THE NORMAL FORM OF THE NAVIER-STOKES EQUATIONS IN SUITABLE NORMED SPACES

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Abstract. We consider solutions to the incompressible Navier–Stokes equations on the periodic domain $\Omega = [0, 2\pi]^3$ with potential body forces. Let $\mathcal{R} \subseteq H^1(\Omega)^3$ denote the set of all initial data that lead to regular solutions. Our main result is to construct a suitable Banach space $S_A^\star$ such that the normalization map $W : \mathcal{R} \to S_A^\star$ is continuous, and such that the normal form of the Navier–Stokes equations is a well-posed system in all of $S_A^\star$. We also show that $S_A^\star$ may be seen as a subset of a larger Banach space $V^\star$ and that the extended Navier–Stokes equations, which are known to have global solutions, are well-posed in $V^\star$.

Contents

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1. Introduction

The Navier–Stokes equations describe the dynamics of incompressible, viscous fluid flows. These equations continue to pose great challenges in mathematics. In particular, the problem of the long time existence of regular solutions is still open. One of the main difficulties in studying the three dimensional Navier–Stokes equations is the analysis of the role of the nonlinear terms in the equations. It is therefore appropriate to consider the simplest case when that role is minimal. One such case occurs when the solutions are periodic in the space variables and the body forces are potential.

The asymptotic behavior of the regular solution $u(t) = S(t)u^0$ of the Navier–Stokes equations in the periodic domain with potential forces where u^0 is the initial data was studied in a series of papers [6, 7, 8]. It was shown that the regular solution $u(t)$ possesses an asymptotic expansion, namely,

$$u(t) \sim \sum_{n=1}^{\infty} W_n(t,u^0)e^{-nt}, \quad \text{as} \quad t \to \infty, \tag{1.1}$$

where $W_n(t,u^0)$ is a polynomial in t whose values are divergence-free trigonometric polynomials. For more details, see (2.10) and the explanations afterward.

Associated to such an asymptotic expansion are the normalization map $W(u^0)$ and the normal form for the Navier–Stokes equations, which is an infinite system of finite dimensional ordinary differential equations. The polynomials $W_n(t,u^0)$ in (1.1) may be explicitly computed from $W(u^0)$ using a recursive formula deduced from the nonlinear terms of the Navier–Stokes equations. The image of the normalization map and the solutions $S_{\rm normal}(t)\bar{\xi}$ of the normal form, where $\bar{\xi}$ is the initial data, were originally studied in a Frechet space S_A. Briefly, $S_A = \oplus_{n=1}^{\infty} R_n H$ where $R_n H$ is the eigenspace of the Stokes operator corresponding to the eigenvalue n or, if n is not an eigenvalue, the trivial linear space $\{0\}$. Since the topology of component-wise convergence associated to the Frechet space S_A is very weak, more precise analysis may be obtained by studying the normalization map and the normal form in a subspace S_A^* of S_A endowed with a stronger norm-induced topology.

In our previous paper [4], among other things, we constructed a suitable Banach space $S_A^\star$, a subspace of S_A, on which the normal form is a well-posed system near the origin. The norm $\|\bar{u}\|_\star$ of $\bar{u} = (u_n)_{n=1}^\infty \in S_A^\star$ is of the form

$$\|\bar{u}\|_\star = \sum_{n=1}^{\infty} \rho_n \|\nabla u_n\|_{L^2(\Omega)}, \tag{1.2}$$

where $\Omega = (0,2\pi)^3$ is the domain of periodicity and $(\rho_n)_{n=1}^\infty$ is a sequence of positive weights. In that study, we also defined a system of the extended Navier–Stokes equations which is appropriate to the study of the asymptotic expansion of $S(t)u^0$ as well as the solution $S_{\rm normal}(t)\bar{\xi}$ of the normal form. In addition, we proved that the semigroup $S_{\rm ext}(t)$ generated by the extended Navier–Stokes equations leaves invariant a Banach space $V^\star$, which is defined similarly to $S_A^\star$. A missing piece of our study in [4] is an affirmative answer to the question whether $S(t)u^0$ belongs to $S_A^\star$. Also the properties and relations of $W, S(t), S_{\rm normal}(t)$ and $S_{\rm ext}(t)$ in the above star spaces had only begun to be addressed and studied.

The current paper is a continuation of [4]. Our main result is to obtain a choice of weights ρ_n such that the norm given by (1.2) yields a Banach space $S_A^\star \subset S_A$ which contains the image of the normalization map. Moreover, we establish the everywhere continuity properties of $S_{\mathrm{normal}}(t), S_{\mathrm{ext}}(t)$ and W with respect to this norm. We summarize our results in the commutative diagram

$$\begin{tikzcd}
& \mathcal{R} \arrow[r, "S(t)"] \arrow[d, "W(\cdot)"] \arrow[ddl, "W(0,\cdot)"'] & \mathcal{R} \arrow[d, "W(\cdot)"'] \arrow[ddr, "W(0,\cdot)"] & \\
& S_A^\star \arrow[r, "S_{\mathrm{normal}}(t)"] \arrow[dl, "Q(0,\cdot)"] & S_A^\star \arrow[dr, "Q(0,\cdot)"'] & \\
V^\star \arrow[rrr, "S_{\mathrm{ext}}(t)"] & & & V^\star
\end{tikzcd}$$

Figure 1. Commutative diagram

where all mappings are continuous. The precise definitions of the maps and spaces in the diagram are given in Section 2.

Our results imply that for each regular solution $u(t)$ the series

$$\sum_{n=1}^{\infty} \rho_n e^{-nt} \|W_n(t, u(0))\| < \infty.$$

However, the weights ρ_n decrease very rapidly and the main question of whether the asymptotic expansion $\sum_{n=1}^{\infty} e^{-nt} W_n(t, u(0))$ actually converges in V to the solution $u(t)$ is still open.

This paper is organized as follows. Section 2 recalls the definitions and properties of the asymptotic expansions, the normalization map and the normal form. In Section 3, we study the extended Navier–Stokes equations and show the conditions on the weights ρ_n given in [4] restated here as Definition 2.1 ensure that $S_{\mathrm{ext}}(t) : V^\star \to V^\star$ is continuous. The estimates obtained in this section will be used later in the study of the normalization map and the normal form. Section 4 contains our study of the normal form. In particular, we show that $S_{\mathrm{normal}}(t) : S_A^\star \to S_A^\star$ and $Q(0, \cdot) : S_A^\star \to V^\star$ are also continuous with respect to any of the norms given by Definition 2.1. In Section 5, we study the normalization map, construct a new norm and prove our main results. Namely, we construct norms satisfying Definition 2.1 of the type specified in Definition 5.2 for which the normalization map $W : \mathcal{R} \to S_A^*$ is continuous. The Appendix provides a number of lemmas on numeric series needed for our norm estimates as well as a useful global estimate for the difference of two regular solutions of the Navier–Stokes equations.

2. Preliminaries

2.1. **Mathematical setting.** The initial value problem for the incompressible Navier–Stokes equations in the three-dimensional space $\mathbb{R}^3$ with a potential body force is

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} = -\nabla p - \nabla\phi, \\ \text{div } \mathbf{u} = 0, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}^0(\mathbf{x}), \end{cases} \tag{2.1}$$

where $\nu > 0$ is the kinematic viscosity, $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ is the unknown velocity field, p is the unknown pressure, $(-\nabla\phi)$ is the body force specified by a given function ϕ and $\mathbf{u}^0(\mathbf{x})$ is the known initial velocity field. We consider only solutions $\mathbf{u}(\mathbf{x},t)$ such that for any $t \geq 0$, $\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x},t)$ satisfies

$$\mathbf{u}(\mathbf{x} + L\mathbf{e}_j) = \mathbf{u}(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^3, \quad j = 1,2,3, \tag{2.2}$$

and

$$\int_\Omega \mathbf{u}(\mathbf{x})d\mathbf{x} = 0 \tag{2.3}$$

where $L > 0$ is fixed and $\Omega = (-L/2, L/2)^3$. We call the functions satisfying (2.2) L-periodic functions. Throughout this paper we take $L = 2\pi$ and $\nu = 1$. The general case is easily recovered by a change of scale.

Let $\mathcal{V}$ be the set of all L-periodic trigonometric polynomials on Ω with values in $\mathbb{R}^3$ which are divergence-free as well as satisfy the condition (2.3). We define

$$\begin{cases} H = \text{ closure of } \mathcal{V} \text{ in } L^2(\Omega)^3, \\ V = \text{ closure of } \mathcal{V} \text{ in } H^1(\Omega)^3, \end{cases}$$

where $H^l(\Omega)$ with $l = 0,1,2,\dots$ denotes the Sobolev space of functions $\varphi \in L^2(\Omega)$ such that for every multi-index α with $|\alpha| \leq l$ the distributional derivative $D^\alpha\varphi \in L^2(\Omega)$.

For $\mathbf{a} = (a_1,a_2,a_3)$ and $\mathbf{b} = (b_1,b_2,b_3)$ in $\mathbb{R}^3$, define $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ and $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$. Let $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the scalar product and norm in $L^2(\Omega)^3$ given by

$$\langle u,v\rangle = \int_\Omega \mathbf{u}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})d\mathbf{x}, \quad |u| = \langle u,u\rangle^{1/2}, \quad u = \mathbf{u}(\cdot), v = \mathbf{v}(\cdot) \in L^2(\Omega)^3.$$

Note that we use $|\cdot|$ for the length of vectors in $\mathbb{R}^3$ as well as the L^2-norm of vector fields in $L^2(\Omega)^3$. In each case the context clarifies the precise meaning of this notation.

Let P_L denote the orthogonal projection in $L^2(\Omega)^3$ onto H. On V we consider the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ and the norm $\|\cdot\|$ defined by

$$\langle\!\langle u,v\rangle\!\rangle = \sum_{j,k=1}^{3} \int_\Omega \frac{\partial u_j(\mathbf{x})}{\partial x_k}\frac{\partial v_j(\mathbf{x})}{\partial x_k}d\mathbf{x} \quad \text{and} \quad \|u\| = \langle\!\langle u,u\rangle\!\rangle^{1/2},$$

for $u = \mathbf{u}(\cdot) = (u_1,u_2,u_3)$ and $v = \mathbf{v}(\cdot) = (v_1,v_2,v_3)$ in V.

Define the Stokes operator A with domain $\mathcal{D}_A = V \cap H^2(\Omega)^3$ by

$$Au = -\Delta u \quad \text{for all} \quad u \in \mathcal{D}_A. \tag{2.4}$$

The inner product of $u, v \in \mathcal{D}_A$ and the norm of $w \in \mathcal{D}_A$ are defined by $\langle Au, Av\rangle$ and $|Aw|$, respectively. Note for $w \in \mathcal{D}_A$ that (2.3) implies the norm $|Aw|$ is equivalent to the usual Sobolev norm of $H^2(\Omega)^3$. We also define the bilinear mapping associated with the nonlinear term in the Navier–Stokes equations by

$$B(u, v) = P_L(u \cdot \nabla v) \quad \text{for all} \quad u, v \in \mathcal{D}_A. \tag{2.5}$$

A classical result tracing back to Leray's pioneering works on the Navier–Stokes equations in the 1930's (see, e.g., [12, 13, 14]) is that for any initial data $\mathbf{u}^0(\mathbf{x})$ in H there exists a weak solution $\mathbf{u}(\mathbf{x}, t)$ defined for all $\mathbf{x} \in \mathbb{R}^3$ and $t > 0$ which eventually becomes analytic in space and time and $\|\mathbf{u}(\cdot, t)\|_{H^1(\Omega)^3}$ converges exponentially to zero as $t \to \infty$ (see also [1, 10, 5]). Thus there is $t_0 \geq 0$ such that the solution $u(t) = \mathbf{u}(\cdot, t)$ is continuous from $[t_0, \infty)$ into V and satisfies the following functional form of the Navier–Stokes equations

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = 0, \quad t > t_0,$$

where the equation holds in H. We say that $u(t), t \geq t_0$, is a regular solution to the Navier–Stokes equations on $[t_0, \infty)$. We denote $\mathcal{R}$ the set of all initial value $u^0 \in V$ such that there is a (unique) solution $u(t), t > 0$, satisfying

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = 0, \quad t > 0, \tag{2.6}$$

with the initial data

$$u(0) = u^0 \in V, \tag{2.7}$$

where the equation holds in H, and $u(t)$ is continuous from $[0, \infty)$ into V. In other words, $\mathcal{R}$ is the set of all initial data $u^0 \in V$ such that the solution $u(t)$ of the Navier–Stokes equations (2.6) is regular (hence also unique) on $[0, \infty)$.

We recall that the spectrum $\sigma(A)$ of the Stokes operator A consists of the eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ of the form $\lambda_j = |\mathbf{k}|^2$ for some $\mathbf{k} \in \mathbb{Z}^3 \setminus \{0\}$ where $j = 1, 2, 3, \ldots$. Note that $\lambda_1 = 1 = |\mathbf{e}_1|^2$ and hence the additive semigroup generated by $\sigma(A)$ coincides with the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of all natural numbers. For $n \in \mathbb{N}$ we denote by R_n the orthogonal projection of H onto the eigenspace of A associated to n. Thus,

$$R_n H = \{\, u \in H : Au = nu \,\}.$$

If n is an eigenvalue of A, then $R_n H$ is generated by functions of the form

$$(a^1_{\mathbf{k}} + i a^2_{\mathbf{k}}) e^{i(\mathbf{k}\cdot\mathbf{x})} + (a^1_{\mathbf{k}} - i a^2_{\mathbf{k}}) e^{-i(\mathbf{k}\cdot\mathbf{x})}, \quad \mathbf{k} \in \mathbb{Z}^3, \quad |\mathbf{k}|^2 = n,$$

where

$$a^1_{\mathbf{k}}, a^2_{\mathbf{k}} \in \mathbb{R}^3 \quad \text{and} \quad a^1_{\mathbf{k}} \cdot \mathbf{k} = a^2_{\mathbf{k}} \cdot \mathbf{k} = 0.$$

Otherwise $R_n = 0$. For example, $R_7 = 0, R_{15} = 0, R_{23} = 0, \ldots$. Define

$$P_n = R_1 + R_2 + \cdots + R_n \qquad \text{and} \qquad Q_n = I - P_n. \tag{2.8}$$

2.2. **The asymptotic behavior of solutions.** Let us recall some known results on the asymptotic expansions and the normal form of the regular solutions to the Navier–Stokes equations (see [6, 7, 8, 11] for more details). First, for any $u^0 \in \mathcal{R}$ there is an eigenvalue n_0 of A such that

$$\lim_{t\to\infty} \frac{\|u(t)\|^2}{|u(t)|^2} = n_0 \quad \text{and} \quad \lim_{t\to\infty} u(t)e^{n_0 t} = w_{n_0}(u^0) \in R_{n_0}H \setminus \{0\}. \tag{2.9}$$

Furthermore, $u(t)$ has the asymptotic expansion

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + \cdots, \tag{2.10}$$

where $q_j(t)$, also denoted by $W_j(t,u^0)$, is a polynomial in t of degree at most $j-1$ with values trigonometric polynomials in H. This means that for any $N \in \mathbb{N}$ the correction term $\tilde{u}_{N+1}(t) = u(t) - \sum_{j=1}^N q_j(t)e^{-jt}$ satisfies

$$|\tilde{u}_{N+1}(t)| = O\big(e^{-(N+\varepsilon)t}\big) \quad \text{as} \quad t \to \infty \quad \text{for some} \quad \varepsilon = \varepsilon_N > 0. \tag{2.11}$$

In fact, $\tilde{u}_{N+1}(t)$ belongs to $C^1\big([0,\infty),V\big) \cap C^\infty\big((0,\infty),C^\infty(\mathbb{R}^3)\big)$, and for each $m \in \mathbb{N}$

$$\|\tilde{u}_{N+1}(t)\|_{H^m(\Omega)} = O\big(e^{-(N+\varepsilon)t}\big) \quad \text{as} \quad t \to \infty \quad \text{for some} \quad \varepsilon = \varepsilon_{N,m} > 0. \tag{2.12}$$

Define the normalization map W by $W(u^0) = W_1(u^0) \oplus W_2(u^0) \oplus \cdots$, where $W_j(u^0) = R_j q_j(0)$ for $j \in \mathbb{N}$. Then W is an one-to-one analytic mapping from $\mathcal{R}$ to the Frechet space $S_A = R_1H \oplus R_2H \oplus \cdots$ endowed with the component-wise topology.

The case (2.9) holds if and only if $W_1(u^0) = W_2(u^0) = \cdots = W_{n_0-1}(u^0) = 0$ and $W_{n_0}(u^0) \neq 0$. In this case

$$q_1 = q_2 = \cdots = q_{n_0-1} = 0 \quad \text{and} \quad q_{n_0} = w_{n_0}(u^0) = W_{n_0}(u^0).$$

If $u^0 \in \mathcal{R}$ then the polynomials $q_j(t)$ are the unique polynomial solutions to the following equations

$$q_j'(t) + (A-j)q_j(t) + \beta_j(t) = 0, \quad t \in \mathbb{R}, \tag{2.13}$$

with

$$R_j q_j(0) = W_j(u^0), \tag{2.14}$$

where the terms $\beta_j(t)$ are defined by

$$\beta_1(t) = 0 \quad \text{and} \quad \beta_j(t) = \sum_{k+l=j} B(q_k(t), q_l(t)) \quad \text{for} \quad j > 1. \tag{2.15}$$

Given arbitrary $\bar{\xi} = (\xi_n)_{n=1}^\infty \in S_A$, the polynomial solutions $q_j(t,\bar{\xi})$ of (2.13) satisfying the initial condition $R_j q_j(0) = \xi_j$, are explicitly given by the recursive formula

(2.16)

$$q_j(t,\bar{\xi}) = \xi_j - \int_0^t R_j\beta_j(\tau)d\tau + \sum_{n\geq 0}(-1)^{n+1}[(A-j)(I-R_j)]^{-n-1}\frac{d^n}{dt^n}(I-R_j)\beta_j,$$

for $j \in \mathbb{N}$. Here $[(A-j)(I-R_j)]^{-n-1}$ is defined by

$$[(A-j)(I-R_j)]^{-n-1}u = \sum_{|\mathbf{k}|^2\neq j} \frac{a_\mathbf{k}}{(|\mathbf{k}|^2-j)^{n+1}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

for $u = \sum_{|\mathbf{k}|^2\neq j} a_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} \in \mathcal{V}$. Above I denotes the identity map on H.

Note that, with our notation, for $u^0 \in \mathcal{R}$, we have $W_j(t, u^0) = q_j(t, W(u^0))$ for all $j \in \mathbb{N}$ and $t \in \mathbb{R}$.

Finally, the S_A-valued function $\bar{\xi}(t) = (\xi_n(t))_{n=1}^{\infty} = \big(W_n(u(t))\big)_{n=1}^{\infty} = W(u(t))$ satisfies the following system of differential equations

$$\begin{cases} \dfrac{d\xi_1(t)}{dt} + A\xi_1(t) = 0, \\[2ex] \dfrac{d\xi_n(t)}{dt} + A\xi_n(t) + \displaystyle\sum_{k+j=n} R_n B\big(q_k(0, \bar{\xi}(t)), q_j(0, \bar{\xi}(t))\big) = 0, \ n > 1. \end{cases} \tag{2.17}$$

This system is the normal form of the Navier–Stokes equations (2.6) associated with the asymptotic expansion (2.10). It is easy to check that the solution of (2.17) with initial data $\bar{\xi}^0 = (\xi_n^0)_{n=1}^{\infty} \in S_A$ is precisely $\big(R_n q_n(t, \bar{\xi}^0) e^{-nt}\big)_{n=1}^{\infty}$. Thus, formula (2.16) yields the normal form and its solutions.

2.3. **Complexification of the Navier–Stokes equations.** We introduce the Navier–Stokes equations with complex times and their analytic solutions (see [1, 9]). Let X be a real Hilbert space with scalar product $(\cdot,\cdot)_X$. Define the complexification of X as

$$X_{\mathbb{C}} = \{\, u + iv : u, v \in X \,\}$$

with the addition and scalar product defined by

$$(u_1 + iu_2) + (v_1 + iv_2) = (u_1 + v_1) + i(u_2 + v_2)$$

and

$$(\zeta_1 + i\zeta_2)(u_1 + iu_2) = \zeta_1 u_1 - \zeta_2 u_2 + i(\zeta_2 u_1 + \zeta_1 u_2)$$

for $u_1, u_2, v_1, v_2 \in X$ and $\zeta_1, \zeta_2 \in \mathbb{R}$. The complexified space $X_{\mathbb{C}}$ is a Hilbert space with respect to the inner product

$$(u + iv, u' + iv')_{X_{\mathbb{C}}} = (u, u')_X + (v, v')_X + i\big[(v, u')_X - (u, v')_X\big],$$

where $u, v, u', v' \in X$. When $X = H$ or $X = V$, we obtain the complexified spaces $H_{\mathbb{C}}$ and $V_{\mathbb{C}}$. We keep the same notation for their corresponding inner products and norms.

