau_0+se^{i\theta})\le(4^{-1}e^{-\tau_0}e^{-s\cos\theta}-\cos\theta)Z_N(\tau_0+se^{i\theta}).\tag{7.11}$$

Combining with (7.10), we obtain

$$\begin{aligned}Z_N(\tau_0+se^{i\theta})&\le Z_N(\tau_0)\exp\Big\{-s\cos\theta+\frac{e^{-\tau_0}}{4\cos\theta}(1-e^{-s\cos\theta})\Big\}\\&\le Z_Ne^{-\tau_0-s\cos\theta}e^{1/8}\exp\Big\{\frac{e^{-\tau_0}}{4\cos\theta}\Big\}.\end{aligned}$$

Proposition 3.8 implies $\tau_0+se^{i\theta}\in D\subset E(\varepsilon_0)$. Therefore

$$\cos\theta>8C_1\varepsilon_0e^{-\tau_0}=\frac{1}{3}e^{-\tau_0}.\tag{7.12}$$

It follows that

$$Z_N(\tau_0+se^{i\theta})\le Z_Ne^{-\tau_0-s\cos\theta}e^{1/8}e^{3/4}=e^{7/8}Z_Ne^{-\tau_0-s\cos\theta}.\tag{7.13}$$

Letting $N\to\infty$ obtains (7.2). Now, using (7.12) in (7.11) gives

$$\frac{d}{ds}Z_N(\tau_0+se^{i\theta})\le-\frac{\cos\theta}{4}Z_N(\tau_0+se^{i\theta}),$$

thus

$$Z_N(\tau_0+se^{i\theta})\le Z_N(\tau_0)e^{-\frac14 s\cos\theta}\le Z_Ne^{-\frac14(\tau_0+s\cos\theta)},$$

where we used (7.9) to estimate $Z_N(\tau_0)$. Then (7.4) follows taking $N\to\infty$. □

We now connect $\|q_n(0)-p_n(0)\|$ with $\|\xi_j-\chi_j\|$, for $j=1,2,\dots,n$ under the same conditions given in Lemma 7.2.

Lemma 7.3. *Suppose $\xi,\chi\in R^\star$ and $\|\xi\|_\star,\|\chi\|_\star<\varepsilon_0/2$. Let $r_n(\zeta)=q_n(\zeta,\xi)-q_n(\zeta,\chi)$, for $\zeta\in\mathbb{C}$. Then*

$$\rho_n\|r_n(\zeta)\|\le e^{7/8}nZ_nC_3^{n-1}(|\zeta|+a_0)^{n-1},\quad\zeta\in\mathbb{C},\tag{7.14}$$

and

$$\|r_n(0)\|\le\|\xi_n-\chi_n\|+\frac{8e^{7/8}\varepsilon_0L_{3,n}}{\sigma_n}Z_n,\tag{7.15}$$

where $Z_n=\sum_{j=1}^n\rho_j\|r_n(0)\|$, the constants C_3 and a_0 are defined in Lemma 5.1 and $L_{3,n}$ is defined in Lemma 6.2.

Proof. Using the same notation as in Lemma 7.2, we have

$$\rho_n\|e^{-(n-1)\zeta}r_n(\zeta)\|\le e^{7/8}Z_n,\quad\zeta\in D,$$

by (7.13) and the fact that $w_n(\zeta) = e^{-n\zeta} r_n(\zeta)$, for $n \geq 1$. Applying Corollary B.6, as in the proof of Lemma 5.1, we obtain (7.14). From (2.11) we derive an evolution equation for r_n analogous to (7.7) expressed as

$$\frac{d}{dt} r_n(t) + (A-n) r_n(t) + \sum_{k+j=n} B(r_k(t), p_j(t)) + B(q_j(t), r_k(t)) = 0, \quad t > 0. \tag{7.16}$$

Estimate as in the proof of Lemma 6.2 writing $r_n = \check{r}_n + R_n r_n + \hat{r}_n$ where $\check{r}_n = P_{n-2} r_n$ and $\hat{r}_n = P_{n^2} Q_{n+1} r_n$. Thus,

$$\check{r}_n(0) = \int_0^{-\infty} e^{\tau(A-n)P_{n-2}} P_{n-2} \sum_{k+j=n} \Big\{ B(r_k(\tau), p_j(\tau)) + B(q_j(\tau), r_k(\tau)) \Big\} d\tau$$

and applying Lemma A.1 gives

$$\sigma_n \|\check{r}_n(0)\| \leq C_1 (n-2)^{3/4} \int_0^{\infty} e^{-2\tau} \sum_{k+j=n} \Big\{ \rho_k \|r_k(\tau)\| (\rho_j \|p_j(\tau)\| + \rho_j \|q_j(\tau)\|) \Big\} d\tau.$$

Using (7.14) and following the calculations in the proof of Lemma 6.2, we have

$$\sigma_n \|\check{r}_n(0)\| \leq L_{1,n} e^{7/8} Z_n \sum_{j=1}^{n-1} (\tilde{\gamma}_{j,\xi} + \tilde{\gamma}_{j,\chi}) \leq 8 e^{7/8} \varepsilon_0 L_{1,n} Z_n,$$

where $\tilde{\gamma}_{j,\xi}$ and $\tilde{\gamma}_{j,\chi}$ for $j \in \mathbb{N}$ are defined in Proposition 6.1 with $u_j^0 = q_j(0)$ and $u_j^0 = p_j(0)$, respectively. Similarly,

$$\sigma_n \|\hat{r}_n(0)\| \leq 8 e^{7/8} \varepsilon_0 L_{2,n} Z_n.$$

Therefore,

$$\begin{aligned} \|r_n(0)\| &\leq \|R_n r_n(0)\| + \|\check{r}_n(0\| + \|\hat{r}_n(0)\| \\ &\leq \|\xi_n - \chi_n\| + \frac{8 e^{7/8} \varepsilon_0 (L_{1,n} + L_{2,n})}{\sigma_n} Z_n, \end{aligned}$$

which proves (7.15), for, $L_{3,n} = L_{1,n} + L_{2,n}$. □

We finish by proving that $F(\xi, \zeta)$ and $G(\xi, \zeta)$ are Lipschitz continuous in ξ, uniformly for $\zeta \in D$.

Theorem 7.4. *Let $\xi, \chi \in R^\star$ and $\|\xi\|_\star < \varepsilon_0/2, \|\chi\|_\star < \varepsilon_0/2$, then*

$$\|F(\xi, 0) - F(\chi, 0)\|_\star \leq 4 \|\xi - \chi\|_\star, \tag{7.17}$$

$$\|G(\xi, t) - G(\chi, t)\|_\star \leq \|F(\xi, t) - F(\chi, t)\|_\star \leq 4 e^{1/8} e^{-t} \|\xi - \chi\|_\star, \tag{7.18}$$

for $t > 0$, and

$$\|G(\xi, \zeta) - G(\chi, \zeta)\|_\star \leq \|F(\xi, \zeta) - F(\chi, \zeta)\|_\star \leq 4 e^{7/8} e^{-\mathbf{Re}\zeta} \|\xi - \chi\|_\star, \tag{7.19}$$

for $\zeta \in D$.

Proof. We adopt the notation used in the previous two lemmas. Since $r_n(0) = w_n(0)$, then summing up (7.15) for $N > 1$ gives

$$\begin{aligned}
Z_N = \sum_{n=1}^N \rho_n \|r_n(0)\| &\le \sum_{n=1}^N \rho_n \|\xi_n - \chi_n\| + \sum_{n=1}^N \frac{8e^{7/8}\varepsilon_0 L_{3,n}\rho_n}{\sigma_n} Z_n \\
&\le \sum_{n=1}^N \rho_n \|\xi_n - \chi_n\| + \frac{e^{7/8}}{2} Z_N \sum_{n=1}^N \alpha_n \\
&\le \sum_{n=1}^N \rho_n \|\xi_n - \chi_n\| + \frac{3}{4} Z_N.
\end{aligned}$$