The Stokes operator may be extended to $\mathcal{D}_{A_{\mathbb{C}}} = (\mathcal{D}_A)_{\mathbb{C}}$ as

$$A(u + iv) = Au + iAv, \quad u, v \in \mathcal{D}_A.$$

Similarly, $B(\cdot,\cdot)$ can be extended to a bounded bilinear map from $V_{\mathbb{C}} \times \mathcal{D}_{A_{\mathbb{C}}}$ to $H_{\mathbb{C}}$ by

$$B(u + iv, u' + iv') = B(u, u') - B(v, v') + i[B(u, v') + B(v, u')]$$

for $u, v \in V,\ u', v' \in \mathcal{D}_A$. Note that unlike the real case we have

$$\langle B(u, v), v\rangle \not\equiv 0, \quad \text{for } u, v \in \mathcal{D}_{A_{\mathbb{C}}}.$$

The Navier–Stokes equations with complex times is defined as

$$\frac{du(\zeta)}{d\zeta} + B\big(u(\zeta), u(\zeta)\big) + Au(\zeta) = 0, \tag{2.18}$$

with the initial condition

$$u(\zeta_0) = u^\star, \tag{2.19}$$

where $\zeta_0 \in \mathbb{C}$ and $u^\star \in V_{\mathbb{C}}$ are given. Here $d/d\zeta$ denotes the complex derivative of $H_{\mathbb{C}}$-valued functions.

2.4. **The extended Navier–Stokes equations.** The $u_n(t) = W_n(t, u^0)e^{-nt}$ must satisfy the following system of equations

$$\frac{du_n(t)}{dt} + Au_n(t) + B_n(t) = 0, \quad t > 0, \tag{2.20}$$

$$u_n(0) = u_n^0, \tag{2.21}$$

where

$$B_1(t) = 0 \quad \text{and} \quad B_n(t) = \sum_{j+k=n} B\big(u_j(t), u_k(t)\big) \quad \text{for} \quad n > 1. \tag{2.22}$$

One can extend $u_n(t)$ for $t > 0$ to $u_n(\zeta)$ for $\zeta \in \mathbb{C}$ with $\mathbf{Re}\zeta > \mathbf{Re}\zeta_0$. Then

$$\frac{du_n(\zeta)}{d\zeta} + Au_n(\zeta) + B_n(\zeta) = 0, \quad \zeta \in \mathbb{C},\ \mathbf{Re}\zeta > \mathbf{Re}\zeta_0, \tag{2.23}$$

$$u_n(\zeta_0) = u_n^\star, \tag{2.24}$$

where $\zeta_0 \in \mathbb{C}$, $u_n^\star \in V_{\mathbb{C}}$ and

$$B_1(\zeta) = 0 \quad \text{and} \quad B_n(\zeta) = \sum_{j+k=n} B\big(u_j(\zeta), u_k(\zeta)\big) \quad \text{for} \quad n > 1.$$

Above, $\mathbf{Re}\zeta$ denotes the real part of the complex number ζ.

2.5. **Prerequisites.** The following spaces are introduced in the previous studies [4, 6, 7, 8] of the normalization map and the normal form of the Navier–Stokes equations. Let $V_{\mathbb{C}}$ and $(R_nH)_{\mathbb{C}}$, for $n \in \mathbb{N}$, be the complexifications of V and R_nH, respectively as in Section 2.3. Define the complex linear Frechet space

$$V_{\mathbb{C}}^\infty = \{\, \bar{u} = (u_n)_{n=1}^\infty : u_n \in V_{\mathbb{C}} \,\},$$

its subspace

$$(S_A)_{\mathbb{C}} = \{\, \bar{u} = (u_n)_{n=1}^\infty : u_n \in (R_nH)_{\mathbb{C}} \,\} \subset V_{\mathbb{C}}^\infty.$$

and recall that the real linear space V^∞ is $V \oplus V \oplus V \oplus \cdots$.

Let $S(t, t_0) : \mathcal{R} \to \mathcal{R}$ be the semigroup generated by the Navier–Stokes equations (2.6) with initial time t_0. We denote $S(t) = S(t, 0)$.

Let $S_{\rm ext}(\zeta, \zeta_0) : V_{\mathbb{C}}^\infty \to V_{\mathbb{C}}^\infty$ be the semigroup generated by the extended Navier–Stokes equations (2.23) with initial complexified time ζ_0. Also denote $S_{\rm ext}(\zeta) = S_{\rm ext}(\zeta, 0)$.

Let $S_{\rm normal}(t) : S_A \to S_A$ be the semigroup generated by the normal form of Navier–Stokes equations (2.17).

Recall $W : u \in \mathcal{R} \mapsto W(u) \in S_A$. Define the following maps

$$W(t, \cdot) : u \in \mathcal{R} \mapsto (W_n(t, u)e^{-nt})_{n=1}^\infty \in V^\infty,$$
$$Q(t, \cdot) : \bar{\xi} \in S_A \mapsto (q_n(t, \bar{\xi})e^{-nt})_{n=1}^\infty \in V^\infty.$$

The studies of W and $S_{\rm normal}(t)$ in the above Frechet spaces can be made more precise by strengthening the topology. To this end we introduce the normed subspaces $V_{\mathbb{C}}^\star$, $V^\star$ and $S_A^\star$ as

$$V_{\mathbb{C}}^\star = \{\, \bar{u} \in V_{\mathbb{C}}^\infty : \|\bar{u}\|_\star < \infty \,\}, \quad V^\star = V^\infty \cap V_{\mathbb{C}}^\star, \quad S_A^\star = S_A \cap V^\star$$

where then norm $\|\bar{u}\|_\star$ has already been defined in (1.2) depending on a sequence of positive weights ρ_n.

Clearly $V^\star_{\mathbb{C}}, V^\star$ and $S^\star_A$ are Banach spaces.

Concerning the choice of ρ_n in defining the weighted norm in (1.2), we recall as Definition 2.1 the particular sequence $(\rho_n)_{n=1}^\infty$ constructed in [4].

Let C_1 be the positive constant introduced in Appendix A and define

$$\varepsilon_0 = \frac{1}{24C_1}. \tag{2.25}$$

Note that ε_0 and C_1 are essentially the same constant. We write ε_0 when we are focusing on something being small and C_1 otherwise.

Definition 2.1. Let $(\alpha_n)_{n=1}^\infty$ be a sequence of numbers satisfying

$$\alpha_1 \geq 0, \quad \alpha_n > 0 \text{ for } n > 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n \leq 1/2. \tag{2.26}$$

Construct the sequence $(\rho_n)_{n=1}^\infty$ as follows: let $\rho_1 = 1$ and for $n > 1$ define

$$\sigma_n = \min\{\, \rho_k \rho_j : k + j = n \text{ and } k, j \in \mathbb{N}\,\}. \tag{2.27}$$

Then let

$$0 < \rho_n = \sigma_n \min\Big\{1, \frac{\alpha_n}{16\varepsilon_0 \max\{1, L_{3,n}, C_1 n^{3/2}\}}\Big\}, \quad n > 1, \tag{2.28}$$

where

$$L_{3,n} = 2C_1 e^{2a_0} (C_2/2)^{n-2} n^2 (n-2)! [(n-2)^{3/4} + n^{3/2}], \tag{2.29}$$

and the positive numbers a_0 and C_2 are defined in Lemma A.2.

We summarize some results from [4]. Let ρ_n be as in Definition 2.1 and the norm $\|\cdot\|_\star$ be defined by (1.2). We have

Fact 2.2. *$S_{\mathrm{ext}}(t)$ leaves $V^\star$ invariant for all $t \geq 0$.*

Fact 2.3. *There is a neighborhood $\mathcal{O}$ of the origin in $S^\star_A$ such that $Q(0, \cdot)\colon \mathcal{O} \to V^\star$ and $S_{\mathrm{normal}}(t)\colon \mathcal{O} \to S^\star_A$ for $t \geq 0$ are well-defined and Lipschitz continuous.*

3. The extended Navier–Stokes equations

In the first part of this section $(\rho_n)_{n=1}^{\infty}$ is a sequence of positive numbers satisfying

$$\rho_n = \kappa_n \min\{\rho_k\rho_j : k+j=n\}, \quad \kappa_n \in (0,1], \quad n \geq 2. \tag{3.1}$$

Note that the particular choice of ρ_n used as the weights given in Definition 2.1 satisfy condition (3.1). After deriving some basic estimates which will be used throughout this paper, we show how they quickly lead to the Lipschitz continuity of each $S_{\rm ext}(t)$ near the origin in $V^\star$. We finish by showing for κ_n satisfying the additional condition (3.39) that $S_{\rm ext}(t)$ is continuous in the whole space $V^\star$, that is, the extended Navier–Stokes equations are well-posed in $V^\star$.

First, we have the following version of Proposition 3.1 in [4]. The proof of the estimates with weights ρ_n is the same as in Lemma 3.3 below in which we only use the fact that

$$\rho_n \leq \min\{\rho_k\rho_j : k+j=n\}, \quad n \geq 2. \tag{3.2}$$

Proposition 3.1. *Let $\zeta_0 \in \mathbb{C}$ and $\bar{u}^\star = (u_n^\star)_{n=1}^{\infty} \in V_{\mathbb{C}}^{\infty}$. Let $\bar{u}(\zeta) = S_{\rm ext}(\zeta,\zeta_0)\bar{u}^\star$. For $s \in (0,\infty)$, $\theta \in (-\pi/2,\pi/2)$ and $n \in \mathbb{N}$, we have*

$$\rho_n\|u_n(\zeta_0+se^{i\theta})\| \leq \gamma_n e^{-s\cos\theta}, \tag{3.3}$$

$$\int_0^s \frac{|\rho_n A u_n(\zeta_0+\rho e^{i\theta})|^2}{\|\rho_n u_n(\zeta_0+\rho e^{i\theta})\|} d\rho \leq \frac{\gamma_n}{\cos\theta}, \tag{3.4}$$

where

$$\gamma_1 = \rho_1\|u_1^\star\|, \quad \gamma_n = \rho_n\|u_n^\star\| + \frac{C_1}{\cos\theta}\sum_{k+j=n}\gamma_k\gamma_j. \tag{3.5}$$

However, taking into account the factor κ_n in (3.1), we derive the following refined version of Proposition 3.1.

Corollary 3.2. *The conclusions (3.3)–(3.4) in Proposition 3.1 hold true for γ_n defined by*

$$\gamma_1 = \rho_1\|u_1^\star\|, \quad \gamma_n = \rho_n\|u_n^\star\| + \frac{C_1\kappa_n}{\cos\theta}\sum_{k+j=n}\gamma_k\gamma_j. \tag{3.6}$$

Proof. Define

$$\gamma_1^{(1)} = \|u_1^\star\|, \quad \gamma_n^{(1)} = \|u_n^\star\| + \frac{C_1}{\cos\theta}\sum_{k+j=n}\gamma_k^{(1)}\gamma_j^{(1)}. \tag{3.7}$$

Proposition 3.1 applied with (3.7) in place of (3.5) and $\rho_n = 1$ for all $n \in \mathbb{N}$ implies

$$\|u_n(\zeta_0+se^{i\theta})\| \leq \gamma_n^{(1)} e^{-s\cos\theta}.$$

Moreover $\rho_1\gamma_1^{(1)} = \gamma_1$ and

$$\rho_n\gamma_n^{(1)} \leq \rho_n\|u_n^\star\| + \frac{C_1\kappa_n}{\cos\theta}\sum_{k+j=n}\rho_k\gamma_k^{(1)}\rho_j\gamma_j^{(1)}$$

implies by induction that $\rho_n\gamma_n^{(1)} \leq \gamma_n$ for all $n \in \mathbb{N}$. Therefore

$$\rho_n\|u_n(\zeta_0+se^{i\theta})\| \leq \rho_n\gamma_n^{(1)}e^{-s\cos\theta} \leq \gamma_n e^{-s\cos\theta}.$$

Similar arguments obtain (3.4) with γ_n defined by (3.6). $\square$

The difference of two solutions of the extended Navier–Stokes equations also satisfy estimates similar to those in Proposition 3.1 and Corollary 3.2.

Lemma 3.3. *Let* $\zeta_0 \in \mathbb{C}$ *and* $\bar{u}^\star = (u_n^\star)_{n=1}^\infty, \bar{v}^\star = (v_n^\star)_{n=1}^\infty \in V_{\mathbb{C}}^\infty$. *Let* $\bar{u}(\zeta) = S_{\text{ext}}(\zeta,\zeta_0)\bar{u}^\star$, $\bar{v}(\zeta) = S_{\text{ext}}(\zeta,\zeta_0)\bar{v}^\star$. *Let* $\bar{w}^\star = \bar{u}^\star - \bar{v}^\star = (w_n^\star)_{n=1}^\infty$ *and* $\bar{w} = \bar{u} - \bar{v} = (w_n)_{n=1}^\infty$. *For* $s \in (0,\infty)$, $\theta \in (-\pi/2, \pi/2)$ *and* $n \in \mathbb{N}$, *we have*

$$\rho_n \|w_n(\zeta_0 + se^{i\theta})\| \le \mu_n e^{-s\cos\theta} \tag{3.8}$$

and

$$\int_0^s \frac{|\rho_n A w_n(\zeta_0 + \rho e^{i\theta})|^2}{\|\rho_n w_n(\zeta_0 + \rho e^{i\theta})\|} d\rho \le \frac{\mu_n}{\cos\theta}, \tag{3.9}$$

where

$$\mu_1 = \rho_1 \|w_1^\star\|, \quad \mu_n = \rho_n \|w_n^\star\| + \frac{C_1}{\cos\theta} \sum_{k+j=n} \mu_k (\gamma_{j,u} + \gamma_{j,v}), \tag{3.10}$$

where $\gamma_{n,u}, \gamma_{n,v}$ *are defined as in Proposition 3.1, namely*

$$\gamma_{1,u} = \rho_1 \|u_1^\star\|, \quad \gamma_{n,u} = \rho_n \|u_n^\star\| + \frac{C_1}{\cos\theta} \sum_{k+j=n} \gamma_{k,u}\gamma_{j,u}, \tag{3.11}$$

$$\gamma_{1,v} = \rho_1 \|v_1^\star\|, \quad \gamma_{n,v} = \rho_n \|v_n^\star\| + \frac{C_1}{\cos\theta} \sum_{k+j=n} \gamma_{k,v}\gamma_{j,v}. \tag{3.12}$$

Proof. We will prove (3.8) and (3.9) by induction. First, when $n = 1$,

$$\frac{dw_1}{d\zeta} + Aw_1 = 0, \quad w_1(\zeta_0) = w_1^\star.$$

Hence $w_1(\zeta_0 + \zeta) = e^{-\zeta A} w_1^\star$ for $\mathbf{Re}\zeta > 0$. It follows that

$$\|w_1(\zeta_0 + \zeta)\| \le e^{-\mathbf{Re}\zeta} \|w_1^\star\|, \quad \mathbf{Re}\zeta > 0. \tag{3.13}$$

Also, $d\|w_1\|/ds + \cos\theta |Aw_1|^2/\|w_1\| = 0$, thus (3.9) holds for $n = 1$.

Now, let $N > 1$ and assume for induction that (3.8) and (3.9) hold for $n = 1, 2, \ldots, N-1$. We have

$$\frac{dw_N}{d\zeta} + Aw_N + \sum_{k+j=N} \big(B(w_k, u_j) + B(v_k, w_j)\big) = 0.$$

Hence, with $\zeta = \zeta_0 + se^{i\theta}$ we obtain

$$\frac{dw_N}{ds} + e^{i\theta} Aw_N + e^{i\theta} \sum_{k+j=N} \big(B(w_k, u_j) + B(v_k, w_j)\big) = 0.$$

It follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d\|w_N\|^2}{ds} + \cos\theta |Aw_N|^2 \\
&\qquad \le C_1 \|w_N\| \sum_{k+j=N} \Big[\|w_k\|^{1/2} |Aw_k|^{1/2} |Au_j| + \|v_k\|^{1/2} |Av_k|^{1/2} |Aw_j| \Big].
\end{aligned}$$

For $k \in \mathbb{N}$ and $s > 0$ denote

$$\hat{u}_k(s) = \rho_n u_k(\zeta_0 + se^{i\theta}), \ \hat{v}_k(s) = \rho_k v_k(\zeta_0 + se^{i\theta}), \ \hat{w}_k(s) = \rho_k w_k(\zeta_0 + se^{i\theta}).$$

Using the fact that $\rho_N \le \rho_k \rho_j$ for $k+j=N$, we obtain

$$\begin{aligned}(3.14)\quad & \frac{1}{2}\frac{d\|\hat{w}_N\|^2}{ds} + \cos\theta |A\hat{w}_N|^2 \\ &\le C_1 \|\hat{w}_N\| \sum_{k+j=N} \Big[\|\hat{w}_k\|^{1/2} |A\hat{w}_k|^{1/2} |A\hat{u}_j| + \|\hat{v}_k\|^{1/2} |A\hat{v}_k|^{1/2} |A\hat{w}_j| \Big].\end{aligned}$$

Then we have

$$\begin{aligned}&\frac{d\|\hat{w}_N\|}{ds} + \cos\theta \frac{|A\hat{w}_N|^2}{\|\hat{w}_N\|} \\ &\le C_1 \sum_{k+j=N} \Big[\frac{|A\hat{w}_k|^{1/2}}{\|\hat{w}_k\|^{1/4}} \|\hat{u}_j\|^{1/4} \Big] \Big[\frac{|A\hat{u}_j|}{\|\hat{u}_j\|^{1/2}} \|\hat{w}_k\|^{1/2} \Big] \Big[\|\hat{w}_k\|^{1/4} \|\hat{u}_j\|^{1/4} \Big] \\ &\quad + C_1 \sum_{k+j=N} \Big[\frac{|A\hat{v}_k|^{1/2}}{\|\hat{v}_k\|^{1/4}} \|\hat{w}_j\|^{1/4} \Big] \Big[\frac{|A\hat{w}_j|}{\|\hat{w}_j\|^{1/2}} \|\hat{v}_k\|^{1/2} \Big] \Big[\|\hat{v}_k\|^{1/4} \|\hat{w}_j\|^{1/4} \Big] \\ &\le C_1 \Big(\sum_{k+j=N} \frac{|A\hat{w}_k|^2}{\|\hat{w}_k\|} \|\hat{u}_j\| \Big)^{1/4} \Big(\sum_{k+j=N} \frac{|A\hat{u}_j|^2}{\|\hat{u}_j\|} \|\hat{w}_k\| \Big)^{1/2} \Big(\sum_{k+j=N} \|\hat{w}_k\| \|\hat{u}_j\| \Big)^{1/4} \\ &\quad + C_1 \Big(\sum_{k+j=N} \frac{|A\hat{v}_k|^2}{\|\hat{v}_k\|} \|\hat{w}_j\| \Big)^{1/4} \Big(\sum_{k+j=N} \frac{|A\hat{w}_j|^2}{\|\hat{w}_j\|} \|\hat{v}_k\| \Big)^{1/2} \Big(\sum_{k+j=N} \|\hat{v}_k\| \|\hat{w}_j\| \Big)^{1/4}.\end{aligned}$$

Using Proposition 3.1 along with the induction hypothesis

$$\begin{aligned}&\frac{d\|\hat{w}_N\|}{ds} + \cos\theta \frac{|A\hat{w}_N|^2}{\|\hat{w}_N\|} \\ &\le C_1 e^{-\frac{5}{4}s\cos\theta} \Big(\sum_{k+j=N} \frac{|A\hat{w}_k|^2}{\|\hat{w}_k\|} \gamma_{j,u} \Big)^{1/4} \Big(\sum_{k+j=N} \frac{|A\hat{u}_j|^2}{\|\hat{u}_j\|} \mu_k \Big)^{1/2} \Big(\sum_{k+j=N} \mu_k \gamma_{j,u} \Big)^{1/4} \\ &\quad + C_1 e^{-\frac{5}{4}s\cos\theta} \Big(\sum_{k+j=N} \frac{|A\hat{v}_k|^2}{\|\hat{v}_k\|} \mu_j \Big)^{1/4} \Big(\sum_{k+j=N} \frac{|A\hat{w}_j|^2}{\|\hat{w}_j\|} \gamma_{k,v} \Big)^{1/2} \Big(\sum_{k+j=N} \gamma_{k,v} \mu_j \Big)^{1/4}.\end{aligned}$$

We have

$$\begin{aligned}\|\hat{w}_N(s)\| &\le e^{-s\cos\theta} \|w_N^\star\| \\ &\quad + e^{-s\cos\theta} C_1 \Big(\sum_{k+j=N} \mu_k \gamma_{j,u} \Big)^{1/4} \Big[\int_0^s \sum_{k+j=N} \frac{|A\hat{w}_k(\rho)|^2}{\|\hat{w}_k(\rho)\|} \gamma_{j,u} d\rho \Big]^{1/4} \\ &\qquad \times \Big[\int_0^s \sum_{k+j=N} \frac{|A\hat{u}_j(\rho)|^2}{\|\hat{u}_j(\rho)\|} \mu_k d\rho \Big]^{1/2} \Big[\int_0^s e^{-\rho\cos\theta} d\rho \Big]^{1/4} \\ &\quad + e^{-s\cos\theta} C_1 \Big(\sum_{k+j=N} \mu_j \gamma_{k,u} \Big)^{1/4} \Big[\int_0^s \sum_{k+j=N} \frac{|A\hat{v}_k(\rho)|^2}{\|\hat{v}_k(\rho)\|} \mu_j d\rho \Big]^{1/4} \\ &\qquad \times \Big[\int_0^s \sum_{k+j=N} \frac{|A\hat{w}_j(\rho)|^2}{\|\hat{w}_j(\rho)\|} \gamma_{k,v} d\rho \Big]^{1/2} \Big[\int_0^s e^{-\rho\cos\theta} d\rho \Big]^{1/4}.\end{aligned}$$

Hence

$$\begin{aligned}\|\hat{w}_N(s)\| &\le e^{-s\cos\theta}\|\hat{w}_N^\star\| \\ &\quad + e^{-s\cos\theta}\frac{C_1}{\cos\theta}\sum_{k+j=N}\mu_k\gamma_{j,u} + e^{-s\cos\theta}\frac{C_1}{\cos\theta}\sum_{k+j=N}\mu_j\gamma_{k,v} \\ &\le e^{-s\cos\theta}\Big\{\|\hat{w}_N^\star\| + \frac{C_1}{\cos\theta}\sum_{k+j=N}\mu_k(\gamma_{j,u}+\gamma_{j,v})\Big\} = e^{-s\cos\theta}\mu_N.\end{aligned}$$

Integrating (3.14), we obtain (3.9) for $n = N$ by similar estimates, thus completing the induction. □

Similar to Corollary 3.2, we derive the following version of Lemma 3.3 in which the estimates depend on κ_n explicitly.

Corollary 3.4. *The statements in Lemma 3.3 hold true for*

$$\mu_1 = \rho_1\|w_1^\star\|, \quad \mu_n = \rho_n\|w_n^\star\| + \frac{C_1\kappa_n}{\cos\theta}\sum_{k+j=n}\mu_k(\gamma_{j,u}+\gamma_{j,v}), \tag{3.15}$$

where $\gamma_{n,u}, \gamma_{n,v}$ *are defined as in Corollary 3.2, namely*

$$\gamma_{1,u} = \rho_1\|u_1^\star\|, \quad \gamma_{n,u} = \rho_n\|u_n^\star\| + \frac{C_1\kappa_n}{\cos\theta}\sum_{k+j=n}\gamma_{k,u}\gamma_{j,u}, \tag{3.16}$$

$$\gamma_{1,v} = \rho_1\|v_1^\star\|, \quad \gamma_{n,v} = \rho_n\|v_n^\star\| + \frac{C_1\kappa_n}{\cos\theta}\sum_{k+j=n}\gamma_{k,v}\gamma_{j,v}. \tag{3.17}$$

Proof. The proof is almost identical to the proof of Corollary 3.2. □

The estimates in Lemma 3.3 are adequate for establishing the Lipschitz continuity of the $S_{\mathrm{ext}}(t)\bar{u}^0$, when $\bar{u}^0$ is close to the origin. We recall as Lemma 3.5 some basic estimates from Proposition 3.9 of [4].

Lemma 3.5 ([4])**.** *Given* $N \in \mathbb{N}$ *and* $u_n^0 \in V, n = 1,\dots,N$. *Let* $u_n(\zeta), n = 1,\dots,N, \mathbf{Re}\zeta > 0$, *be the solutions to the extended Navier–Stokes equations* (2.23) *with* $\zeta_0 = 0$ *satisfying* $u_n(0) = u_n^0$. *Denote*

$$S_n(\zeta) = \sum_{j=1}^n \rho_j\|u_j(\zeta)\|, \quad S_n = S_n(0), \quad n = 1,\dots,N, \quad \mathbf{Re}\zeta > 0.$$

Suppose $S_N < \varepsilon_0$. *Let*

$$\gamma_1 = \rho_1\|u_1^0\|, \quad \gamma_n = \rho_n\|u_n^0\| + 6C_1\sum_{k+j=n}\gamma_k\gamma_j, \quad 1 < n \le N, \tag{3.18}$$

$$\tilde{\gamma}_1 = \gamma_1, \quad \tilde{\gamma}_n = \gamma_n + 3C_1\sum_{k+j=n}\tilde{\gamma}_k\tilde{\gamma}_j, \quad 1 < n \le N. \tag{3.19}$$

Then

$$\rho_n\|u_n(t)\| \le \gamma_n e^{-t}, \quad t > 0,\ 1 \le n \le N, \tag{3.20}$$

$$\rho_n\|u_n(\zeta)\| \le \tilde{\gamma}_n e^{-\mathbf{Re}\zeta}, \quad \zeta \in E,\ 1 \le n \le N, \tag{3.21}$$

where the domain E is defined in (A.5)*, and*

$$\sum_{k=1}^{n} \tilde\gamma_j \le 2\sum_{k=1}^{n}\gamma_j \le 4S_n, \quad 1\le n\le N. \tag{3.22}$$

In particular,

$$S_n(t) \le 2S_n e^{-t}, \quad t\ge 0,\ 1\le n\le N, \tag{3.23}$$

and

$$S_n(\zeta) \le 4S_n e^{-\mathbf{Re}\zeta}, \quad \zeta\in E,\ 1\le n\le N. \tag{3.24}$$

Theorem 3.6. *Let $B_{V^\star}(\varepsilon_0/2)$ be the open ball in $V^\star$ of radius $\varepsilon_0/2$ centered at the origin. Then the map $S_{\rm ext}(\zeta): B_{V^\star}(\varepsilon_0/2)\to V^\star$ is Lipschitz continuous for all $\zeta\in E$. More precisely,*

$$\|S_{\rm ext}(t)\bar u^0 - S_{\rm ext}(t)\bar v^0\|_\star \le 2e^{-t}\|\bar u^0-\bar v^0\|_\star, \quad t>0, \tag{3.25}$$

$$\|S_{\rm ext}(\zeta)\bar u^0 - S_{\rm ext}(\zeta)\bar v^0\|_\star \le 4e^{-\mathbf{Re}\zeta}\|\bar u^0-\bar v^0\|_\star, \quad \zeta\in E. \tag{3.26}$$

for any $\bar u^0,\bar v^0\in V^\star$ such that $\|\bar u^0\|_\star<\varepsilon_0/2$ and $\|\bar v^0\|_\star<\varepsilon_0/2$.

Proof. Let $\bar u(\zeta)=S_{\rm ext}(\zeta)\bar u^0$, $\bar v(\zeta)=S_{\rm ext}(\zeta)\bar v^0$. Let $\bar w^0=\bar u^0-\bar v^0=(w_n^0)_{n=1}^\infty$ and $\bar w=\bar u-\bar v=(w_n)_{n=1}^\infty$. Let

$$\mu_1=\rho_1\|w_1^0\|, \quad \mu_n=\rho_n\|w_n^0\|+C_1\sum_{k+j=n}\mu_k(\gamma_{j,u}+\gamma_{j,v}), \quad n>1, \tag{3.27}$$

and

$$\tilde\mu_1=\mu_1, \quad \tilde\mu_n=\mu_n+3C_1\sum_{k+j=n}\tilde\mu_k(\tilde\gamma_{j,u}+\tilde\gamma_{j,v}), \quad n>1, \tag{3.28}$$

where $\gamma_{j,u},\tilde\gamma_{j,u}$ (respectively $\gamma_{j,v},\tilde\gamma_{j,v}$) are defined as in Lemma 3.5 for $\bar u^0$ (respectively $\bar v^0$).
Claims:

$$\rho_n\|w_n(t)\|\le\mu_n e^{-t}, \quad t>0, \quad n\ge1, \tag{3.29}$$

$$\rho_n\|w_n(\zeta)\|\le\tilde\mu_n e^{-\mathbf{Re}\zeta}, \quad \zeta\in E, \quad n\ge1, \tag{3.30}$$

$$\sum_{n=1}^\infty\tilde\mu_n\le 2\sum_{n=1}^\infty\mu_n\le 4\|\bar w^0\|_\star. \tag{3.31}$$

After proving these claims, then inequality (3.25) (respectively inequality (3.26)) follows from (3.29) (respectively (3.30)) and (3.31).
Proof of the claims. Let $\bar u=(u_n)_{n=1}^\infty$, $\bar u^0=(u_n^0)_{n=1}^\infty$, $\bar v=(v_n)_{n=1}^\infty$ and $\bar v^0=(v_n^0)_{n=1}^\infty$. Let

$$S_{n,u}=\sum_{j=1}^n\rho_j\|u_j^0\|,\ S_{n,v}=\sum_{j=1}^n\rho_j\|v_j^0\|,\ S_{n,w}=\sum_{j=1}^n\rho_j\|w_j^0\|.$$

Since $S_{n,u},S_{n,v}<\varepsilon_0/2$ for all $n\in\mathbb{N}$, Lemma 3.5 implies

$$\sum_{n=1}^\infty\gamma_{n,u}\le\varepsilon_0,\ \sum_{n=1}^\infty\gamma_{n,u}\le\varepsilon_0,\ \sum_{n=1}^\infty\tilde\gamma_{n,u}\le2\varepsilon_0 \text{ and } \sum_{n=1}^\infty\tilde\gamma_{n,u}\le2\varepsilon_0.$$

We also have

$$\sum_{j=1}^{n}\mu_j \le S_{n,w} + C_1\Big(\sum_{j=1}^{n-1}\mu_j\Big)\sum_{j=1}^{n-1}(\gamma_{j,u}+\gamma_{j,v})$$
$$\le S_{n,w} + C_1\Big(\sum_{j=1}^{n}\mu_j\Big)(2\varepsilon_0) = S_{n,w} + \frac{1}{12}\sum_{j=1}^{n}\mu_j,$$

thus $\sum_{j=1}^{n}\mu_j \le 2S_{n,w}$. Letting $n\to\infty$ gives the second inequality of (3.31). Now, summing up (3.28),

$$\sum_{j=1}^{n}\tilde\mu_j \le \sum_{j=1}^{n}\mu_j + 3C_1\Big(\sum_{j=1}^{n-1}\tilde\mu_j\Big)\sum_{j=1}^{n-1}(\tilde\gamma_{j,u}+\tilde\gamma_{j,v})$$
$$\le \sum_{j=1}^{n}\mu_j + 3C_1\Big(\sum_{j=1}^{n}\tilde\mu_j\Big)(4\varepsilon_0)$$
$$= \sum_{j=1}^{n}\mu_j + \frac{1}{2}\sum_{j=1}^{n}\tilde\mu_j.$$

Hence $\sum_{j=1}^{n}\tilde\mu_j \le 2\sum_{j=1}^{n}\mu_j \le 4S_{n,w}$. Letting $n\to\infty$ yields the first inequality in (3.31).

Applying Lemma 3.3 with $\zeta_0 = 0$, $s = t$ and $\theta = 0$, we have

$$\rho_n\|w_n(t)\| \le \mu_n^0 e^{-t}, \quad t>0, \quad n\ge 1, \tag{3.32}$$

where

$$\mu_1^0 = \rho_1\|w_1^0\|, \quad \mu_n^0 = \rho_n\|w_n^0\| + C_1\sum_{k+j=n}\mu_k^0(\gamma_{j,u}^0+\gamma_{j,v}^0), \quad n>1, \tag{3.33}$$

$$\gamma_{1,u}^0 = \rho_1\|u_1^0\|, \quad \gamma_{n,u}^0 = \rho_n\|u_n^0\| + C_1\sum_{k+j=n}\gamma_{k,u}^0\gamma_{j,u}^0, \tag{3.34}$$

$$\gamma_{1,v}^0 = \rho_1\|v_1^0\|, \quad \gamma_{n,v}^0 = \rho_n\|v_n^0\| + C_1\sum_{k+j=n}\gamma_{k,v}^0\gamma_{j,v}^0. \tag{3.35}$$

Note that the above $\gamma_{n,u}^0, \gamma_{n,v}^0$ given in Proposition 3.1 and used in Lemma 3.3 are not the same as $\gamma_{n,u}, \gamma_{n,v}$ given in Lemma 3.5 and used in the current theorem. However, based on their formulas, we have $\gamma_{n,u}^0 \le \gamma_{n,u}$ and $\gamma_{n,v}^0 \le \gamma_{n,v}$. Hence $\mu_n^0 \le \mu_n$, and therefore (3.29) follows from (3.32).

Let $\zeta = t_0 + se^{i\theta} \in E$, then $\cos\theta > 1/(3e^{t_0})$. Applying Lemma 3.3 with $\zeta_0 = t_0$, we have

$$\rho_n\|w_n(t_0+se^{i\theta})\| \le \tilde\mu_n(t_0)e^{-s\cos\theta} \tag{3.36}$$

where $\tilde\mu_1(t_0) = \rho_1\|w_1(t_0)\|$;

$$\tilde\mu_n(t_0) = \rho_n\|w_n(t_0)\| + 3C_1e^{t_0}\sum_{k+j=n}\tilde\mu_k(t_0)(\tilde\gamma_{j,u}(t_0)+\tilde\gamma_{j,v}(t_0)), \quad n>1, \tag{3.37}$$

where $\tilde\gamma_{1,u}(t_0) = \rho_1\|u_1(t_0)\|$;

$$\tilde\gamma_{n,u}(t_0) = \rho_n\|u_n(t_0)\| + 3C_1e^{t_0}\sum_{k+j=n}\tilde\gamma_{k,u}(t_0)\tilde\gamma_{j,u}(t_0), \quad n>1, \tag{3.38}$$

and $\tilde{\gamma}_{j,v}(t_0)$ are defined similarly using $v_n(t_0)$.

It is known that (cf. proof of Theorem 3.7 in [4])

$$\tilde{\gamma}_{j,u}(t_0) \le \tilde{\gamma}_{j,u} e^{-t_0} \text{ and } \tilde{\gamma}_{j,v}(t_0) \le \tilde{\gamma}_{j,v} e^{-t_0}.$$

Hence by (3.29) and by induction, one can show $\tilde{\mu}_1(t_0) \le \mu_1 e^{-t_0} = \tilde{\mu}_1 e^{-t_0}$,

$$\tilde{\mu}_n(t_0) \le \mu_n e^{-t_0} + 3C_1 e^{t_0} \sum_{k+j=n} \mu_k e^{-t_0} (\tilde{\gamma}_{j,u} e^{-t_0} + \tilde{\gamma}_{j,v} e^{-t_0}) = \tilde{\mu}_n e^{-t_0},\ n > 1.$$

Therefore (3.30) follows from (3.36). The proof is complete. □

Our next goal is to study the extended Navier–Stokes equations in the whole space $V^\star$ rather than only near the origin. For that purpose we require that the positive numbers κ_n in (3.1) satisfy

$$\lim_{n\to\infty} \kappa_n^{1/n} = 0. \tag{3.39}$$

Note that (3.39) holds true for the weights defining $V^\star$ given in Definition 2.1 because of the rapid growth of $L_{3,n}$.

First recall Theorem 4.3 in [4], the existence theorem in $V^\star$ for the extended Navier–Stokes equations. Note that the constant M_0 appearing here results from using Lemma A.3 in place of Lemma 4.2 in [4].

Theorem 3.7 ([4])**.** *Let $\bar{u}^0 \in V^\star$. Then $S_{\rm ext}(t)\bar{u}^0 \in V^\star$ for all $t > 0$. More precisely,*

$$\|S_{\rm ext}(t)\bar{u}^0\|_\star \le M e^{-t},\ t > 0, \tag{3.40}$$

where

$$M = \|\bar{u}^0\|_\star + C_1 \sum_{n=2}^{\infty} \kappa_n (n-1) M_0^n, \tag{3.41}$$

$$M_0 = \max\{1, 2C_1\kappa_n(n-1)\} \max\{1, 2\|\bar{u}^0\|_\star\}. \tag{3.42}$$

We establish the well-posedness of the extended Navier–Stokes equations now.

Theorem 3.8. *$S_{\rm ext}(t)$ is continuous from $V^\star$ to $V^\star$, for $t \in [0,\infty)$. More precisely, for any $\bar{u}^0 \in V^\star$ and $\varepsilon > 0$, there is $\delta > 0$ such that*

$$\|S_{\rm ext}(t)\bar{v}^0 - S_{\rm ext}(t)\bar{u}^0\|_\star < \varepsilon e^{-t},$$

for all $\bar{v}^0 \in V^\star$ satisfying $\|\bar{v}^0 - \bar{u}^0\|_\star < \delta$ and for all $t \ge 0$.