Therefore $Z_N \le 4\sum_{n=1}^N \rho_n \|\xi_n - \chi_n\|$. Letting $N \to \infty$, we obtain (7.17). We then apply Lemma 7.2 to obtain (7.18) and (7.19). □

Remark 7.5. According to Theorems 7.1 and 7.4, the formal inverse $F(\xi, 0)$ of the normalization map W is Lipschitz continuous from a small neighborhood $B_{R^\star}(0, \varepsilon_0/2) = \{\xi \in R^\star : \|\xi\|_\star < \varepsilon_0/2\}$ of the origin in $R^\star$ into $V^\star$. Similarly, the formal solution $F(\xi, t)$ to the Navier–Stokes equations generated by ξ in $B_{R^\star}(0, \varepsilon_0/2)$ belongs to $V^\star$ for all $t \ge 0$ and depends continuously on the generator ξ.

Also, the solution $G(\xi, t)$ of the normal form (2.15) depends continuously on such small initial data $\xi \in B_{R^\star}(0, \varepsilon_0/2)$. Hence the normal form (2.15) is a well-posed system in a neighborhood of zero in $R^\star$.

Appendix A. Auxiliary results

For the reader's convenience, we prove the following estimates for the bilinear form $B(\cdot,\cdot)$ (see (2.5) for its definition) which were used in this paper.

Lemma A.1. *There is an absolute constant $C_1 > 0$ such that*

$$|P_n B(u, v)| \le C_1 n^{1/4} \|u\| \|v\|, \quad u, v \in V, \tag{A.1}$$

$$\|P_n B(u, v)\| \le C_1 n^{3/4} \|u\| \|v\|, \quad u, v \in V, \tag{A.2}$$

$$|B(u, v)| \le C_1 \|u\|^{1/2} |Au|^{1/2} \|v\|, \quad u \in \mathcal{D}_A, v \in V, \tag{A.3}$$

$$\|B(u, v)\| \le C_1 \|u\|^{1/2} |Au|^{1/2} |Av|, \quad u, v \in \mathcal{D}_A. \tag{A.4}$$

Moreover, we can choose C_1 such that the complexified versions of the above inequalities still hold, i.e., when the operators P_n, B, A and the spaces $V, \mathcal{D}_A$ are replaced by $(P_n)_\mathbb{C}, B_\mathbb{C}, A_\mathbb{C}$ and $V_\mathbb{C}, (\mathcal{D}_A)_\mathbb{C}$, respectively.

Proof. These inequalities are the consequences of elementary inequalities and the Sobolev and Agmon inequalities (see, for example, [1, 2]). In the estimates below, the positive constant C is generic.

For $u, v, w \in \mathcal{V}$, we have

$$\begin{aligned} |\langle P_n B(u,v), w\rangle| &\le C\|u\|_{L^6}\|v\|\|P_n w\|_{L^3} \\ &\le C\|u\|\|v\|\|P_n w\|_{L^2}^{1/2}\|P_n w\|_{L^6}^{1/2} \\ &\le C\|u\|\|v\||P_n w|^{1/2}\|P_n w\|^{1/2} \\ &\le C\|u\|\|v\||P_n w|^{1/2}|P_n w|^{1/2}(n^{1/2})^{1/2} \\ &\le Cn^{1/4}\|u\|\|v\||w|. \end{aligned}$$

Hence we obtain (A.1). Since $\|P_n B(u,v)\| = |A^{1/2}P_n B(u,v)|$, inequality (A.2) immediately implies (A.2).

Inequality (A.3) follows from

$$\begin{aligned} \big|\big\langle B(u,v), w\big\rangle\big| &\le C\|u\|_{L^\infty}|A^{1/2}v||w| \\ &\le C\|u\|^{1/2}|Au|^{1/2}\|v\||w|. \end{aligned}$$

To prove (A.4), consider $u = \sum_{0\neq\mathbf{k}\in\mathbb{Z}^3}\hat{u}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}$, define the scalar function

$$u' = \sum_{0\neq\mathbf{k}\in\mathbb{Z}^3}\hat{u}'_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} \text{ where } \hat{u}'_{\mathbf{k}} = |\hat{u}_{\mathbf{k}}|.$$

Similarly define v' and w'. We estimate

$$\begin{aligned} \big|\big\langle B(u,v), Aw\big\rangle\big| &= \Big|\int_\Omega B(u,v)\cdot(Aw)^\star\Big| \\ &= L^3\Big|\sum_{\mathbf{h}+\mathbf{k}=\mathbf{l}}(\hat{u}_{\mathbf{h}}\cdot i\mathbf{k})\hat{v}_{\mathbf{k}}\cdot|\mathbf{l}|^2\hat{w}_{\mathbf{l}}^\star\Big| \\ &\le L^3\sum_{\mathbf{h}+\mathbf{k}=\mathbf{l}}|\hat{u}_{\mathbf{h}}||\mathbf{k}||\hat{v}_{\mathbf{k}}||\mathbf{l}|^2|\hat{w}_{\mathbf{l}}| \\ &\le L^3\sum_{\mathbf{h}+\mathbf{k}=\mathbf{l}}\big\{|\hat{u}_{\mathbf{h}}||\mathbf{k}|^2|\hat{v}_{\mathbf{k}}||\mathbf{l}||\hat{w}_{\mathbf{l}}| + |\mathbf{h}||\hat{u}_{\mathbf{h}}||\mathbf{k}||\hat{v}_{\mathbf{k}}||\mathbf{l}||\hat{w}_{\mathbf{l}}|\big\}. \end{aligned}$$

Converting the last sum back to the integral form, we obtain

$$\begin{aligned} \big|\big\langle B(u,v), Aw\big\rangle\big| &\le \int_\Omega u'(-\Delta v')[(-\Delta)^{1/2}w'] \\ &\quad + \int_\Omega [(-\Delta)^{1/2}u'][(-\Delta)^{1/2}v'][(-\Delta)^{1/2}w'] \\ &\le \|u'\|_{L^\infty}\|\Delta v'\|_{L^2}|\nabla w'| + \|\nabla u'\|_{L^3}\|\nabla v'\|_{L^6}|\nabla w'| \end{aligned}$$

Then by using the Sobolev and Agmon inequalities, we continue

$$\begin{aligned} \big|\big\langle B(u,v), Aw\big\rangle\big| &\le c_\infty|\nabla u'|^{1/2}\|\Delta u'\|_{L^2}^{1/2}\|\Delta v'\|_{L^2}|\nabla w'| \\ &\quad + c_6|\nabla u'|^{1/2}\|\Delta u'\|_{L^2}^{1/2}\|\Delta v'\|_{L^2}|\nabla w'| \\ &\le c_1|\nabla u'|^{1/2}\|\Delta u'\|_{L^2}^{1/2}\|\Delta v'\|_{L^2}|\nabla w'| \\ &= c_1\|u\|^{1/2}|Au|^{1/2}|Av|\|w\|, \end{aligned}$$

where $c_1 = c_\infty + c_6$ involving constants of Sobolev and Agmon inequalities. □

We next present some supplementary properties of the polynomials $q_j(t)$ in the asymptotic expansion (2.9). Let F be the space of formal Fourier series $\big\{ u \sim$

$\sum_{\mathbf{k}\in\mathbb{Z}^3\setminus\{0\}} \hat{u}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} : \hat{u}_{\mathbf{k}} \in \mathbb{C}^3 \Big\}$. For $n \in \mathbb{N}$ define

$$F_n = \Big\{ u \in F : \hat{u}_{\mathbf{k}} = 0 \text{ for } |\mathbf{k}|^2 \neq n \bmod 2 \Big\}. \tag{A.5}$$

The following result is from [4].