Proof. Given $\bar{u}^0 \in V^\star$. Let $\bar{v}^0 \in V^\star$ such that $\|\bar{u}^0 - \bar{v}^0\|_\star < 1$. Let $\bar{u}(t) = S_{\rm ext}(t)\bar{u}^0$, $\bar{v}(t) = S_{\rm ext}(t)\bar{v}^0$ and $\bar{w} = \bar{u} - \bar{v} = (w_n)_{n=1}^\infty$. Let $\gamma_{n,u}$, $\gamma_{n,v}$ and μ_n be defined by

$$\gamma_{1,u} = \rho_1\|u_1^0\|, \quad \gamma_{n,u} = \rho_n\|u_n^0\| + C_1\kappa_n \sum_{k+j=n} \gamma_{k,u}\gamma_{j,u},\ n > 1, \tag{3.43}$$

$$\gamma_{1,v} = \rho_1\|v_1^0\|, \quad \gamma_{n,v} = \rho_n\|v_n^0\| + C_1\kappa_n \sum_{k+j=n} \gamma_{k,v}\gamma_{j,v},\ n > 1, \tag{3.44}$$

and

$$\mu_1 = \rho_1\|w_1^0\|, \quad \mu_n = \rho_n\|w_n^0\| + C_1\kappa_n \sum_{k+j=n} \mu_k(\gamma_{j,u} + \gamma_{j,v}). \tag{3.45}$$

Taking $\zeta_0 = 0$, $s = t$ and $\theta = 0$ in Corollaries 3.2 and 3.4 we obtain

$$\rho_n\|u_n(t)\| \le \gamma_{n,u}e^{-t}, \quad \rho_n\|v_n(t)\| \le \gamma_{n,v}e^{-t}, \quad t \ge 0 \tag{3.46}$$

and

$$\rho_n\|w_n(t)\| \le \mu_n e^{-t}, \quad t \ge 0. \tag{3.47}$$

Let

$$h_1 = \rho_1(\|u_1^0\| + \|v_1^0\|),\ h_n = \rho_n(\|u_n^0\| + \|v_n^0\|) + C_1\kappa_n \sum_{k+j=n} h_k h_j,\ n > 1. \tag{3.48}$$

Noting that $\mu_1 \le \gamma_{1,u}+\gamma_{1,v} = h_1$ and $\gamma_{k,u}\gamma_{j,u}+\gamma_{k,v}\gamma_{j,v} \le (\gamma_{k,u}+\gamma_{k,v})(\gamma_{j,u}+\gamma_{j,v})$, one can prove by induction that

$$\gamma_{n,u} + \gamma_{n,v} \le h_n \text{ and then } \mu_n \le h_n, \quad n \in \mathbb{N}.$$

By Lemma A.3, we have

$$\sum_{n=1}^{\infty} h_n \le \|\bar{u}^0\|_\star + \|\bar{v}^0\|_\star + C_1 \sum_{n=1}^{\infty} \kappa_n(n-1){M_0}^n,$$

where $M_0 = K\max\{1, 2(\|\bar{u}^0\|_\star + \|\bar{v}^0\|_\star)\}$ and $K = \max\{1, 2C_1\kappa_n(n-1) : n > 1\}$. Hence $\|\bar{v}_0\|_\star \le \|\bar{u}_0\|_\star + 1$ implies

$$\sum_{n=1}^{\infty} h_n \le M \overset{\text{def}}{=\!=} 2\|\bar{u}^0\|_\star + 1 + C_1 \sum_{n=1}^{\infty} \kappa_n(n-1)\Big[2K(2\|\bar{u}^0\|_\star + 1)\Big]^n,$$

which is finite and independent of $\bar{v}^0$. For $N > 0$, considering $\sum_{n>N}\mu_n$, we have

$$\begin{aligned}
\sum_{n>N} \mu_n &\le \sum_{n>N} \rho_n\|w_n^0\| + C_1 \sum_{n>N} \kappa_n \sum_{k+j=n} \mu_k(\gamma_{j,u} + \gamma_{j,v}) \\
&\le \sum_{n=1}^{\infty} \rho_n\|w_n^0\| + C_1 \sum_{n>N} \kappa_n(\sum_{k+j=n} h_k h_j) \\
&\le \|w_n^0\|_\star + M^2 C_1 \sum_{n>N} \kappa_n.
\end{aligned}$$

Given $\varepsilon > 0$. Let $N = N(\|\bar{u}^0\|_\star)$ be sufficiently large such that

$$C_1 M^2 \sum_{n>N} \kappa_n < \varepsilon/2.$$

By virtue of (3.45), $\mu_1 \le \|\bar{w}^0\|_\star$ and $\mu_n \le \|\bar{w}^0\|_\star + \tilde{M}\sum_{j=1}^{n-1}\mu_j, n > 1$, where $\tilde{M} = C_1 M(\sup_{n\ge 2}\kappa_n) > 0$. Therefore $\mu_n \le \theta_n\|\bar{w}^0\|_\star$, for all $n \in \mathbb{N}$, where θ_n are positive numbers defined recursively by

$$\theta_1 = 1, \quad \theta_n = 1 + \tilde{M}\sum_{j=1}^{n-1}\theta_j, \quad n > 1.$$

Take $\delta = \varepsilon/[2(1+\sum_{n=1}^N \theta_n)]$. For $\|\bar w^0\|_\star = \|\bar u^0 - \bar v^0\|_\star < \delta$, we have

$$\begin{aligned}\sum_{n=1}^\infty \mu_n &= \sum_{n=1}^N \mu_n + \sum_{n>N} \mu_n\\ &\le \|\bar w^0\|_\star \Big(1+\sum_{n=1}^N \theta_n\Big) + C_1 M^2 \sum_{n>N} \kappa_n < \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Therefore, inequality (3.47) implies $\|\bar w(t)\|_\star \le e^{-t}\sum_{n=1}^\infty \mu_n < \varepsilon e^{-t}$ for all $t \ge 0$. □

For the complexified extended Navier–Stokes equations the following estimates are important for our study of the normal form of the Navier–Stokes equations and the normalization map in subsequent sections. Combining the estimates in Proposition 3.6 and the proof of Theorem 3.7 in [4] we obtain

Proposition 3.9. *Let $(u_n(\zeta))_{n=1}^\infty$ for $\mathbf{Re}\zeta > 0$ be the solution to the extended Navier–Stokes equations (2.23) with $\zeta_0 = 0$ and initial condition $u_n(0) = u_n^0 \in V_{\mathbb C}$ for $n \in \mathbb N$. Then*

$$\rho_n \|u_n(t)\| \le \gamma_n e^{-t},\ \ t \ge 0, \tag{3.49}$$

$$\rho_n \|u_n(\zeta)\| \le \tilde\gamma_n e^{-\mathbf{Re}\zeta},\ \ \zeta \in E, \tag{3.50}$$

where

$$\gamma_1 = \rho_1\|u_1^0\|,\ \ \gamma_n = \rho_n\|u_n^0\| + C_1\kappa_n \sum_{k+j=n} \gamma_k\gamma_j, \tag{3.51}$$

$$\tilde\gamma_1 = \gamma_1,\ \ \tilde\gamma_n = \gamma_n + 3C_1\kappa_n \sum_{k+j=n} \tilde\gamma_k\tilde\gamma_j, \tag{3.52}$$

Proof. Inequality (3.49) was obtained in Corollary 3.2. Given $\zeta = t + se^{i\theta} \in E$, let

$$\tilde\gamma_1(t) = \rho_1\|u_1(t)\|,\ \ \tilde\gamma_n(t) = \rho_n\|u_n(t)\| + \frac{C_1\kappa_n}{\cos\theta}\sum_{k+j=n} \tilde\gamma_k(t)\tilde\gamma_j(t). \tag{3.53}$$

It follows from Corollary 3.2 that

$$\rho_n\|u_n(t+se^{i\theta})\| \le \tilde\gamma_n(t) e^{-s\cos\theta}. \tag{3.54}$$

Claim:

$$\tilde\gamma_n(t) \le \tilde\gamma_n e^{-t}. \tag{3.55}$$

Clearly $\tilde\gamma_1(t) = \rho_1\|u_1(t)\| \le \gamma_1 e^{-t} = \tilde\gamma_1 e^{-t}$. For induction, assume $\tilde\gamma_k(t) \le \tilde\gamma_k e^{-t}$ for all $k < n$. Since $\zeta = t + se^{i\theta} \in E$, we have $e^{-t}/\cos\theta < 3$. Therefore

$$\tilde\gamma_n(t) \le e^{-t}\Big\{\gamma_n + \frac{C_1\kappa_n e^{-t}}{\cos\theta}\sum_{k+j=n}\tilde\gamma_k\tilde\gamma_j\Big\} \le \tilde\gamma_n e^{-t}. \tag{3.56}$$

Combining (3.54) and (3.55) yields (3.50) for all $n \in \mathbb N$. □

Similar arguments using Corollary 3.4 yield

Proposition 3.10. *Let $\bar{u}^0, \bar{v}^0 \in V_{\mathbb{C}}^{\infty}$ and $\bar{w}(\zeta) = (w_n)_{n=1}^{\infty} = S_{\mathrm{ext}}(\zeta)\bar{u}^0 - S_{\mathrm{ext}}(\zeta)\bar{v}^0$. Then we have*

$$\rho_n \|w_n(t)\| \le \mu_n e^{-t}, \ t \ge 0, \tag{3.57}$$

$$\rho_n \|w_n(\zeta)\| \le \tilde{\mu}_n e^{-\mathbf{Re}\zeta}, \ \zeta \in E, \tag{3.58}$$

where

$$\mu_1 = \rho_1 \|w_1^0\|, \ \mu_n = \rho_n \|w_n^0\| + C_1 \kappa_n \sum_{k+j=n} \mu_k (\gamma_{j,u} + \gamma_{j,v}), \tag{3.59}$$

$$\tilde{\mu}_1 = \mu_1, \ \tilde{\mu}_n = \mu_n + 3C_1 \kappa_n \sum_{k+j=n} \tilde{\mu}_k (\tilde{\gamma}_{j,u} + \tilde{\gamma}_{j,v}), \tag{3.60}$$

and $\gamma_{j,u}$ and $\tilde{\gamma}_{j,u}$ (respectively $\gamma_{j,v}$ and $\tilde{\gamma}_{j,v}$) are defined as in Proposition 3.9 for $\bar{u}^0$ (respectively $\bar{v}^0$).

The complex version of Theorem 3.8 then easily follows.

Theorem 3.11. *For any $\zeta \in E$, $S_{\mathrm{ext}}(\zeta)$ maps $V_{\mathbb{C}}^{\star}$ to $V_{\mathbb{C}}^{\star}$ and is continuous.*

As a consequence of the study of the extended Navier–Stokes equations in this section, we show the explicit continuous dependence on the initial data of the regular solutions $S(\zeta)u^0$ with small norm $\|u^0\|$. Our explicit estimates will be used in proving the continuity of the normalization map in Section 4.

Corollary 3.12. *Suppose $u^0, v^0 \in V_{\mathbb{C}}$ satisfy $\|u^0\| < \varepsilon_0/2$ and $\|v^0\| < \varepsilon_0/2$. Let $w(\zeta) = S(\zeta)u^0 - S(\zeta)v^0$ for $\zeta \in E$. Then*

$$\|w(t)\| \le 2\|w^0\| e^{-t}, \quad t > 0, \tag{3.61}$$

$$\|w(\zeta)\| \le 4\|w^0\| e^{-\mathbf{Re}\zeta}, \quad \zeta \in E. \tag{3.62}$$

Proof. Let $\bar{u}^0 = (u^0, 0, 0, \ldots), \bar{v}^0 = (v^0, 0, 0, \ldots), \bar{w}^0 = \bar{u}^0 - \bar{v}^0$ and let $\rho_n = 1$ for all $n \in \mathbb{N}$. Let $\bar{u}(\zeta) = (u_n(\zeta))_{n=1}^{\infty} = S_{\mathrm{ext}}(\zeta)\bar{u}^0$ and $\bar{v}(\zeta) = (v_n(\zeta))_{n=1}^{\infty} = S_{\mathrm{ext}}(\zeta)\bar{v}^0$. It is known from [4] that $S(\zeta)u^0 = \sum_{n=1}^{\infty} u_n(\zeta)$ and $S(\zeta)v^0 = \sum_{n=1}^{\infty} v_n(\zeta)$. Let $S_n^0 = \sum_{k=1}^{n} \|w_k^0\|$, $S_n = \sum_{k=1}^{n} \mu_k$ and $\tilde{S}_n = \sum_{k=1}^{n} \tilde{\mu}_k$. By Proposition 3.10

$$\sum_{k=1}^{n} \mu_k \le \sum_{k=1}^{n} \|w_k^0\| + C_1 \Big(\sum_{k=1}^{n-1} \mu_k \Big) \Big(\sum_{j=1}^{n-1} \gamma_{j,u} + \gamma_{j,v} \Big). \tag{3.63}$$

By Lemma 3.5, we have

$$\sum_{j=1}^{n} \gamma_{j,u^0} \le 2\|u^0\| < \varepsilon_0, \quad \sum_{j=1}^{n} \tilde{\gamma}_{j,u^0} \le 4\|u^0\| < 2\varepsilon_0, \tag{3.64}$$

and similarly,

$$\sum_{j=1}^{n} \gamma_{j,v^0} < \varepsilon_0, \quad \sum_{j=1}^{n} \tilde{\gamma}_{j,v^0} < 2\varepsilon_0. \tag{3.65}$$

Therefore

$$S_n \le S_n^0 + 2\varepsilon_0 C_1 S_{n-1} \le S_n^0 + (1/2) S_{n-1}.$$

since $2\varepsilon_0 C_1 = 1/12$. By induction, we obtain $S_n \le 2S_n^0$. Letting $n \to \infty$ we obtain (3.61).

Similarly, using the formula of $\tilde{\mu}_n$,

$$\tilde{S}_n \le S_n + 12\varepsilon_0 C_1 \tilde{S}_{n-1} = S_n + (1/2)\tilde{S}_{n-1}.$$

By induction, one can prove $\tilde{S}_n \le 2S_n \le 4S_n^0$. Letting $n \to \infty$ gives (3.62). $\square$

4. Solutions to the normal form

In [4] we proved that there are positive numbers $\rho_n, n \in \mathbb{N}$, such that the solution $S_{\mathrm{normal}}(t)\bar{\xi}$ of the normal form (2.17) is in $S_A^\star$ for all $t > 0$ when $\|\bar{\xi}\|_\star$ is small. The numbers $\rho_n, n > 1$, are of the form (3.1), namely,

$$\rho_1 = 1, \quad \rho_n = \kappa_n \min\{\rho_k \rho_j : k + j = n\}, \quad n \geq 2,$$

for some particular numbers $\kappa_n \in (0, 1]$.

In this section, we find a condition on κ_n under which $S_{\mathrm{normal}}(t)\bar{\xi}$ belongs to $S_A^\star$ for all $t \geq 0$, whenever $\bar{\xi} \in S_A^\star$. This says that the semigroup $S_{\mathrm{normal}}(t), t > 0$, generated by the solutions of the normal form of the Navier–Stokes equations leaves invariant the whole space $S_A^\star$. Furthermore, we establish the continuity (but not necessarily Lipschitz continuity) of each $S_{\mathrm{normal}}(t)$ as a map form $S_A^\star$ to $S_A^\star$, which means that the normal form is a well-posed system.

We take $\kappa_n \in (0, 1]$ satisfying

$$\lim_{n\to\infty} n\kappa_n^{1/n} = 0. \tag{4.1}$$

Note that condition (4.1) is more stringent than (3.39). Moreover, the weights explicitly defined in Definition 2.1 also satisfy this more stringent condition.

Theorem 4.1. *Let $\bar{\xi} = (\xi_n)_{n=1}^\infty \in S_A^\star$. Then $S_{\mathrm{normal}}(t)\bar{\xi} \in S_A^\star$ for all $t \geq 0$. Moreover,*

$$\|S_{\mathrm{normal}}(t)\bar{\xi}\|_\star \leq M e^{-t}, \quad t > 0, \tag{4.2}$$

where M is a positive number depending on $\|\bar{\xi}\|_\star$ and the sequence $(\rho_n)_{n=1}^\infty$.

Proof. Let

$$x_1 = \eta_1 = \gamma_1 = \tilde{\gamma}_1 = \rho_1 \|\xi_1\|.$$

and $n > 1$, we recursively define

$$x_n = \rho_n \|\xi_n\|, \tag{4.3}$$

$$\eta_n = x_n + \kappa_n D_n \sum_{k+j=n} \tilde{\gamma}_k \tilde{\gamma}_j, \tag{4.4}$$

$$\gamma_n = \eta_n + C_1 \kappa_n \sum_{k+j=n} \gamma_k \gamma_j, \tag{4.5}$$

$$\tilde{\gamma}_n = \gamma_n + 3C_1 \kappa_n \sum_{k+j=n} \tilde{\gamma}_k \tilde{\gamma}_j, \tag{4.6}$$

$$D_n = C_1 e^{2a_0} n^{3/2} C_2^{n-2} 2^{-n} n!, \tag{4.7}$$

where the constants a_0 and C_2 are defined in Lemma A.2.

Claims:

$$\rho_n \|q_n(0, \bar{\xi})\| \leq \eta_n, \quad n \in \mathbb{N}, \tag{4.8}$$

$$\rho_n \|u_n(t)\| = \rho_n \|q_n(t, \bar{\xi}) e^{-nt}\| \leq \gamma_n e^{-t}, \quad n \in \mathbb{N}, \quad t > 0, \tag{4.9}$$

$$\rho_n\|u_n(\zeta)\| = \rho_n\|q_n(t,\bar\xi)e^{-n\zeta}\| \le \tilde\gamma_n e^{-\mathbf{Re}\zeta}, \quad n\in\mathbb{N}, \quad \zeta\in E. \tag{4.10}$$

Indeed, following the proof of Lemma 6.2 in [4], we obtain

$$\rho_n\|q_n(0,\bar\xi)\| \le x_n + \kappa_n(D_{1,n}+D_{2,n})\sum_{k+j=n}\tilde\gamma_k\tilde\gamma_j,$$

where

$$D_{1,n} = C_1(n-2)^{3/4}\frac{n^2}{4}C_2^{n-2}\int_0^\infty e^{-2\tau}(\tau+a_0)^{n-2}d\tau,$$
$$D_{2,n} = C_1 n^{3/2}\frac{n^2}{4}C_2^{n-2}\int_{-\infty}^0 e^{2\tau}(|\tau|+a_0)^{n-2}d\tau.$$

Elementary calculation shows $D_{1,n}+D_{2,n}\le D_n$ and hence (4.8) follows. Then (4.9) and (4.10) follow by the virtue of Proposition 3.9 with $u_n^0 = q_n(0,\bar\xi)$.

Note that

$$x_n \le \eta_n \le \gamma_n \le \tilde\gamma_n.$$

For each n, summing up (4.4)–(4.6) gives

$$\tilde\gamma_n \le x_n + \kappa_n(D_n+4C_1)\sum_{k+j=n}\tilde\gamma_k\tilde\gamma_j.$$

From (4.1),

$$\lim_{n\to\infty}\Big\{\kappa_n(D_n+4C_1)\Big\}^{1/n} = 0. \tag{4.11}$$

By Lemma A.3 and (4.11), we have $\sum_{n=1}^\infty \tilde\gamma_n = M < \infty$. Then it follows from (4.9) that

$$\|S_{\rm normal}(t)\bar\xi\|_\star \le \sum_{n=1}^\infty \|e^{-nt}q_n(t,\bar\xi)\| \le e^{-t}\sum_{n=1}^\infty\gamma_n \le Me^{-t},$$

for all $t>0$. □

Theorem 4.2. *The map $S_{\rm normal}(t)\bar\xi$ is continuous in $\bar\xi$ for each $t\ge 0$.*

Proof. Let $\bar\xi = (\xi_n)_{n\in\mathbb{N}}$ and $\bar\chi = (\chi_n)_{n\in\mathbb{N}}$ be in $S_A^\star$. Let

$$y_1 = \nu_1 = \mu_1 = \tilde\mu_1 = \rho_1\|\xi_1-\chi_1\|. \tag{4.12}$$

For $n>1$, let

$$y_n = \rho_n\|\xi_n-\chi_n\|, \tag{4.13}$$

$$\nu_n = y_n + \kappa_n D_n\sum_{k+j=n}\tilde\mu_k(\tilde\gamma_{j,\xi}+\tilde\gamma_{j,\chi}), \tag{4.14}$$

$$\mu_n = \nu_n + C_1\kappa_n\sum_{k+j=n}\mu_k(\gamma_{j,\xi}+\gamma_{j,\chi}), \tag{4.15}$$

$$\tilde\mu_n = \mu_n + 3C_1\kappa_n\sum_{k+j=n}\tilde\mu_k(\tilde\gamma_{j,\xi}+\tilde\gamma_{j,\chi}), \tag{4.16}$$

where $\gamma_{j,\xi}$ and $\tilde{\gamma}_{j,\xi}$ (respectively $\gamma_{j,\chi}$ and $\tilde{\gamma}_{j,\chi}$) are defined by (4.5) and (4.6) for $\bar{\xi}$ (respectively $\bar{\chi}$). Let $u_n(\zeta) = q_n(\zeta, \bar{\xi})e^{-n\zeta}$ and $v_n(\zeta) = q_n(\zeta, \bar{\chi})e^{-n\zeta}$, $n \in \mathbb{N}$. Following the proof of Lemma 7.3 in [4], we obtain

$$\rho_n \|u_n(0) - v_n(0)\| \le \nu_n. \tag{4.17}$$

Then by Proposition 3.10

$$\rho_n \|u_n(t) - v_n(t)\| \le \mu_n e^{-t}, \quad t > 0, \tag{4.18}$$

$$\rho_n \|u_n(\zeta) - v_n(\zeta)\| \le \tilde{\mu}_n e^{-\mathbf{Re}\zeta}, \quad \zeta \in E. \tag{4.19}$$

Since $\gamma_{j,\xi} \le \tilde{\gamma}_{j,\xi}, \gamma_{j,\chi} \le \tilde{\gamma}_{j,\chi}$ and $y_n \le \nu_n \le \mu_n \le \tilde{\mu}_n$, it follows from (4.13)–(4.16) that

$$\tilde{\mu}_n \le y_n + \tilde{\kappa}_n \sum_{k+j=n} \tilde{\mu}_k (\tilde{\gamma}_{j,\xi} + \tilde{\gamma}_{j,\chi}), \tag{4.20}$$

where

$$\tilde{\kappa}_n = \kappa_n (D_n + 4C_1). \tag{4.21}$$

The proof now proceeds as in Theorem 3.8. □

We close this section by remarking that the proofs of Theorems 4.1 and 4.2 also show

Proposition 4.3. *The map $Q(0, \cdot) : S_A^\star \to V^\star$ is well-defined and continuous.*

5. The normalization map

In [4], we did not know whether $W(u^0)$ belongs to some $S_A^\star$ with appropriate ρ_n for even small $\|u^0\|$. In this section we show provided the weights ρ_n satisfy the additional conditions in Definition 5.2 below for $n \in \mathbb{N}$ that $W(u^0) \in S_A^\star$. The continuity of W as a map from $\mathcal{R}$ to such a space $S_A^\star$ is also established.