Lemma A.2. *Let $\xi \in S_A$. Then the polynomials $q_n(t) = q_n(t,\xi)$ and $\beta_n(t)$ defined by (2.14) and (2.13) belong to F_n for all $n \in \mathbb{N}$ and $t \geq 0$.*

In addition, we have

Lemma A.3. *Let $\xi \in S_A$. Then $q_n(t,\xi) \in P_{n^2}H$, for all $n \in \mathbb{N}, t \geq 0$,*

Proof. Based on the formula of $q_n(t,\xi)$ in (2.14) and $B(u,v)$ in Subsection 2.4 and by induction we can prove the following: for each $n \geq 1$, suppose

$$q_n(t,\xi) = \sum_{\mathbf{k}\in\mathbb{Z}^3} a(\mathbf{k},t) e^{i\mathbf{k}\cdot\mathbf{x}},$$

then we have that $|\mathbf{k}| \leq n$ whenever $a(\mathbf{k},t) \neq 0$. □

Appendix B. Some Phragmen-Linderlöf type estimates

First we recall the following direct consequence of problem 325, page 168 in [16].

Theorem B.1. *Let $f(\zeta)$ be analytic on the right half plane $H_0 = \{\zeta \in \mathbb{C} : \mathbf{Re}\zeta > 0\}$, bounded by a constant M and*

$$\sup_{x>0} e^{\alpha x}|f(x)| < \infty, \tag{B.1}$$

where α is a positive number. Then

$$|f(\zeta)| \leq M e^{-\alpha \mathbf{Re}\zeta}, \ \zeta \in H_0. \tag{B.2}$$

We will establish a version of this theorem for some special domains arising in the study of the analytic solutions of the Navier–Stokes equations. By virtue of Proposition 3.8, we consider the following domains

$$D(c,\alpha) = \{\tau + i\sigma : \tau > 0, |\sigma| < c\tau e^{\alpha\tau}\}, \tag{B.3}$$

where c and α are positive numbers. To take the advantage of Theorem B.1, we need to convert $D(c,\alpha)$ to the right half plane. The following analytic transformation is what we need

$$\phi_\alpha(\zeta) = \zeta - \frac{1}{\alpha}\log(1+\alpha\zeta). \tag{B.4}$$

Lemma B.2. *Let $c \geq \sqrt{2}, \alpha > 0$, then function $\phi_\alpha(\zeta)$ conformally maps $D(c,\alpha)$ to a set containing the right half plane H_0. Moreover, $\phi_\alpha([0,\infty)) = [0,\infty)$.*

Proof. For $\zeta \neq 0$ we have that

$$\phi'_\alpha(\zeta) = 1 - \frac{1}{1+\alpha\zeta} \neq 0.$$

It follows that ϕ_α is one-to-one on D. Moreover, if $\zeta \in \partial D(c,\alpha)$ then

$$\mathbf{Re}\phi_\alpha(\zeta) = \tau - \frac{1}{2\alpha}\log\left((1+\alpha\tau)^2 + (\alpha\sigma)^2\right) \leq 0$$

if and only if

$$e^{2\alpha\tau} \leq 1 + 2\alpha\tau + \alpha^2\tau^2 + \alpha^2c^2\tau^2 e^{2\alpha\tau}. \tag{B.5}$$

Since $c \geq \sqrt{2}$, then (B.5) is implied by

$$e^{2\alpha\tau} < 1 + 2\alpha\tau + 2\alpha^2\tau^2 e^{2\alpha\tau}, \quad \tau > 0.$$

It follows that $\phi_\alpha^{-1}(H_0) \subset D(c,\alpha)$. For real τ we have

$$\phi_\alpha(\tau) = \tau - \alpha^{-1}\log(1+\alpha\tau).$$

Therefore $\phi_\alpha(0) = 0$ and $\phi_\alpha\big([0,\infty)\big) = [0,\infty)$. □