5.1. **The range of the normalization map.** Let $u^0 \in \mathcal{R}$ and let $u(t) = S(t)u^0$ be the regular solution to the Navier–Stokes equations with the initial data u^0. Let $u_n(t) = W_n(t, u^0)e^{-nt}$ for $n \in \mathbb{N}$ and $t \geq 0$. Then the asymptotic expansion of $u(t)$ is

$$u(t) \sim \sum u_n(t) = \sum W_n(t, u^0)e^{-nt} \quad \text{as} \quad t \to \infty. \tag{5.1}$$

For $n \geq 2$, denote

$$\tilde{u}_n(t) = u(t) - \sum_{k=1}^{n-1} u_k(t) = S(t)u^0 - \sum_{k=1}^{n-1} W_k(t, u^0)e^{-kt}.$$

Let $\beta_n(t)$ be defined as in (2.15) with $q_n(t) = q_n(t, W(u^0)) = W_n(t, u^0)$. Explicitly, $\beta_1(t) = 0$ and for $n > 1$

$$\beta_n(t) = \sum_{k+j=n} B(W_k(t, u^0), W_j(t, u^0)) = e^{nt} \sum_{k+j=n} B(u_k(t), u_j(t)). \tag{5.2}$$

Let $u_n^0 = W_n(0, u^0)$ for $n \in \mathbb{N}$ and $\bar{u}^0 = (u_n^0)_{n=1}^\infty$. Since $W_n(t, u^0) = q_n(t, W(u^0))$ for $n \in \mathbb{N}$ we obtain

$$\bar{u}^0 = Q(0, W(u^0)) \quad \text{and} \quad (u_n(t))_{n=1}^\infty = S_{\text{ext}}(t)\bar{u}^0, \quad t \geq 0. \tag{5.3}$$

Similarly, for complex times $\zeta \in E$ we write $u_n(\zeta) = W_n(\zeta, u^0)e^{-n\zeta}$ for $n \in \mathbb{N}$ so that $(u_n(\zeta))_{n=1}^\infty = S_{\text{ext}}(\zeta)\bar{u}^0$. Recall that the set E is defined in (A.5).

We start with a recursive formula of $W_n(u^0)$.

Lemma 5.1. *Let $u^0 \in \mathcal{R}$ and $u(t) = S(t)u^0$. Then*

$$W_1(u^0) = \lim_{t\to\infty} e^t u(t) = \lim_{t\to\infty} e^t R_1 u(t), \tag{5.4}$$

where the limits are taken in the V norm.

For $n \in \sigma(A)$ and $n \geq 2$ we have

$$\begin{aligned} W_n(u^0) = R_n\tilde{u}_n(0) &- \int_0^\infty e^{n\tau} \sum_{\substack{k,j\leq n-1\\ k+j\geq n+1}} R_n B(u_k, u_j)d\tau \\ &- \int_0^\infty e^{n\tau} R_n\big[B(u, \tilde{u}_n) + B(\tilde{u}_n, u) - B(\tilde{u}_n, \tilde{u}_n)\big]d\tau. \end{aligned} \tag{5.5}$$

Proof. Equation (5.4) follows from the asymptotic expansion of the solution $u(t)$. In particular, from (2.12) we obtain $\|\tilde{u}_2(t)\| = O(e^{-(1+\epsilon)t})$ and from (2.16) we obtain $W_1(u^0) = R_1 W_1(u^0, 0) = W_1(u^0, t)$ for all $t \geq 0$. Therefore

$$\begin{aligned} \big\|W_1(u^0) - e^t R_1 u(t)\big\| &= \big\|R_1 W_1(0, u^0) - e^t R_1 u(t)\big\| \\ &\leq \big\|W_1(0, u^0) - e^t u(t)\big\| \\ &\leq \big\|W_1(0, u^0) - W_1(t, u^0)\big\| + \big\|e^t \tilde{u}_2(t)\big\| \\ &\leq 0 + O(e^{-\epsilon t}) \to 0 \quad \text{as} \quad t \to \infty \end{aligned}$$

Similarly $\|W_1(u^0) - e^t u(t)\| \to 0$ as $t \to \infty$.

Let $n \geq 2$ and $n \in \sigma(A)$. We know from (5.1) given $\epsilon \in (0,1)$ that $\tilde{u}_n(t) = W_n(t,u^0)e^{-nt} + O(e^{-(n+\epsilon)t})$ as $t \to \infty$. By (2.16),

$$\begin{aligned} W_n(u^0) &= R_n W_n(t,u^0) + \int_0^t R_n \beta_n(\tau) d\tau \\ &= e^{nt} R_n \tilde{u}_n(t) + O(e^{-\epsilon t}) + \int_0^t R_n \beta_n(\tau) d\tau. \end{aligned}$$

Letting $t \to \infty$ gives

$$W_n(u^0) = \lim_{t\to\infty} \left[e^{nt} R_n \tilde{u}_n(t) + \int_0^t R_n \beta_n(\tau) d\tau \right]. \tag{5.6}$$

Recall from (2.20) and (2.6) that

$$\begin{gathered} \frac{du}{dt} + Au + B(u,u) = 0, \\ \frac{du_m}{dt} + Au_m + \sum_{k+j=m} B(u_k,u_j) = 0, \quad m = 1,2,\ldots,n-1. \end{gathered}$$

It follows that the remainder $\tilde{u}_n(t) = u(t) - \sum_{k=1}^{n-1} u_k(t)$ satisfies

$$\frac{d\tilde{u}_n}{dt} + A\tilde{u}_n + B(u,\tilde{u}_n) + B(\tilde{u}_n,u) - B(\tilde{u}_n,\tilde{u}_n) + \sum_{\substack{k,j\leq n-1\\ k+j\geq n}} B(u_k,u_j) = 0. \tag{5.7}$$

Then $R_n A = nR_n$ on $\mathcal{D}_A$ implies

$$\begin{aligned} \frac{d}{dt}(e^{nt}R_n\tilde{u}_n) + e^{nt} \sum_{k+j=n} R_n B(u_k,u_j) + e^{nt} \sum_{\substack{k,j\leq n-1\\ k+j\geq n+1}} R_n B(u_k,u_j) & \\ + e^{nt} R_n[B(u,\tilde{u}_n) + B(\tilde{u}_n,u) - B(\tilde{u}_n,\tilde{u}_n)] &= 0 \end{aligned}$$

after applying the projection R_n to (5.7) and multiplying it by e^{nt}. Integrating yields

$$\begin{aligned} e^{nt}R_n\tilde{u}_n(t) + \int_0^t R_n\beta_n(\tau)d\tau = R_n\tilde{u}_n(0) - \int_0^t e^{n\tau} \sum_{\substack{k,j\leq n-1\\ k+j\geq n+1}} R_n B(u_k,u_j)d\tau & \\ - \int_0^t e^{n\tau} R_n\big[B(u,\tilde{u}_n) + B(\tilde{u}_n,u) - B(\tilde{u}_n,\tilde{u}_n)\big]d\tau. & \end{aligned} \tag{5.8}$$

Since $u_n = W_n(t,u^0)e^{-nt}$ where $W_n(t,u_0)$ is a polynomial in t then (A.2) implies

$$e^{-nt} \sum_{\substack{k,j\leq n-1\\ k+j\geq n+1}} \|R_n B(u_k,u_j)\| \leq \sum_{\substack{k,j\leq n-1\\ k+j\geq n+1}} C_1 e^{-nt} n^{3/4} \|u_k\|\|u_j\| \leq q(t)e^{-t}$$

where $q(t)$ is a polynomial in t. Thus, the limit

$$\int_0^\infty e^{n\tau} \sum_{\substack{k,j\leq n-1\\ k+j\geq n+1}} R_n B(u_k,u_j)d\tau$$

converges. The argument that the last integral in (5.8) converges as $t \to \infty$ follows from the estimates (5.36)–(5.38). In fact, explicit bounds for each of the terms appearing on the right hand side of (5.8) will be given in the proof of Lemma 5.6. Now, letting $t \to \infty$ in (5.8) and using (5.6) we obtain (5.5). $\square$

To estimate the integrals on the right hand side of (5.5), we not only need to have good estimates of the integrands for large τ, but for small τ as well. Therefore the energy inequality for regular solutions to the Navier–Stokes equations will play a crucial role in our estimates.

We recall from [4] that if $\|u^0\| < \varepsilon_0$ then $u^0 \in \mathcal{R}$ and

$$\|u(t)\| \le 2\|u^0\|e^{-t}, \quad t > 0, \tag{5.9}$$

$$\|u(\zeta)\| \le 4\|u^0\|e^{-\mathbf{Re}\zeta}, \quad \zeta \in E. \tag{5.10}$$

For general $u^0 \in \mathcal{R}$, the energy estimate is

$$|u(t')|^2 + 2\int_t^{t'} \|u(\tau)\|^2 d\tau \le |u(t)|^2, \quad t' > t \ge 0. \tag{5.11}$$

By Poincare's and Gronwall's inequalities

$$|u(t)|^2 \le e^{-2t}|u^0|^2, \quad t \ge 0, \tag{5.12}$$

hence

$$2\int_t^{t'} \|u(\tau)\|^2 d\tau \le |u(t)|^2 \le e^{-2t}|u^0|^2, \quad t' \ge t \ge 0. \tag{5.13}$$

In particular,

$$2\int_0^{\infty} \|u(\tau)\|^2 d\tau \le |u^0|^2. \tag{5.14}$$

Denote $\log^+ \alpha = \log(\max\{1, \alpha\})$ and let

$$t_0 = t_0(u^0) = \log^+(2|u^0|/\varepsilon_0) + 1. \tag{5.15}$$

Take $t = t_0 - 1$, $t' = t_0$ in (5.13). It follows that there is a $t_1 \in (t_0 - 1, t_0)$ such that $\|u(t_1)\| < |u(t_0 - 1)| \le e^{-t_0+1}|u^0| \le \varepsilon_0/2$. Then by (5.9),

$$\|u(t_0)\| \le 2\|u(t_1)\| < \varepsilon_0. \tag{5.16}$$

Hence it follows from (5.9) and (5.10) that

$$\|u(t_0 + \tau)\| \le 2\varepsilon_0 e^{-\tau}, \quad \tau > 0. \tag{5.17}$$

$$\|u(t_0 + \zeta)\| \le 4\varepsilon_0 e^{-\mathbf{Re}\zeta}, \quad \zeta \in E. \tag{5.18}$$

If $\|u^0\| < \varepsilon_0$, we simply take $t_0 = 0$. Note that

$$e^{t_0} \le g_0 = g_0(u^0) \stackrel{\text{def}}{=\!=} \max\{e, 2e\|u^0\|/\varepsilon_0\}. \tag{5.19}$$

Definition 5.2. Let $(\kappa_n')_{n=2}^{\infty}$ be a fixed sequence of real numbers in the interval $(0, 1]$ satisfying

$$\lim_{n\to\infty} (\kappa_n')^{1/2^n} = 0. \tag{5.20}$$

We define the sequence of positive weights $(\rho_n)_{n=1}^{\infty}$ by

$$\rho_1 = 1, \quad \rho_n = \frac{\kappa_n' \rho_{n-1}^2}{\max\{\tilde{L}_n, \tilde{L}_n'\}}, \quad n > 1, \tag{5.21}$$

where $(\tilde{L}_n)_{n=2}^{\infty}$ and $(\tilde{L}'_n)_{n=2}^{\infty}$ are defined by (5.56) and (5.90), respectively, and depend only on the constants C_1, C_2 and a_0.

Note that $\tilde{L}_n \geq 3$ and $\tilde{L}'_n \geq 1$ for all $n \geq 2$. Therefore the sequence $(\rho_n)_{n=1}^{\infty}$ is decreasing and $\rho_n \leq \kappa'_n$ for $n > 1$. Define

$$\kappa_n = \frac{\rho_n}{\min\{\rho_k\rho_j : k+j=n\}}, \quad n \geq 2.$$

Then $\kappa_n \leq \rho_n\rho_{n-1}^{-2} \leq \kappa'_n\tilde{L}_n^{-1} \leq \kappa'_n \leq 1$. Thus (3.1) holds as in Sections 3 and 4. Moreover,

$$\lim_{n\to\infty} \kappa_n^{1/2^n} = 0 \quad \text{and subsequently} \quad \lim_{n\to\infty} n\kappa_n^{1/n} = 0. \tag{5.22}$$

Therefore ρ_n and κ_n satisfy (3.1), (3.39) and (4.1) for any choices of sequences $(\tilde{L}_n)_{n=2}^{\infty}$ and $(\tilde{L}'_n)_{n=2}^{\infty}$ such that $\tilde{L}_n \geq 3$ and $\tilde{L}'_n \geq 1$, in particular, for the choices given by (5.56) and (5.90). Note also that

$$\sum_{n=1}^{\infty} \kappa_n \quad \text{and} \quad \sum_{n=1}^{\infty} \rho_n \quad \text{are finite.} \tag{5.23}$$

Moreover,

$$D_n^{1/2^n} \leq (\text{const} \cdot n^n)^{1/2^n} \to 1 \quad \text{as} \quad n \to \infty$$

implies that

$$\sum_{n=1}^{\infty} \kappa'_n D_n < \infty. \tag{5.24}$$

Definition 5.3. We denote for each $u^0 \in \mathcal{R}$,

$$\eta_n = \rho_n \|W_n(0,u^0)\|, \quad n \in \mathbb{N}, \tag{5.25}$$

$$\gamma_1 = \eta_1, \quad \gamma_n = \eta_n + C_1\kappa_n \sum_{k+j=n} \gamma_k\gamma_j, \ n > 1 \tag{5.26}$$

$$\tilde{\gamma}_1 = \gamma_1, \quad \tilde{\gamma}_n = \gamma_n + 3C_1\kappa_n \sum_{k+j=n} \tilde{\gamma}_k\tilde{\gamma}_j, \ n > 1. \tag{5.27}$$

It follows from Proposition 3.9 that

$$\rho_n\|u_n(t)\| = \rho_n\|W_n(t,u^0)e^{-nt}\| \leq \gamma_n e^{-t}, \quad t > 0, \tag{5.28}$$

$$\rho_n\|u_n(\zeta)\| = \rho_n\|W_n(\zeta,u^0)e^{-n\zeta}\| \leq \tilde{\gamma}_n e^{-\mathbf{Re}\zeta}, \quad \zeta \in E. \tag{5.29}$$

Consequently, by Lemma A.2,

$$\rho_n\|u_n(t)\| \leq n\tilde{\gamma}_n e^{-nt}[C_2(t+a_0)]^{n-1}, \quad t > 0. \tag{5.30}$$

Definition 5.4. For $u^0 \in \mathcal{R}$ and $n > 1$, denote

$$M_n(u^0) = \|u^0\| + \frac{1}{\rho_{n-1}} \sum_{k=1}^{n-1} \eta_k, \tag{5.31}$$

$$M'_n(u^0) = \left\{ |u^0|^2 + \frac{1}{\rho_{n-1}^2} \Big(\sum_{k=1}^{n-1} \gamma_k \Big)^2 \right\}^{1/2}, \tag{5.32}$$

$$\tilde{M}_n(u^0) = 4\varepsilon_0 + \frac{e^{-t_0}}{\rho_{n-1}} \sum_{k=1}^{n-1} \tilde{\gamma}_k. \tag{5.33}$$

We need the following recursive inequalities for estimating the right hand side of (5.5).

Lemma 5.5. *Let $u^0 \in \mathcal{R}$ and $N > 1$. Then*

$$\|\tilde{u}_N(0)\| \le M_N(u^0), \tag{5.34}$$

$$\|\tilde{u}_N(t_0+\tau)\| \le \tilde{M}_N(u^0) e^{-N\tau} \Big|1 + \frac{\tau}{2}\Big|^{2N}, \quad \tau > 0, \tag{5.35}$$

$$\int_0^{t_0} \|\tilde{u}_N(t)\|^2 dt \le |u^0|^2 + \frac{1}{\rho_{N-1}^2} \Big(\sum_{k=1}^{N-1} \gamma_k \Big)^2 = M_N'^2(u^0), \tag{5.36}$$

$$\int_{t_0}^{\infty} e^{Nt} \|u(t)\| \|\tilde{u}_N(t)\| dt \le 2e^{Nt_0} L_N \|u(t_0)\| \tilde{M}_N(u^0), \tag{5.37}$$

$$\int_{t_0}^{\infty} e^{Nt} \|\tilde{u}_N(t)\|^2 dt \le e^{Nt_0} L_N' \tilde{M}_N^2(u^0), \tag{5.38}$$

where

$$L_n = \frac{e^2 (2n)!}{2^{2n}}, \quad L_n' = \frac{e^{2n}(4n)!}{2^{4n} n^{4n+1}}, \quad n \in \mathbb{N}. \tag{5.39}$$

Proof. First, note that

$$\|\tilde{u}_N(t)\| \le \|u(t)\| + \sum_{k=1}^{N-1} \|u_k(t)\| \le \|u(t)\| + \frac{1}{\rho_{N-1}} \sum_{k=1}^{N-1} \rho_k \|u_k(t)\|, \tag{5.40}$$

since ρ_n is decreasing. Taking $t = 0$ in (5.40) yields (5.34).

Second, since $\zeta \in E$ implies $t_0 + \zeta \in E$, then by (5.18) and (5.29) for $n = 1, 2, \ldots, N-1$, we have

$$\|\tilde{u}_N(t_0+\zeta)\| \le \|u(t_0+\zeta)\| + \sum_{k=1}^{N-1} \|u_k(t_0+\zeta)\| \le e^{-\mathbf{Re}\zeta} \Big(4\varepsilon_0 + \frac{e^{-t_0}}{\rho_{N-1}} \sum_{k=1}^{N-1} \tilde{\gamma}_k \Big)$$

for $\zeta \in E$. Given any $\epsilon > 0$ then

$$\begin{aligned} \|\tilde{u}_N(t_0+\tau)\| &\le \|\tilde{u}_{N+1}(t_0+\tau)\| + \|u_N(t_0+\tau)\| \\ &\le O(e^{-N\tau}) + \rho_n^{-1} N \tilde{\gamma}_N e^{-N(t_0+\tau)} [C_2(t_0+\tau+a_0)]^{N-1} \\ &= O(e^{-(N-\epsilon)\tau}) \end{aligned}$$

as $\tau \to \infty$. Now Lemma A.1 implies

$$\|\tilde{u}_N(t_0+\tau)\| \le (4\varepsilon_0 + \frac{e^{-t_0}}{\rho_{N-1}} \sum_{k=1}^{N-1} \tilde{\gamma}_k) e^{-(N-\epsilon)\tau} \Big|1 + \frac{\tau}{2}\Big|^{2(N-\epsilon)} \tag{5.41}$$

and taking $\epsilon \to 0$ yields (5.35).

Third, squaring (5.40) and integrating yields

$$\int_0^{t_0} \|\tilde{u}_N(t)\|^2 dt \le \int_0^{t_0} \Big\{ 2\|u(t)\|^2 + 2\Big[\frac{e^{-t}}{\rho_{N-1}} \sum_{k=1}^{N-1} \gamma_k\Big]^2 \Big\} dt.$$

Then (5.36) follows from the energy inequality (5.14).

Fourth, from (5.9) and (5.35) we obtain

$$\begin{aligned}\int_{t_0}^{\infty} e^{Nt}\|u(t)\|\|\tilde{u}_N(t)\|dt &= e^{Nt_0}\int_0^{\infty} e^{N\tau}\|u(t_0+\tau)\|\|\tilde{u}_N(t_0+\tau)\|d\tau \\ &\le e^{Nt_0}\int_0^{\infty} e^{N\tau}\Big(2\|u(t_0)\|e^{-\tau}\Big)\Big(\tilde{M}_N(u^0)e^{-N\tau}\Big|1+\frac{\tau}{2}\Big|^{2N}\Big)d\tau \\ &\le 2e^{Nt_0}L_N\|u(t_0)\|\tilde{M}_N(u^0),\end{aligned}$$

since

$$\int_0^{\infty} e^{-\tau}\Big|1+\frac{\tau}{2}\Big|^{2n} d\tau \le \int_0^{\infty} (\tau'/2)^{2n} e^{-\tau'+2} d\tau' = \frac{e^2}{2^{2n}}\Gamma(2n+1) = L_n.$$

Fifth,

$$\begin{aligned}\int_{t_0}^{\infty} e^{Nt}\|\tilde{u}_N(t)\|^2 dt &\le e^{Nt_0}\int_0^{\infty} e^{N\tau}\Big\{\tilde{M}_N(u^0)e^{-N\tau}\Big|1+\frac{\tau}{2}\Big|^{2N}\Big\}^2 d\tau \\ &= e^{Nt_0}L'_N\tilde{M}_N^2(u^0),\end{aligned}$$

since

$$\int_0^{\infty} e^{-n\tau}\Big|1+\frac{\tau}{2}\Big|^{4n} d\tau \le \frac{e^{2n}}{2^{4n}}\frac{\Gamma(4n+1)}{n^{4n+1}} = L'_n.$$

The proof is complete. □

Our main recursive step is

Lemma 5.6. *Let $u^0 \in \mathcal{R}$. For $N > 1$ let $M_N = M_N(u^0)$, $M'_N = M'_n(u^0)$ and $\tilde{M}_N = \tilde{M}_N(u^0)$. We then have*

(5.42)

$$\|W_N(u^0)\| \le M_N + \frac{L(0,N)}{\rho_{N-1}^2}\Big(\sum_{k=1}^{N-1}\tilde{\gamma}_k\Big)^2 + e^{Nt_0}L(0,N)\Big\{1+|u^0|^2+{M'_N}^2+\tilde{M}_N^2\Big\},$$

where $L(0,n)$ defined by (5.43) below for $n > 1$ are positive constants independent of u^0 and ρ_n.

Proof. If $N \notin \sigma(A)$, then $W_N(u^0) = 0$ and (5.42) holds. We now consider the case $N \in \sigma(A)$. By (5.5),

$$\begin{aligned}
\|W_N(u^0)\| \le \|\tilde{u}_N(0)\| + &\underbrace{\int_0^\infty e^{N\tau} \sum_{\substack{k,j\le N-1\\ k+j\ge N+1}} \|R_N B(u_k, u_j)\| d\tau}_{J_1} \\
&+ \underbrace{\int_0^\infty e^{N\tau} \left(\|R_N B(u, \tilde{u}_N)\| + \|R_N B(\tilde{u}_N, u)\|\right) d\tau}_{J_2} \\
&+ \underbrace{\int_0^\infty e^{N\tau} \|R_N B(\tilde{u}_N, \tilde{u}_N)\| d\tau}_{J_3} \\
= \|\tilde{u}_N(0)\| + J_1 + J_2 + J_3.
\end{aligned}$$

According to Lemma 5.5, we first have $\|\tilde{u}_N(0)\| \le M_N(u^0)$.

Estimate of J_1. By using inequality (A.2), the fact $\rho_{N-1}^2 \le \min\{\rho_k\rho_j : k, j \le N-1\}$, and the estimates (5.30) of $\|u_k(t)\|$, we obtain

$$\begin{aligned}
J_1 &\le \rho_{N-1}^{-2} C_1 N^{3/4} \int_0^\infty e^{N\tau} \sum_{\substack{k,j\le N-1\\ k+j\ge N+1}} \tilde{\gamma}_k \tilde{\gamma}_j kj [C_2(\tau + a_0)]^{k+j-2} e^{-(k+j)\tau} d\tau \\
&\le \rho_{N-1}^{-2} C_1 N^{3/4} \int_0^\infty e^{N\tau} (N-1)^2 \sum_{\substack{k,j\le N-1\\ k+j\ge N+1}} \tilde{\gamma}_k \tilde{\gamma}_j [C_2(\tau + a_0)]^{2(N-1)-2} e^{-(k+j)\tau} d\tau \\
&\le \rho_{N-1}^{-2} C_1 N^{3/4} (N-1)^2 \Big(\sum_{k=1}^{N-1} \tilde{\gamma}_k\Big)^2 C_2^{2N-4} \int_0^\infty e^{-\tau} (\tau + a_0)^{2N-4} d\tau.
\end{aligned}$$

For $n \in \mathbb{N}$,

$$\begin{aligned}
&C_1 n^{3/4} (n-1)^2 C_2^{2n-4} \int_0^\infty e^{-\tau} (\tau + a_0)^{2n-4} d\tau \\
&\qquad\qquad \le C_1 C_2^{2n-4} n^{3/4} (n-1)^2 \int_0^\infty e^{-\tau' + a_0} \tau'^{2n-4} d\tau' = L(1, n)
\end{aligned}$$

where $L(1, n) = e^{a_0} C_1 C_2^{2n-4} n^{3/4} (n-1)^2 \Gamma(2n-3)$. Hence

$$J_1 \le \rho_{N-1}^{-2} L(1, N) \Big(\sum_{k=1}^{N-1} \tilde{\gamma}_k\Big)^2.$$

Estimate of J_2. By (A.2),

$$\begin{aligned}
J_2 &\le \int_0^\infty 2C_1 N^{3/4} e^{Nt} \|u(t)\| \|\tilde{u}_N(t)\| dt \\
&= \Big(\int_0^{t_0} + \int_{t_0}^\infty\Big) 2C_1 N^{3/4} e^{Nt} \|u(t)\| \|\tilde{u}_N(t)\| dt = J_{2,1} + J_{2,2}.
\end{aligned}$$

For $J_{2,1}$ we use (5.36) and (5.14)

$$\begin{aligned} J_{2,1} &\le 2C_1 N^{3/4} e^{Nt_0}\Big(\int_0^{t_0} \|u(t)\|^2 dt\Big)^{1/2}\Big(\int_0^{t_0} \|\tilde{u}_N(t)\|^2 dt\Big)^{1/2} \\ &\le \sqrt{2} C_1 N^{3/4} e^{Nt_0} |u^0| M'_N \\ &\le L(2,N) e^{Nt_0} |u^0| M'_N \quad \text{where} \quad L(2,n) = \sqrt{2} C_1 n^{3/4},\ n \in \mathbb{N}. \end{aligned}$$

For $J_{2,2}$ we use (5.37) and (5.16)

$$\begin{aligned} J_{2,2} &\le 4C_1 N^{3/4} e^{Nt_0} L_N \tilde{M}_N(u^0) \|u(t_0)\| \\ &\le 4C_1 \varepsilon_0 N^{3/4} e^{Nt_0} \tilde{M}_N L_N \\ &= L(3,N) e^{Nt_0} \tilde{M}_N \quad \text{where} \quad L(3,n) = 4C_1 \varepsilon_0 n^{3/4} L_n,\ n \in \mathbb{N}. \end{aligned}$$

Estimate of J_3. By (A.2)

$$J_3 \le C_1 N^{3/4}\Big(\int_0^{t_0} + \int_{t_0}^{\infty}\Big) e^{Nt} \|\tilde{u}_N(t)\|^2 dt = J_{3,1} + J_{3,2}.$$

For $J_{3,1}$, we use (5.36)

$$J_{3,1} \le C_1 N^{3/4} e^{Nt_0} M_N'^2 = L(4,N) e^{Nt_0} M_N'^2 \quad \text{where} \quad L(4,n) = C_1 n^{3/4},\ n \in \mathbb{N}.$$

For $J_{3,2}$, we use (5.38)

$$J_{3,2} \le C_1 N^{3/4} e^{Nt_0} \tilde{M}_N^2 L'_N = L(5,N) e^{Nt_0} \tilde{M}_N^2 \text{ where } L(5,n) = C_1 n^{3/4} L'_n,\ n \in \mathbb{N}.$$

Combining the above estimates we obtain

$$\begin{aligned} \|W_N(u^0)\| &\le M_N + \frac{L(1,N)}{\rho_{N-1}^2}\Big(\sum_{k=1}^{N-1} \tilde{\gamma}_k\Big)^2 \\ &\quad + e^{Nt_0}\Big\{L(2,N)|u^0| M'_N + L(3,N)\tilde{M}_N + L(4,N) M_N'^2 + L(5,N)\tilde{M}_N^2\Big\}, \end{aligned}$$

Inequality (5.42) easily follows with

$$L(0,n) = \max\Big\{L(1,n), \frac{1}{2}L(2,n) + L(4,n), \frac{1}{2}L(3,n) + L(5,n)\Big\} \tag{5.43}$$

for $n > 1$. □

Definition 5.7. Given $u^0 \in \mathcal{R}$. Let

$$x_1 = \eta_1^* = \gamma_1^* = \tilde{\gamma}_1^* = \rho_1 \|u^0\|. \tag{5.44}$$

For $n > 1$, let

$$x_n = \rho_n \|u^0\| + \kappa'_n\Big(\sum_{k=1}^{n-1} \tilde{\gamma}_k^*\Big) + \kappa'_n\Big(\sum_{k=1}^{n-1} \tilde{\gamma}_k^*\Big)^2 + \kappa'_n g_0^n \Big\{1 + \|u^0\|^2 + (\sum_{k=1}^{n-1} \tilde{\gamma}_k^*)^2\Big\}, \tag{5.45}$$

$$\eta_n^* = x_n + \kappa'_n D_n \sum_{k+j=n} \tilde{\gamma}_k^* \tilde{\gamma}_j^*, \tag{5.46}$$

$$\gamma_n^* = \eta_n^* + C_1 \kappa'_n \sum_{k+j=n} \gamma_k^* \gamma_j^*, \tag{5.47}$$

$$\tilde{\gamma}_n^* = \gamma_n^* + 3C_1 \kappa'_n \sum_{k+j=n} \tilde{\gamma}_k^* \tilde{\gamma}_j^* \tag{5.48}$$

where D_n are defined as in (4.7) and g_0 is given in (5.19).

Our next goal is to prove that $\rho_n\|\xi_n\| \le x_n$ for all $n\in\mathbb{N}$. In fact, we have

Lemma 5.8. *Let $u^0\in\mathcal{R}$ and let $x_n, \eta_n^*, \gamma_n^*, \tilde\gamma_n^*$ be defined as in Definition 5.7. For all $n\in\mathbb{N}$, we have*

$$\rho_n\|W_n(u^0)\| \le x_n, \tag{5.49}$$

$$\rho_n\|u_n(0)\| = \rho_n\|W_n(0,u^0)\| \le \eta_n^*, \tag{5.50}$$

$$\rho_n\|u_n(t)\| = \rho_n\|W_n(t,u^0)e^{-nt}\| \le \gamma_n^* e^{-t}, \quad t>0, \tag{5.51}$$

$$\rho_n\|u_n(\zeta)\| = \rho_n\|W_n(\zeta,u^0)e^{-n\zeta}\| \le \tilde\gamma_n^* e^{-\mathbf{Re}\zeta}, \quad \zeta\in E. \tag{5.52}$$

Proof. Let $\xi_n = W_n(u^0), n\in\mathbb{N}$ and $W(u^0)=\bar\xi=(\xi_n)_{n=1}^\infty$. Using the notation in Definition 5.3 and inequalities (5.28)–(5.29), it suffices to prove that

$$\rho_n\|\xi_n\|\le x_n,\ \eta_n\le\eta_n^*,\ \gamma_n\le\gamma_n^*,\ \tilde\gamma_n\le\tilde\gamma_n^*, \quad n\in\mathbb{N}. \tag{5.53}$$

For $n=1$, we have by (2.16) that

$$u_1(0)=W_1(0,u^0)=q_1(0,\bar\xi)=\xi_1. \tag{5.54}$$

Therefore (2.23) implies

$$u_1(t)=\xi_1 e^{-t} \quad \text{and} \quad u_1(\zeta)=\xi_1 e^{-\zeta}. \tag{5.55}$$

From Lemma 5.1, $\|W_1(u^0)\| = |W_1(u^0)| = \lim_{t\to\infty} e^t|u(t)| \le |u^0| \le \|u^0\|$. Therefore (5.53) holds for $n=1$.

For induction, let $N>1$ and assume (5.53) holds for $n=1,\dots,N-1$. Let $T_n = (\sum_{k=1}^n \tilde\gamma_k)/\rho_n,\ n>1$. Then, from Definitions 5.3 and 5.4 we immediately have

$$M_n \le \|u^0\| + T_{n-1}, {M_n'}^2 \le \|u^0\|^2 + T_{n-1}^2, \tilde M_n \le 4\varepsilon_0 + T_{n-1}, \tilde M_n^2 \le 32\varepsilon_0^2 + 2T_{n-1}^2.$$

It follows from Lemma 5.6 that

$$\begin{aligned}\|\xi_N\| &\le \{\|u^0\| + T_{N-1}\} + \{L(0,N)T_{N-1}^2\} \\ &\quad + e^{Nt_0}L(0,N)\Big\{1+\|u^0\|^2 + (\|u^0\|^2+T_{N-1}^2) + (32\varepsilon_0^2 + 2T_{N-1}^2)\Big\} \\ &\le \|u^0\| + T_{N-1} + \tilde L_N T_{N-1}^2 + \tilde L_N g_0^N\Big\{1+\|u^0\|^2+T_{N-1}^2\Big\},\end{aligned}$$

where

$$\tilde L_n = \max\{1, L(0,n)\}\max\{1+32\varepsilon_0^2, 3\}, \quad n>1, \tag{5.56}$$

are positive constants greater than or equal 3, independent of u^0 and depending only on C_1, C_2 and a_0. Multiplying by ρ_N, we derive

$$\begin{aligned}\rho_N\|\xi_N\| &\le \rho_N\|u^0\| + \kappa'_N\Big(\sum_{k=1}^{N-1}\tilde\gamma_k\Big)\\ &\qquad + \kappa'_N\Big(\sum_{k=1}^{N-1}\tilde\gamma_k\Big)^2 + \kappa'_N g_0^N\Big\{1+\|u^0\|^2+\Big(\sum_{k=1}^{N-1}\tilde\gamma_k\Big)^2\Big\}\\ &\le \rho_N\|u^0\| + \kappa'_N\Big(\sum_{k=1}^{N-1}\tilde\gamma_k^*\Big)\\ &\qquad + \kappa'_N\Big(\sum_{k=1}^{N-1}\tilde\gamma_k^*\Big)^2 + \kappa'_N g_0^N\Big\{1+\|u^0\|^2+\Big(\sum_{k=1}^{N-1}\tilde\gamma_k^*\Big)^2\Big\},\end{aligned}$$

by the induction hypothesis. Thus the first inequality of (5.53) holds for $n = N$. Applying the arguments used to obtain (4.8) in the proof of Theorem 4.1 with equations (5.45)–(5.48) in place of (4.3)–(4.6) we obtain

$$\eta_n = \rho_n\|W_n(0,u_0)\| = \rho_n\|q_n(0,\bar\xi)\| \le \eta_n^*,$$

the second inequality of (5.53) for $n = N$. Now the last two inequalities of (5.53) follow easily. The induction is hence complete. □

Theorem 5.9. *For any $u^0 \in \mathcal{R}$, $W(u^0) \in S_A^\star$. In other words, the range of the normalization map is contained in the Banach space $S_A^\star$.*

Proof. It is clear from Definition 5.7 that $x_n \le \eta_n^* \le \gamma_n^* \le \tilde\gamma_n^*$ for all $n \in \mathbb{N}$. For each n, summing up (5.45)–(5.48) and noting that $g_0 \ge 1, \kappa'_n \le 1$, we obtain:

$$\begin{aligned}\tilde\gamma_n^* &\le \rho_n\|u^0\| + \kappa'_n\Big(\sum_{k=1}^{n-1}\tilde\gamma_k^*\Big) + \kappa'_n\Big(\sum_{k=1}^{n-1}\tilde\gamma_k^*\Big)^2\\ &\quad + \kappa'_n g_0^n\Big\{1+\|u^0\|^2+\Big(\sum_{k=1}^{n-1}\tilde\gamma_k^*\Big)^2\Big\} + \kappa'_n(D_n+4C_1)\Big(\sum_{k=1}^{n-1}\tilde\gamma_k^*\Big)^2\\ &\le \rho_n\|u^0\| + \tilde\kappa_n^* g_0^n\Big\{1+\|u^0\|^2+\Big(\sum_{k=1}^{n-1}\tilde\gamma_k^*\Big)^2\Big\},\end{aligned}$$

where $\tilde\kappa_n^* = \kappa'_n(D_n + 4C_1 + 3)$ is independent of u^0. Let $a_n = \rho_n\|u^0\|$ and $X = \sqrt{1+\|u^0\|^2}$. We then have

$$\tilde\gamma_n^* \le a_n + \tilde\kappa_n^* g_0^n\Big\{X^2+\Big(\sum_{k=1}^{n-1}\tilde\gamma_k^*\Big)^2\Big\}. \tag{5.57}$$

Note that $\sum_{n=1}^\infty a_n$ is finite by (5.24) and (5.20) implies $\lim_{n\to\infty}(\tilde\kappa_n^*)^{1/2^n} = 0$. Applying Lemma A.5 we obtain that $\sum_{n=1}^\infty \tilde\gamma_n^*$ is finite. Thus (5.49) in Lemma 5.8 implies

$$\|W(u^0)\|_\star = \sum_{n=1}^\infty \rho_n\|W_n(u^0)\| \le \sum_{n=1}^\infty x_n \le \sum_{n=1}^\infty \tilde\gamma_n^* < \infty.$$

Therefore $W(u^0) \in S_A^\star$. □

Regarding the commutative diagram in the Introduction, we also obtain

Corollary 5.10. *For any $u^0 \in \mathcal{R}$ then $\big(W_n(0,u^0)\big)_{n=1}^{\infty} \in V^\star$, i.e., $W(0,\mathcal{R}) \subset V^\star$.*

Proof. This follows from the proof of Theorem 5.9 since

$$\sum_{n=1}^{\infty} \rho_n \|W_n(0,u^0)\| = \sum_{n=1}^{\infty} \eta_n \le \sum_{n=1}^{\infty} \tilde{\gamma}_n^* < \infty.$$

□

Remark 5.11. Note that the results up to now, in particular, those of Theorem 5.9 and Corollary 5.10, are valid for any choice of $\tilde{L}_n'$ in Definition 5.2 such that $\tilde{L}_n'$ is independent of u_0.

In our study of the continuity of the normalization map below, it will be necessary to specify a bound $K(u^0)$ on $\sum \tilde{\gamma}_n^*$ in terms of $\|u^0\|$ only. For this purpose, we introduce the following function $K_1(r,s)$ based on (5.57), Lemmas A.4 and A.5.

Definition 5.12. Given $r \ge 0$ and $s \ge 1$. Let $k_n' = \kappa_n'(D_n + 4C_1 + 3)s^n$. Let $a_1' = \rho_1 r$ and $a_n' = \rho_n r + \kappa_n'(D_n + 4C_1 + 3) + k_n'(1 + r^2)$ for $n > 1$. Define

$$K_1(r,s) = \sum_{n=1}^{\infty} a_n' + \alpha^2 \sum_{n=1}^{\infty} k_n' M^{2(2^n-1)}, \tag{5.58}$$

where $\alpha = \sup\{a_n' : n \in \mathbb{N}\}$ and $M = 3\sup\{1, \alpha, k_n'\alpha : n > 1\}$.

Note that $K_1(r,s)$ is finite and is increasing in each variable r and s. In addition, if we let

$$d_1 = a_1' \quad \text{and} \quad d_n = a_n' + k_n'\Big(\sum_{k=1}^{n-1} d_k\Big)^2 \quad \text{for} \quad n > 1, \tag{5.59}$$

then $\sum_{n=1}^{\infty} d_n \le K_1(r,s)$ by virtue of Lemma A.4.

Lemma 5.13. *Given $u^0 \in \mathcal{R}$. Let $t_0 \ge 0$ and $g_0 \ge 1$ satisfy*

$$\|u(t_0)\| < \varepsilon_0 \quad \textit{and} \quad e^{t_0} \le g_0. \tag{5.60}$$

Then all the results in section 5.1 hold true for these particular values of t_0 and g_0. Furthermore, using the notation set in Definitions 5.3, 5.7 and 5.4, we have

$$\sum_{n=1}^{\infty} \tilde{\gamma}_n^* \le K = K_1(\|u^0\|, g_0). \tag{5.61}$$

Consequently,

$$M_n(u^0), M_n'(u^0) \le \|u^0\| + \frac{K}{\rho_{n-1}}, \quad \tilde{M}_n(u^0) \le 4\varepsilon_0 + \frac{K}{\rho_{n-1}}. \tag{5.62}$$

Proof. Due to (5.57), Lemma A.5 and Definition 5.12, $\sum_{n=1}^{\infty} \tilde{\gamma}_n^* \le \sum_{n=1}^{\infty} d_n \le K$, where the d_n are defined by (5.59) for $r = \|u^0\|$ and $s = g_0$. Hence inequality (5.61) holds true. The other inequalities in (5.62) follow easily. □

Remark 5.14. A bound $K(u^0)$ for $\sum \tilde{\gamma}_n^*$ can be obtained from Lemma 5.13 by taking $K(u^0) = K_1(\|u^0\|, g_0(u^0))$ where $g_0(u^0)$ is given by (5.19).