Corollary B.3. *Suppose $u(\zeta)$ is analytic in $D(c,\alpha)$ where $c \geq \sqrt{2}, \alpha > 0$,*

$$|u(\zeta)| \leq M, \quad \zeta \in D(c,\alpha) \tag{B.6}$$

and

$$|u(t)| \leq Ce^{-nt}, \quad t > 0, \tag{B.7}$$

where C and n are positive constants. Then

$$|u(\zeta)| \leq Me^{-n\mathbf{Re}\zeta}|1+\alpha\zeta|^{n/\alpha}, \quad \zeta \in \phi_\alpha^{-1}(H_0). \tag{B.8}$$

Proof. Let $v(\eta) = u(\phi_\alpha^{-1}(\eta)), \eta \in H_0$, then

$$|v(\eta)| \leq M$$

and

$$|v(\phi_\alpha(\tau))| = |u(\tau)| \leq Ce^{-n\tau} \leq Ce^{-n(\tau - \frac{1}{\alpha}\log(1+\alpha\tau))} = Ce^{-n\phi_\alpha(\tau)}, \quad \tau > 0.$$

Then

$$|v(\tau)| \leq Ce^{-n\tau}, \quad \tau > 0.$$

By Theorem B.1, we have

$$|v(\eta)| \leq Me^{-n\mathbf{Re}\eta}, \quad \eta \in H_0.$$

Hence for $\zeta \in \phi_\alpha^{-1}(H_0)$,

$$\begin{aligned} |u(\zeta)| = |v(\phi_\alpha(\zeta))| &\leq Me^{-n\mathbf{Re}(\zeta - \frac{1}{\alpha}\log(1+\alpha\zeta))} \\ &= Me^{-n\mathbf{Re}\zeta}e^{\frac{n}{\alpha}\log|1+\alpha\zeta|} = Me^{-n\mathbf{Re}\zeta}|1+\alpha\zeta|^{n/\alpha}. \end{aligned}$$

□

We next describe subdomains of $\phi_\alpha^{-1}(H_0)$ consisting of disks with certain specified centers and radii.

Lemma B.4. *Given $c \geq \sqrt{2}, \alpha > 0$, let*

$$T^\star = \frac{1+\sqrt{2}}{\alpha} \quad \textit{and} \quad g(\tau) = \frac{\sqrt{3}e^{\alpha\tau}}{2\alpha}.$$

For each $\tau > T^\star$, let $t(\tau) = \tau + \alpha g(\tau)^2$ and $r_t(\tau) = g(\tau)\sqrt{1+\alpha^2 g(\tau)^2}$. Then the disk $B(t(\tau), r_t(\tau)) = \{\zeta \in \mathbb{C} \ : \ |\zeta - t(\tau)| < r_t(\tau)\}$ is a subset of $\phi_\alpha^{-1}(H_0) \subset D(c,\alpha)$.

Proof. Let $\zeta = \tau + i\sigma,\ \tau > 0$ and $\sigma \in \mathbb{R}$, the condition for $\phi_\alpha(\zeta) \in H_0$ is that

$$\tau - \frac{1}{2\alpha}\log\left[(1+\alpha\tau)^2 + (\alpha\sigma)^2\right] > 0, \text{ or equivalently, } |\sigma| < \frac{\sqrt{e^{2\alpha\tau} - (1+\alpha\tau)^2}}{\alpha}.$$

Take $\tau^\star > 0$ such that $e^{2\tau\alpha} - (1+\alpha\tau)^2 \geq (3/4)e^{2\tau\alpha}$ for all $\tau \geq \tau^\star$. Equivalently, we require that

$$e^{\alpha\tau} \geq 2(1+\alpha\tau) \quad \text{for all} \quad \tau \geq \tau^\star. \tag{B.9}$$

We obtain

$$U \overset{\text{def}}{=\!=} \{\tau + i\sigma \in D(c,\alpha) : \tau > \tau^\star, |\sigma| < g(\tau)\} \subset \phi_\alpha^{-1}(H_0). \tag{B.10}$$

Since $e^y \geq 1 + y + y^2/2$ for $y > 0$, then the fact that

$$\frac{y^2}{2} \geq 1 + y \iff (y-1)^2 \geq 3$$

implies that (B.9) holds whenever $1 + \alpha\tau^\star \geq 2$, or equivalently, when $\tau^\star \geq 1/\alpha$. Moreover, if $\tau > \tau^\star \geq \sqrt{3}/(2\alpha c)$, then $g(\tau) \leq c\tau e^{\alpha\tau}$ and hence $\{\tau + i\sigma : \tau > \tau^\star, |\sigma| < g(\tau)\} \subset U$. Since $\sqrt{3}/(2\alpha c) < 1/\alpha$, we choose $\tau^\star = 1/\alpha$.

Now, for each $t > \tau^\star$ we find the maximum radius r_t such that the disk $B(t, r_t)$ centered at t with radius r_t is in U. Let $\tau > \tau^\star$. The line orthogonal to the curve

$$\{\tau' + ig(\tau') : \tau' > 0\} \tag{B.11}$$

at $\tau + ig(\tau)$ is

$$\{\tau' + i\sigma' : \sigma' = -\frac{1}{g'(\tau)}(\tau' - \tau) + g(\tau),\ \tau' \in \mathbb{R}\}. \tag{B.12}$$

Note that $g'(\tau) = \alpha g(\tau)$. The straight line (B.12) intersects the real axis at

$$t(\tau) = \tau + g'(\tau)g(\tau) = \tau + \alpha g(\tau)^2.$$

The distance from this $t(\tau)$ to the curve (B.11) is

$$r_t(\tau) = \sqrt{(g(\tau)g'(\tau))^2 + g(\tau)^2} = g(\tau)\sqrt{1 + \alpha^2 g(\tau)^2}.$$

We have

$$\begin{aligned} t(\tau) - r_t(\tau) &= \frac{t^2(\tau) - r_t^2(\tau)}{t(\tau) + r_t(\tau)} \\ &= \frac{\tau^2 + g'(\tau)^2 g(\tau)^2 + 2g'(\tau)g(\tau) - g'(\tau)^2 g(\tau)^2 - g(\tau)^2}{t(\tau) + r_t(\tau)} \\ &= \frac{\tau^2 + (2\alpha\tau - 1)g(\tau)^2}{\tau + \alpha g(\tau)^2 + g(\tau)\sqrt{1 + \alpha^2 g(\tau)^2}}. \end{aligned}$$

Since $\alpha\tau > 1 > \log(2/\sqrt{3})$, it follows that $1 < \alpha^2 g(\tau)^2$ and

$$t(\tau) - r_t(\tau) \geq \frac{\tau^2 + \alpha\tau g^2(\tau)}{\tau + (1+\sqrt{2})\alpha g^2(\tau)} \geq \frac{\tau}{1+\sqrt{2}}. \tag{B.13}$$

Now, if $\tau > T^\star = (1+\sqrt{2})\tau^\star$, then $\tau > \tau^\star$, $t(\tau) - r_t(\tau) > \tau^\star$, and hence $B(t(\tau), r_t(\tau)) \subset \{\tau' + i\sigma' : \tau' > \tau^\star, |\sigma'| < g(\tau')\} \subset U \subset \phi_\alpha^{-1}(H_0)$. $\square$

We now give an explicit estimate of a polynomial based on its behavior in the domain $D(c, \alpha)$.

Lemma B.5. *Let $a > r > 0$ and $B(a, r)$ be the disk $\{\zeta \in \mathbb{C} : |\zeta - a| < r\}$. Suppose $q(\zeta)$ is a polynomial of degree less than or equal to p and*

$$|q(\zeta)| \leq M|1 + \alpha\zeta|^N, \quad \zeta \in B(a, r), \tag{B.14}$$

where M and N are positive numbers. Then

$$|q(\zeta)| \leq M(p+1)(1 + \alpha a + \alpha r)^N (\frac{|\zeta| + a}{r})^p, \quad \zeta \in \mathbb{C}. \tag{B.15}$$