5.2. **The continuity of the normalization map.** We now study the continuity of the normalization map $W : \mathcal{R} \to S_A^\star$. Let $u^0 \in \mathcal{R}$ be fixed. We show that W is continuous at u^0. For the rest of this section we assume, unless otherwise stated, that

$$v^0 \in \mathcal{R} \quad \text{and} \quad \|v^0 - u^0\| < 1. \tag{5.63}$$

Let

$$u(t) = S(t)u^0,\ u_j(t) = W_j(t,u^0)e^{-jt},\ \tilde{u}_n(t) = u(t) - \sum_{j=1}^{n-1} u_j(t),$$

$$v(t) = S(t)v^0,\ v_j(t) = W_j(t,v^0)e^{-jt},\ \tilde{v}_n(t) = v(t) - \sum_{j=1}^{n-1} v_j(t).$$

We have for any $\epsilon > 0$ that

$$\|\tilde{u}_n(t)\| = O(e^{-(n-\epsilon)t}) \text{ and } \|\tilde{v}_n(t)\| = O(e^{-(n-\epsilon)t}), \quad t \to \infty.$$

Let

$$u_n^0 = u_n(0), \quad v_n^0 = v_n(0), \quad \bar{u}^0 = (u_n^0)_{n=1}^\infty, \quad \bar{v}^0 = (v_n^0)_{n=1}^\infty, \tag{5.64}$$

$$w = u - v, \quad w_n = u_n - v_n, \quad \tilde{w}_n = \tilde{u}_n - \tilde{v}_n, \tag{5.65}$$

$$w^0 = u^0 - v^0, \quad w_n^0 = w_n(0) = u_n^0 - v_n^0. \tag{5.66}$$

Note that $\bar{u}^0 = Q(0, W(u^0))$, $\bar{v}^0 = Q(0, W(v^0))$ and $(u_n(t))_{n=1}^\infty = S_{\text{ext}}(t)\bar{u}^0$, $(v_n(t))_{n=1}^\infty = S_{\text{ext}}(t)\bar{v}^0$. Corollary 5.10 implies that $\bar{u}^0, \bar{v}^0 \in V^\star$.

To estimate $\|W(u^0) - W(v^0)\|_\star$ we begin with a recursive formula for the difference $W_n(u^0) - W_n(v^0)$ similar to (5.5).

Lemma 5.15. *Let $u^0, v^0 \in \mathcal{R}$ and $n \in \sigma(A)$ with $n > 1$. Then we have*

$$\begin{aligned}(5.67)\quad & W_n(u^0) - W_n(v^0) = R_n\tilde{w}_n(0) \\ & -\int_0^\infty e^{n\tau} \sum_{\substack{k,j\le n-1\\ k+j\ge n+1}} R_n[B(w_k,u_j)+B(v_k,w_j)]d\tau - \int_0^\infty e^{n\tau} R_n[B(w,\tilde{u}_n)+B(\tilde{u}_n,w)]d\tau \\ & - \int_0^\infty e^{n\tau} R_n[B(v,\tilde{w}_n)+B(\tilde{w}_n,v)]d\tau + \int_0^\infty e^{n\tau} R_n[B(\tilde{w}_n,\tilde{u}_n)+B(\tilde{v}_n,\tilde{w}_n)]d\tau.\end{aligned}$$

Proof. We have from (5.6),

$$W_n(u^0) - W_n(v^0) = \lim_{t\to\infty}\left\{e^{nt}R_n\tilde{w}_n(t) + \int_0^t R_n[\beta_{n,u^0}(\tau) - \beta_{n,v^0}(\tau)]d\tau\right\}, \tag{5.68}$$

where $\beta_{n,u^0}(\tau)$ and $\beta_{n,v^0}(\tau)$ are defined by (5.2) for u^0 and v^0 respectively.

Recall from (5.7) that for $n \ge 2$,

$$\frac{d\tilde{u}_n}{dt} + A\tilde{u}_n + \sum_{\substack{k,j\le n-1\\ k+j\ge n}} B(u_k,u_j) + B(u,\tilde{u}_n) + B(\tilde{u}_n,u) - B(\tilde{u}_n,\tilde{u}_n) = 0,$$

$$\frac{d\tilde{v}_n}{dt} + A\tilde{v}_n + \sum_{\substack{k,j\le n-1\\ k+j\ge n}} B(v_k,v_j) + B(v,\tilde{v}_n) + B(\tilde{v}_n,v) - B(\tilde{v}_n,\tilde{v}_n) = 0.$$

Then $\tilde{w}_n$ satisfies the equation

$$\frac{d\tilde{w}_n}{dt} + A\tilde{w}_n + \sum_{\substack{k,j\le n-1\\ k+j\ge n}} [B(u_k,u_j) - B(v_k,v_j)]$$
$$+ B(w,\tilde{u}_n) + B(v,\tilde{w}_n) + B(\tilde{u}_n,w) + B(\tilde{w}_n,v) - [B(\tilde{w}_n,\tilde{u}_n) + B(\tilde{v}_n,\tilde{w}_n)] = 0.$$

Hence

$$\begin{aligned} e^{nt}R_n\tilde{w}_n(t) &+ \int_0^t R_n[\beta_{n,u^0}(\tau) - \beta_{n,v^0}(\tau)]d\tau \\ &= R_n\tilde{w}_n(0) - \int_0^t e^{n\tau} \sum_{\substack{k,j\le n-1\\ k+j\ge n+1}} R_n[B(w_k,u_j) + B(v_k,w_j)]d\tau \\ &\quad - \int_0^t e^{n\tau}R_n[B(w,\tilde{u}_n) + B(\tilde{u}_n,w)]d\tau \\ &\quad - \int_0^t e^{n\tau}R_n[B(v,\tilde{w}_n) + B(\tilde{w}_n,v)]d\tau \\ &\quad + \int_0^t e^{n\tau}R_n[B(\tilde{w}_n,\tilde{u}_n) + B(\tilde{v}_n,\tilde{w}_n)]d\tau. \end{aligned}$$

Letting $t\to\infty$ and using (5.68) give (5.67). □

In estimating the integrands on the right hand side of (5.67), we use the estimates obtained in Section 5.1 applied to both u^0 and v^0. However, $t_0(v^0)$ and $g_0(v^0)$ given by formulas (5.15) and (5.19) respectively, may vary for different v^0. For our convenience, we fix for the rest of this section

$$t_0 = \log^+\Big(8\big(\|u^0\|+1\big)/\varepsilon_0\Big) + 1 > 0, \tag{5.69}$$

$$g_0 = \max\{e, 8e(\|u^0\|+1)/\varepsilon_0\}. \tag{5.70}$$

Similar to (5.16), we have

$$\|u(t_0)\|, \|v(t_0)\| < \varepsilon_0/2. \tag{5.71}$$

Since $e^{t_0}\le g_0$ and by (5.71), the condition (5.60) in Lemma 5.13 is satisfied for both u^0 and v^0. Therefore any results in Section 5.1 applied to u^0 or v^0 are understood with t_0 and g_0 taking the values in (5.69) and (5.70) respectively.

By (5.71) and Corollary 3.12,

$$\|w(t_0+\tau)\| \le 2\|w(t_0)\|e^{-\tau}, \quad \tau>0, \tag{5.72}$$

$$\|w(t_0+\zeta)\| \le 4\|w(t_0)\|e^{-\mathbf{Re}\zeta}, \quad \zeta\in E. \tag{5.73}$$

Definition 5.16. Define $\gamma_{j,u}$, $\tilde{\gamma}_{j,u}$ and $\gamma_{j,v}$, $\tilde{\gamma}_{j,v}$ as in Definition 5.3 for u^0 and v^0 respectively. Let

$$\nu_n = \rho_n\|w_n^0\| = \rho_n\|W_n(0,u^0) - W_n(0,v^0)\|, \quad n\in\mathbb{N}, \tag{5.74}$$

$$\mu_1=\nu_1, \quad \mu_n = \nu_n + C_1\kappa_n\sum_{k+j=n}\mu_k(\gamma_{j,u}+\gamma_{j,v}),\ n>1, \tag{5.75}$$

$$\tilde{\mu}_1 = \mu_1, \quad \tilde{\mu}_n = \mu_n + 3C_1\kappa_n \sum_{k+j=n} \tilde{\mu}_k(\tilde{\gamma}_{j,u} + \tilde{\gamma}_{j,v}),\ n > 1. \tag{5.76}$$

Applying Proposition 3.10 with $\bar{u}^0$ and $\bar{v}^0$ given by (5.64), we have

$$\rho_n\|w_n(t)\| \le \mu_n e^{-t}, \quad t > 0, \tag{5.77}$$

$$\rho_n\|w_n(\zeta)\| \le \tilde{\mu}_n e^{-\mathbf{Re}\zeta}, \quad \zeta \in E. \tag{5.78}$$

Consequently, by Lemma A.2,

$$\rho_n\|w_n(t)\| \le n\tilde{\mu}_n e^{-nt}[C_2(t+a_0)]^{n-1}, \quad t > 0. \tag{5.79}$$

The following are some estimates similar to those in Lemma 5.5.

Lemma 5.17. *Let $n > 1$, we have*

$$\|\tilde{w}_n(0)\| \le K_n, \tag{5.80}$$

$$\|\tilde{w}_n(t_0+\tau)\| \le \tilde{K}_n e^{-n\tau}\Big(1+\frac{\tau}{2}\Big)^{2n}, \quad \tau > 0, \tag{5.81}$$

$$\int_0^{t_0} \|\tilde{w}_n(t)\|^2 dt \le {K'}_n^2, \tag{5.82}$$

where

$$K_n = \|w(0)\| + \frac{1}{\rho_{n-1}}\sum_{k=1}^{n-1}\nu_k, \tag{5.83}$$

$$\tilde{K}_n = 4\|w(t_0)\| + \frac{e^{-t_0}}{\rho_{n-1}}\sum_{k=1}^{n-1}\tilde{\mu}_k, \tag{5.84}$$

$$K'_n = \Big\{2N_2|w(0)|^2 + \frac{1}{\rho_{n-1}^2}\Big(\sum_{k=1}^{n-1}\mu_k\Big)^2\Big\}^{1/2}, \tag{5.85}$$

where the positive number $N_2 = N_2(u^0)$ is defined in Lemma A.8.

Proof. The derivations of inequalities (5.80) and (5.81) are almost exactly like (5.34) and (5.35). Inequality (5.82) follows from Lemma A.8 using similar techniques. □

The analogue of Lemma 5.6 is

Lemma 5.18. *Given $u^0 \in \mathcal{R}$. Let $v^0 \in \mathcal{R}$ such that $\|u^0 - v^0\| < 1$. Then there is $M = M(u^0) > 1$ such that for each $n > 1$,*

$$\begin{aligned}&\rho_n\|W_n(u^0) - W_n(v^0)\| \\ &\quad \le \rho_n\|w^0\| + \kappa'_n M^n\Big\{|w^0| + \|w(t_0)\| + 3\sum_{k=1}^{n-1}\tilde{\mu}_k\Big\}.\end{aligned} \tag{5.86}$$

Proof. We derive from (5.67)

$$\begin{aligned}
\|W_n(u^0)-W_n(v^0)\| &\le \|R_n\tilde w_n(0)\| \\
&\quad + \underbrace{\int_0^\infty e^{nt}\sum_{\substack{k,j\le n-1\\ k+j\ge n+1}}\|R_nB(w_k,u_j)+R_nB(v_k,w_j)\|dt}_{J_1} \\
&\quad + \underbrace{\int_0^\infty e^{nt}\|R_nB(w,\tilde u_n)+R_nB(\tilde u_n,w)\|dt}_{J_2} \\
&\quad + \underbrace{\int_0^\infty e^{nt}\|R_nB(v,\tilde w_n)+R_nB(\tilde w_n,v)\|dt}_{J_3} \\
&\quad + \underbrace{\int_0^\infty e^{nt}\|R_nB(\tilde w_n,\tilde u_n)+R_nB(\tilde v_n,\tilde w_n)\|dt}_{J_4} \\
&= \|R_n\tilde w_n(0)\| + J_1+J_2+J_3+J_4.
\end{aligned}$$

First, $\|R_n\tilde w(0)\|\le K_n$, by (5.80).

Estimate of J_1. Using inequalities (A.2), (5.30) and (5.79), we obtain

$$J_1 \le \frac{L(6,n)}{\rho_{n-1}^2}\sum_{k,j<n,k+j>n}\tilde\mu_k(\tilde\gamma_{j,u}+\tilde\gamma_{j,v}) \le \frac{L(6,n)}{\rho_{n-1}^2}\big(K(u^0)+K(v^0)\big)\sum_{k=1}^{n-1}\tilde\mu_k$$

where $L(6,n)=L(1,n)$.

Estimate of J_2. Using (A.2), we have

$$\begin{aligned}
J_2 &\le \Big(\int_0^{t_0}+\int_{t_0}^\infty\Big)2C_1n^{3/4}e^{nt}\|w(t)\|\|\tilde u_n(t)\|dt \\
&= J_{2,1}+J_{2,2}.
\end{aligned}$$

For $J_{2,1}$, use (A.21) and (5.36)

$$\begin{aligned}
J_{2,1} &\le 2C_1n^{3/4}e^{nt_0}\Big\{\int_0^{t_0}\|w(t)\|^2dt\Big\}^{1/2}\Big\{\int_0^{t_0}\|\tilde u_n(t)\|^2dt\Big\}^{1/2} \\
&\le 2C_1n^{3/4}e^{nt_0}\sqrt{N_2}|w(0)|M_n'(u^0) = L(7,n)e^{nt_0}\sqrt{N_2}|w(0)|M_n'(u^0),
\end{aligned}$$

where $N_2=N_2(u^0)$ is defined in Lemma A.8. For $J_{2,2}$, use (5.72) and (5.35)

$$\begin{aligned}
J_{2,2} &\le 2C_1n^{3/4}\int_0^\infty e^{nt_0}e^{n\tau}\Big(2\|w(t_0)\|e^{-\tau}\Big)\Big\{\tilde M_n(u^0)e^{-n\tau}(1+\frac{\tau}{2})^{2n}\Big\}d\tau \\
&\le 4C_1n^{3/4}e^{nt_0}\|w(t_0)\|\|\tilde M_n(u^0)L_n = L(8,n)e^{nt_0}\|w(t_0)\|\tilde M_n(u^0).
\end{aligned}$$

Estimate of J_3. By inequality (A.2),

$$\begin{aligned}
J_3 &\le 2C_1n^{3/4}\Big(\int_0^{t_0}+\int_{t_0}^\infty\Big)e^{nt}\|v(t)\|\|\tilde w_n(t)\|dt \\
&= J_{3,1}+J_{3,2}.
\end{aligned}$$

For $J_{3,1}$, use (5.14) and (5.82)

$$\begin{aligned} J_{3,1} &\le 2C_1 n^{3/4} e^{nt_0} \Big\{ \int_0^{t_0} \|v(t)\|^2 dt \Big\}^{1/2} \Big\{ \int_0^{t_0} \|\tilde{w}_n(t)\|^2 dt \Big\}^{1/2} \\ &\le \sqrt{2} C_1 n^{3/4} e^{nt_0} |v^0| K_n' = L(9,n) e^{nt_0} |v^0| K_n'. \end{aligned}$$

For $J_{3,2}$, use (5.9) and (5.81)

$$\begin{aligned} J_{3,2} &\le 2C_1 n^{3/4} e^{nt_0} \int_0^\infty e^{n\tau} \Big\{ 2\|v(t_0)\| e^{-\tau} \Big\} \Big\{ \tilde{K}_n e^{-n\tau} (1 + \frac{\tau}{2})^{2n} \Big\} d\tau \\ &\le 2C_1 n^{3/4} e^{nt_0} \|v(t_0)\| L_n \tilde{K}_n = L(10,n) e^{nt_0} \|v(t_0)\| \tilde{K}_n. \end{aligned}$$

Estimate of J_4. By inequality (A.2),

$$\begin{aligned} J_4 &\le C_1 n^{3/4} \Big(\int_0^{t_0} + \int_{t_0}^\infty \Big) e^{nt} (\|\tilde{u}_n(t)\| + \|\tilde{v}_n(t)\|) \|\tilde{w}_n(t)\| dt \\ &= J_{4,1} + J_{4,2}. \end{aligned}$$

For $J_{4,1}$, use (5.36) and (5.82)

$$\begin{aligned} J_{4,1} &\le C_1 n^{3/4} e^{nt_0} \Big\{ \int_0^{t_0} 2\|\tilde{u}_n(t)\|^2 + 2\|\tilde{v}_n(t)\|^2 dt \Big\}^{1/2} \Big\{ \int_0^{t_0} \|\tilde{w}_n(t)\|^2 dt \Big\}^{1/2} \\ &\le \sqrt{2} C_1 n^{3/4} e^{nt_0} (M_n'(u^0) + M_n'(v^0)) K_n' \\ &= L(11,n) e^{nt_0} (M_n'(u^0) + M_n'(v^0)) K_n'. \end{aligned}$$

For $J_{4,2}$, use (5.35) and (5.81),

$$\begin{aligned} J_{4,2} &\le C_1 n^{3/4} e^{nt_0} \int_0^\infty e^{-n\tau} (\tilde{M}_n(u^0) + \tilde{M}_n(v^0)) \tilde{K}_n |1 + \frac{\tau}{2}|^{4n} d\tau \\ &= L(12,n) e^{nt_0} (\tilde{M}_n(u^0) + \tilde{M}_n(v^0)) \tilde{K}_n. \end{aligned}$$

Combining the inequalities above, we obtain

$$\begin{aligned} \|W_n(u^0) - W_n(v^0)\| &\le K_n + \frac{L(6,n)}{\rho_{n-1}^2} \big(K(u^0) + K(v^0) \big) \sum_{k=1}^{n-1} \tilde{\mu}_k \\ &\quad + L(13,n) e^{nt_0} \Big\{ \sqrt{N_2} |w^0| M_n'(u^0) + \|w(t_0)\| \tilde{M}_n(u^0) + |v^0| K_n' \\ &\quad + \|v(t_0)\| \tilde{K}_n + K_n'(M_n'(u^0) + M_n'(v^0)) + \tilde{K}_n(\tilde{M}_n(u^0) + \tilde{M}_n(v^0)) \Big\}. \end{aligned} \tag{5.87}$$

Note that all constants which depend on v^0 depend on it through the norms $|v^0|$ or $\|v^0\|$. Since $\|u^0 - v^0\| < 1$, these constants may all be estimated in terms of $|u^0| + 1$ and $\|u^0\| + 1$. More specifically, let $M_1 = K_1(\|u^0\| + 1, g_0)$, then Lemma 5.13 yields $K(u^0), K(v^0) \le M_1$ and

$$M_n(u^0), M_n'(u^0), M_n(v^0), M_n'(v^0) \le \|u^0\| + 1 + \frac{M_1}{\rho_{n-1}} \le \frac{M_2}{\rho_{n-1}}, \tag{5.88}$$

$$\tilde{M}_n(u^0), \tilde{M}_n(v^0) \le 4\varepsilon_0 + \frac{M_1}{\rho_{n-1}} \le \frac{M_2}{\rho_{n-1}}, \tag{5.89}$$

where $M_2 = M_1+\|u^0\|+1+4\varepsilon_0$. Therefore, there are positive numbers $M_0 = g_0(u^0)$, M_1 and M_2 depending only on u^0 so that inequality (5.87) gives

$$\|W_n(u^0) - W_n(v^0)\| \le K_n + \frac{L(6,n)(2M_1)}{\rho_{n-1}^2}\sum_{k=1}^{n-1}\tilde\mu_k$$
$$+ L(13,n)M_0^n\Big\{\sqrt{N_2}|w^0|\frac{M_2}{\rho_{n-1}} + \|w(t_0)\|\frac{M_2}{\rho_{n-1}} + \frac{\varepsilon_0\tilde K_n}{2} + K_n'\frac{3M_2}{\rho_{n-1}} + \tilde K_n\frac{2M_2}{\rho_{n-1}}\Big\}.$$

Hence there is $M = M(u^0) > 1$ such that

$$\|W_n(u^0) - W_n(v^0)\| \le \|w^0\| + \frac{L(6,n)M}{\rho_{n-1}^2}\sum_{k=1}^{n-1}\tilde\mu_k$$
$$+ L(13,n)M^n\Big\{\frac{|w^0| + \|w(t_0)\|}{\rho_{n-1}} + \frac{1}{\rho_{n-1}^2}\Big(\sum_{k=1}^{n-1}\nu_k + \sum_{k=1}^{n-1}\mu_k + \sum_{k=1}^{n-1}\tilde\mu_k\Big)\Big\},$$
$$\le \|w^0\| + \frac{\tilde L_n' M^n}{\rho_{n-1}^2}\Big\{|w^0| + \|w(t_0)\| + 3\sum_{k=1}^{n-1}\tilde\mu_k\Big\}$$

where

$$\tilde L_n' = \max\{1, L(6,n) + L(13,n)\} \tag{5.90}$$

does not depend on u^0, v^0 or ρ_n. After multiplying by ρ_n and using (5.21) we obtain inequality (5.86). □

Remark 5.19. Unlike inequality (5.42) in Lemma 5.6, estimate (5.86) in Lemma 5.18 involves the term $\|w(t_0)\|$ where t_0 may well be nonzero. Hence $\|w(t_0)\|$, in general, does not depend on $\|w^0\|$ explicitly and neither does $\|W(u^0) - W(v^0)\|_\star$. However, for the continuity of W, we only need

$$\|w(t_0)\| \to 0 \quad \text{as} \quad v^0 \to u^0 \quad \text{in} \quad V. \tag{5.91}$$

Note that t_0 is fixed for all v^0 satisfying (5.63) and the regular solutions of the Navier–Stokes equations in our context depend continuously on the initial data (cf. [15]). Therefore (5.91) holds true.