Proof. Write $q(\zeta) = d_0 + d_1(\zeta - a) + \cdots + d_p(\zeta - a)^p$. Estimate the coefficients using the Cauchy formula

$$\begin{aligned} |d_j| &= \Big|\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{(\zeta-a)^{j+1}}d\zeta\Big| \\ &\le \frac{1}{2\pi}\int_0^{2\pi}\frac{|q(a+re^{i\theta})|}{r^{j+1}}rd\theta \\ &\le \frac{M}{2\pi r^j}\int_0^{2\pi}|1+\alpha(a+re^{i\theta})|^N d\theta \\ &\le \frac{M(1+\alpha(a+r))^N}{r^j}. \end{aligned}$$

This implies

$$|q(\zeta)| \le M(1+\alpha a+\alpha r)^N\sum_{j=0}^{p}\frac{|\zeta-a|^j}{r^j}, \quad \zeta\in\mathbb{C}.$$

Since $a/r > 1$ we obtain

$$\begin{aligned} |q(\zeta)| &\le M(1+\alpha a+\alpha r)^N\sum_{j=0}^{p}\Big(\frac{|\zeta|+a}{r}\Big)^p \\ &\le M(p+1)(1+\alpha a+\alpha r)^N\Big(\frac{|\zeta|+a}{r}\Big)^p. \end{aligned}$$

□

Combining Corollary B.3 and Lemma B.5, we obtain

Corollary B.6. *Given $c \ge \sqrt{2}, \alpha > 0$. Let $a > r > 0$ such that the disk $B(a, r)$ is a subset of $\phi_\alpha^{-1}(H_0)$. Suppose $q(\zeta)$ is a polynomial of degree less than or equal to p and*

(B.16) $$|e^{-N\zeta}q(\zeta)| \le M, \quad \zeta\in D(c,\alpha).$$

Then

(B.17) $$|q(\zeta)| \le M|1+\alpha\zeta|^{N/\alpha}, \quad \zeta\in\phi_\alpha^{-1}(H_0)$$

and

(B.18) $$|q(\zeta)| \le M(p+1)(1+\alpha a+\alpha r)^{N/\alpha}\Big(\frac{|\zeta|+a}{r}\Big)^p, \quad \zeta\in\mathbb{C}.$$

In particular, when $p = N$,

(B.19) $$|q(\zeta)| \le M(N+1)C(\alpha,a,r)^N(|\zeta|+a)^N, \quad \zeta\in\mathbb{C},$$

where

(B.20) $$C(\alpha,a,r) = r^{-1}(1+\alpha a+\alpha r)^{1/\alpha}.$$

Proof. Let $\varepsilon > 0$ be arbitrary. Apply Corollary B.3 to $u(\zeta) = e^{-N\zeta}q(\zeta)$ using the fact that $|u(\zeta)| \le M$ for $\zeta \in D(c, \alpha)$ and $|u(t)| = O(e^{-(N-\varepsilon)t})$ as $t \to \infty$. Hence,

$$|u(\zeta)| \le Me^{-(N-\varepsilon)\mathbf{Re}\zeta}|1+\alpha\zeta|^{(N-\varepsilon)/\alpha}, \quad \zeta\in\phi_\alpha^{-1}(H_0).$$

Taking limits as $\varepsilon \to 0$ gives

$$|q(\zeta)| \le M|1+\alpha\zeta|^{N/\alpha}, \quad \zeta\in\phi_\alpha^{-1}(H_0).$$

Apply Lemma B.5 to finish the proof. □

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[1] Department of Mathematics, 3368 TAMU, Texas A&M University, College Station, TX 77843-3368, U.S.A.

[2] Department of Mathematics, Indiana University, Bloomington, IN 47405, U.S.A.
E-mail address: `lhoang@indiana.edu`

[3] Department of Mathematics, University of Nevada, Reno, NV 89503, U.S.A.
E-mail address: `ejolson@unr.edu`

[4] USC Department of Mathematics, Kaprielian Hall, Room 108, 3620 Vermont Avenue, Los Angeles, CA 90089-2532, U.S.A.
E-mail address: `ziane@math.usc.edu`