Lemma 5.20. *Let $\gamma_{j,u}^*, \tilde\gamma_{j,u}^*$ (respectively $\gamma_{j,v}^*, \tilde\gamma_{j,v}^*$) be defined by* (5.44)–(5.48) *for u^0 (respectively v^0). Note again that g_0 is given by* (5.70) *instead of* (5.15). *Let*

$$y_1 = \nu_1^* = \mu_1^* = \tilde\mu_1^* = \rho_1\|w^0\| + N_3\rho_1|w^0|^{1/2}, \tag{5.92}$$

where $N_3 = N_3(u^0)$ is defined in Lemma A.8. For $n > 1$ let

$$y_n = \rho_n\|w^0\| + \kappa_n' M^n\Big\{|w^0| + \|w(t_0)\| + 3\sum_{k=1}^{n-1}\tilde\mu_k^*\Big\}, \tag{5.93}$$

$$\nu_n^* = y_n + \kappa_n' D_n \sum_{k+j=n}\tilde\mu_k^*(\tilde\gamma_{j,u}^* + \tilde\gamma_{j,v}^*), \tag{5.94}$$

$$\mu_n^* = \nu_n^* + C_1\kappa_n' \sum_{k+j=n}\mu_k^*(\gamma_{j,u}^* + \gamma_{j,v}^*), \tag{5.95}$$

$$\tilde{\mu}_n^* = \mu_n^* + 3C_1\kappa_n' \sum_{k+j=n} \tilde{\mu}_k^*(\tilde{\gamma}_{j,u}^* + \tilde{\gamma}_{j,v}^*), \tag{5.96}$$

where $M = M(u^0) > 0$ be given in Lemma 5.18. Then

$$\rho_n\|W_n(u^0) - W_n(v^0)\| \le y_n, \quad \nu_n \le \nu_n^*, \quad \mu_n \le \mu_n^*, \quad \tilde{\mu}_n \le \tilde{\mu}_n^*, \quad n \in \mathbb{N}. \tag{5.97}$$

Proof. By induction. Case $n = 1$, similar to (5.54) and (5.55) in Lemma 5.8 we have $w_1(\zeta) = \Big(W_1(u^0) - W_1(v^0)\Big)e^{-\zeta}$ for $\mathbf{Re}\zeta > 0$. Thus it suffices to prove

$$\|W_1(u^0) - W_1(v^0)\| = |W_1(u^0) - W_1(v^0)| \le \|w^0\| + N_3|w^0|^{1/2}.$$

This indeed follows from $|W_1(u^0) - W_1(v^0)| = \lim_{t\to\infty} e^t|R_1w(t)|$, by (5.4), and inequality (A.22). For the induction step with $n > 1$, the key estimate is

$$\rho_n\|W_n(u^0) - W_n(v^0)\| \le y_n. \tag{5.98}$$

This follows from Lemma 5.18 and the induction hypothesis $\tilde{\mu}_k \le \tilde{\mu}_k^*$, for $k < n$. By (4.17) in Theorem 4.2, we have

$$\nu_n \le \rho_n\|W_n(u^0) - W_n(v^0)\| + \kappa_n D_n \sum_{k+j=n} \tilde{\mu}_k(\tilde{\gamma}_{j,u} + \tilde{\gamma}_{j,v})$$

which is less than or equal to ν_n^* by (5.98), the induction hypothesis, the relations $\kappa_n \le \kappa_n'$ and $\tilde{\gamma}_{n,u} \le \tilde{\gamma}_{n,u}^*$, $\tilde{\gamma}_{n,v} \le \tilde{\gamma}_{n,v}^*$ where the latter two are analogues of (5.53). The last two inequalities of (5.97) follow easily. □

Theorem 5.21. *The normalization map $W : \mathcal{R} \to S_A^\star$ is continuous.*

Proof. Given $u^0 \in \mathcal{R}$, let $v^0 \in \mathcal{R}$ satisfying $\|u^0 - v^0\| < 1$. We set the notation as in Lemma 5.20. Then by Lemma 5.13 the sums $\sum_{n=1}^\infty \tilde{\gamma}_{n,u}^*$ and $\sum_{n=1}^\infty \tilde{\gamma}_{n,v}^*$ are bounded by a positive constant depending only on $\|u^0\|$, namely, $K_1(\|u^0\| + 1, g_0)$ (see Definition 5.12). Summing up (5.93)–(5.96), we have

$$\tilde{\mu}_n^* \le \rho_n\|w^0\| + \tilde{\kappa}_n^*\tilde{M}^n\Big\{|w^0| + \|w(t_0)\| + \sum_{k=1}^{n-1}\tilde{\mu}_k^*\Big\}, \tag{5.99}$$

where $\tilde{M} = \tilde{M}(u^0) > 1$ and $\tilde{\kappa}_n^* = \kappa_n'(3+D_n+4C_1)$. Note that $|w^0|, \|w^0\| \le 2\|u^0\|+1$ and $\|w(t_0)\| \le \varepsilon_0$. By (5.20) $\lim_{n\to\infty}(\tilde{\kappa}_n^*)^{1/2^n} = 0$; hence, Lemma A.7 implies $\sum_{n=1}^\infty \tilde{\mu}_n^* < M^* = M^*(u^0) < \infty$.

Now, by using Remark 5.19, an induction argument shows that $\tilde{\mu}_n^* \to 0$ as $\|u^0 - v^0\| \to 0$ for each n. The same arguments as in Theorem 3.8 show that for any $\varepsilon > 0$, there is $\delta > 0$ such that if $\|u^0 - v^0\| < \delta$ then $\sum_{n=1}^\infty \tilde{\mu}_n^* < \varepsilon$. Consequently $\|W(u^0) - W(v^0)\|_\star < \varepsilon$. Therefore the normalization map W is continuous at u^0. □

Since $W(0,\cdot) = Q(0,\cdot) \circ W$, Theorem 5.21 and Proposition 4.3 imply

Proposition 5.22. *The map $W(0,\cdot) : \mathcal{R} \to S_A^\star$ is continuous.*

Remark 5.23. Combining the estimates obtained in this section with the techniques used in Theorem 7.4 of [4] one can impose stricter conditions on the ρ_n than given in Definition 5.2 and also show that the normalization map is Lipschitz continuous near the origin of V. We leave the subject of this and finer properties of the normalization map for our future research.

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Appendix A.

First, we recall some inequalities involving the nonlinear terms in the Navier–Stokes equations (see e.g. [4] for their proofs). There is an absolute constant $C_1 > 0$ such that

$$|P_n B(u,v)| \le C_1 n^{1/4} \|u\| \|v\|, \quad u, v \in V_{\mathbb{C}}, \tag{A.1}$$

$$\|P_n B(u,v)\| \le C_1 n^{3/4} \|u\| \|v\|, \quad u, v \in V_{\mathbb{C}}, \tag{A.2}$$

$$|B(u,v)| \le C_1 \|u\|^{1/2} |Au|^{1/2} \|v\|, \quad u \in (\mathcal{D}_A)_{\mathbb{C}}, v \in V, \tag{A.3}$$

$$\|B(u,v)\| \le C_1 \|u\|^{1/2} |Au|^{1/2} |Av|, \quad u, v \in (\mathcal{D}_A)_{\mathbb{C}}. \tag{A.4}$$

The following region in the complex plane is used in [4] to describe the domains of analyticity of the solutions of the complexified extended Navier–Stokes equations:

$$E = \{t + se^{i\theta} : \cos\theta > 1/(3e^t), t > 0, s \ge 0, |\theta| < \pi/2\}. \tag{A.5}$$

The next two lemmas are some Phragmen-Linderlöf type estimates obtained in [4]. The first is Corollary B.3 of [4].

Lemma A.1. *Suppose $u(\zeta)$ is analytic in E,*

$$|u(\zeta)| \le M, \quad \zeta \in E, \quad \text{and} \quad |u(t)| \le Ce^{-nt}, \quad t > 0, \tag{A.6}$$

for some positive numbers M, C and n. Then

$$|u(t)| \le Me^{-nt} \Big|1 + \frac{t}{2}\Big|^{2n}, \quad t > 0. \tag{A.7}$$

Combining Corollary B.6 and Lemma 5.1 of [4] we have

Lemma A.2. *Let $n \in \mathbb{N}$. Suppose $q(\zeta), \zeta \in \mathbb{C}$, is a polynomial of degree less than or equal to $(n-1)$ and*

$$|e^{-n\zeta} q(\zeta)| \le Me^{-\mathbf{Re}\zeta}, \quad \zeta \in E, \tag{A.8}$$

where M is a positive number. Then

$$|q(\zeta)| \le MnC_2^{n-1} \big(|\zeta| + a_0\big)^{n-1}, \quad \zeta \in \mathbb{C}, \tag{A.9}$$

where C_2 and a_0 are fixed positive constants.

Estimates on a_0 and C_2 given in Lemma 5.1 and Lemma B.4 of [4] indicate that

$$a_0 = \tau + \frac{1}{2} g(\tau)^2 \quad \text{and} \quad C_2 \ge \frac{\big(4 + 2\tau + g(\tau)^2 + g(\tau)\sqrt{4 + g(\tau)^2}\,\big)^2}{8g(\tau)\sqrt{4 + g(\tau)^2}}$$

where $g(\tau) = \sqrt{3}e^{\tau/2}$ and $\tau > 2 + 2\sqrt{2}$. Thus, we may take $a_0 = 192$ and $C_2 = 196$.

Lemma 4.2 of [4] is used repeatedly in this paper. We recall this result with a simplified proof which can be easily adapted for the subsequent lemmas.

Lemma A.3 ([4])**.** *Let $a_n \ge 0$, $n \in \mathbb{N}$, and let $(k_n)_{n=2}^\infty$ be a sequence of positive numbers satisfying*

$$\lim_{n\to\infty} k_n^{1/n} = 0. \tag{A.10}$$

Let $d_1 = a_1$ and $d_n = a_n + k_n \sum_{k+j=n} d_k d_j$, $n > 1$. If $\sum_{n=1}^\infty a_n$ is finite, then

$$\sum_{n=1}^\infty d_n \le \sum_{n=1}^\infty a_n + \sum_{n=2}^\infty k_n (n-1) M_0^n < \infty, \tag{A.11}$$

where $M_0 = \max\{1, 2\kappa_n(n-1) : n \geq 2\} \max\{1, 2\sum_{n=1}^{\infty} a_n\}$.

Proof. We claim that

$$d_n \leq M^n/K, \quad n \in \mathbb{N}, \tag{A.12}$$

$$\sum_{n=1}^{\infty} d_n \leq \sum_{n=1}^{\infty} a_n + \frac{1}{K^2} \sum_{n=2}^{\infty} k_n(n-1)M^n < \infty, \tag{A.13}$$

where K and M are positive numbers satisfying

$$K \geq 2\max\big\{(n-1)k_n : n \geq 2\big\}, \quad M \geq \max\big\{(2Ka_n)^{1/n} : n \geq 1\big\}.$$

We prove (A.12) by induction. Clearly, the inequality holds when $n = 1$. Let $n > 1$ and assume that (A.12) holds for $k = 1, 2, \cdots, n-1$. Then

$$d_n \leq a_n + k_n \sum_{k+j=n} \frac{M^k}{K} \frac{M^j}{K} = \frac{M^n}{K} \Big\{ \frac{Ka_n}{M^n} + \frac{k_n(n-1)}{K} \Big\} \leq \frac{M^n}{K}.$$

Therefore (A.12) holds for all $n \in \mathbb{N}$. Then (A.13) follows immediately. To prove (A.11), take $K = \max\{1, 2\kappa_n(n-1) : n \geq 2\}$ and $M = K\max\{1, 2\sum_{n=1}^{\infty} a_n\}$. Note that $K \geq 1$ and $M \geq 1$, hence $M^n \geq M \geq 2Ka_n$, for all $n \in \mathbb{N}$. Thus (A.11) follows from (A.13) and (A.10). □

The following lemmas generalize Lemma A.3 to other numeric series.

Lemma A.4. *Let $(a_n)_{n=1}^{\infty}$ and $(k_n)_{n=2}^{\infty}$ be two sequences of positive numbers. Let $d_1 = a_1$ and $d_n = a_n + k_n(\sum_{k=1}^{n-1} d_k)^2$, for $n > 1$. Suppose*

$$\lim_{n\to\infty} k_n^{1/2^n} = 0. \tag{A.14}$$

If $\sum_{n=1}^{\infty} a_n$ is finite, so is $\sum_{n=1}^{\infty} d_n$. More precisely,

$$\sum_{n=1}^{\infty} d_n \leq \sum_{n=1}^{\infty} a_n + \alpha^2 \sum_{n=1}^{\infty} k_n M^{2(2^n-1)} < \infty, \tag{A.15}$$

where $\alpha = \sup\{a_n : n \in \mathbb{N}\}$ and $M = 3\sup\{1, \alpha, k_n\alpha : n > 1\}$.

Proof. Let $S_n = \sum_{k=1}^{n} d_k$. Note that $S_1 = a_1$ and

$$S_n = a_n + S_{n-1} + k_n S_{n-1}^2, \quad n > 1. \tag{A.16}$$

We prove by induction that

$$S_n \leq \alpha M^{2^n-1}, \quad n \in \mathbb{N}. \tag{A.17}$$

One can see that (A.17) holds when $n = 1$. Let $N > 1$ and assume (A.17) holds for all $n < N$. Using the induction hypothesis and the fact that $2^{N-1} \leq 2^N - 1$, $\alpha_N \leq \alpha$, $M \geq 3$ and $k_N\alpha/M \leq 1/3$, we then have

$$\begin{aligned} S_N &\leq a_N + \alpha M^{2^{N-1}-1} + k_N\alpha^2 M^{2^N-2} \\ &= \alpha M^{2^N-1}\Big(\frac{a_N}{\alpha M^{2^N-1}} + \frac{1}{M} + \frac{k_N\alpha}{M}\Big) \\ &\leq \alpha M^{2^N-1}\Big(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\Big) = \alpha M^{2^N-1}. \end{aligned}$$

Thus (A.17) is true. Using (A.17), we obtain

$$\sum_{n=1}^{\infty} d_n \le \sum_{n=1}^{\infty} a_n + \sum_{n=2}^{\infty} k_n \alpha^2 M^{2^{n+1}-2}.$$

The last sum is finite due to (A.14), hence (A.15) follows. □

For the numeric sequences appearing in the study of the range of the normalization map, we have

Lemma A.5. *Let $X, Y > 0$, $(a_n)_{n=1}^{\infty}$ and $(k_n)_{n=2}^{\infty}$ be two sequences of positive numbers. Let $d_1 = a_1$ and*

$$d_n = a_n + k_n Y^n \Big\{ X^2 + \Big(\sum_{k=1}^{n-1} d_k \Big)^2 \Big\}$$

for $n > 1$. Suppose $\lim_{n\to\infty} k_n^{1/2^n} = 0$. If $\sum_{n=1}^{\infty} a_n$ is finite, so is $\sum_{n=1}^{\infty} d_n$.

Proof. Assume $\sum_{n=1}^{\infty} a_n < \infty$. Take $a'_1 = a_1$ and $a'_n = a_n + k_n Y^n X^2$, $n > 1$. Also take $k'_n = k_n Y^n$, $n > 1$. Note that we still have $\sum_{n=1}^{\infty} a'_n$ is finite and $(k'_n)^{1/2^n} \to 0$. By the preceding lemma, $\sum_{n=1}^{\infty} d_n < \infty$. □

The following are similar results to Lemmas A.4 and A.5 applied to different types of sequences arising from the study the continuity of the normalization map.

Lemma A.6. *Let $(a_n)_{n=1}^{\infty}$ and $(k_n)_{n=2}^{\infty}$ be two sequences of positive numbers . Let $d_1 = a_1$ and $d_n = a_n + k_n \sum_{k=1}^{n-1} d_k$ for $n > 1$. Suppose $\lim_{n\to\infty} k_n^{1/n} = 0$. If $\sum_{n=1}^{\infty} a_n$ is finite, so is $\sum_{n=1}^{\infty} d_n$. More precisely,*

$$d_n \le M^n, \quad n \in \mathbb{N}, \tag{A.18}$$

$$\sum_{n=1}^{\infty} d_n \le \sum_{n=1}^{\infty} a_n + \sum_{n=2}^{\infty} k_n (n-1) M^{n-1} < \infty, \tag{A.19}$$

where $M \ge \max\{1, 2k_n(n-1) : n > 1\}$ and $M \ge \max\{(2a_n)^{1/n} : n \in \mathbb{N}\}$.

Proof. We see that (A.18) holds for $n = 1$. Let $N > 1$ and assume (A.18) holds for $n \le N - 1$. We have

$$d_N \le a_N + k_n(n-1)M^{n-1} \le \frac{M^n}{2} + \frac{k_n(n-1)}{M} M^N \le M^N.$$

Hence (A.18) is true and (A.19) follows obviously. □

Lemma A.7. *Let $X > 0$, $(a_n)_{n=1}^{\infty}$ and $(k_n)_{n=2}^{\infty}$ be two sequences of positive numbers. Let $d_1 = a_1$ and*

$$d_n = a_n + k_n \Big(X + \sum_{k=1}^{n-1} d_k \Big)$$

for $n > 1$. Suppose $\lim_{n\to\infty} k_n^{1/n} = 0$. If $\sum_{n=1}^{\infty} a_n$ is finite, so is $\sum_{n=1}^{\infty} d_n$.

Proof. Let $a'_1 = a_1$ and $a'_n = a_n + k_n X$, $n > 1$. We have $\sum_{n=1}^{\infty} a'_n < \infty$. Then apply Lemma A.6. □

Finally, for the sake of the completeness of this paper, we recall with a proof some commonly known facts about the regular solutions of the Navier–Stokes equations.

Lemma A.8. *Given $u^0 \in \mathcal{R}$. Let $v^0 \in \mathcal{R}$ such that $\|u^0 - v^0\| < 1$. Let $w(t) = S(t)u^0 - S(t)v^0$. Then there are positive numbers $N_1 = N_1(u^0)$, $N_2 = N_2(u^0)$ and $N_3 = N_3(u^0)$ independent of v^0 such that*

$$|w(t)| \le N_1 |w(0)| e^{-t/2}, \quad t > 0, \tag{A.20}$$

$$\int_0^\infty \|w(\tau)\|^2 d\tau \le N_2 |w(0)|^2, \tag{A.21}$$

$$|R_1 w(t)| \le e^{-t}(|R_1 w(0)| + N_3 |w(0)|^{1/2}), \quad t > 0. \tag{A.22}$$

Proof. The positive constant C in this proof is generic and is independent of u^0 and v^0. First we know that

$$\int_0^\infty |Au(\tau)|^2 d\tau \le C\|u^0\|^2 (1 + \max_{t\ge 0} \|u(t)\|^4) = N_0(u^0) < \infty. \tag{A.23}$$

The equation for w is

$$\frac{dw}{dt} + Aw + B(w,u) + B(v,w) = 0. \tag{A.24}$$

From (A.24) we have

$$\frac{d}{dt}|w|^2 + 2\|w\|^2 \le C|w|^{3/2}\|w\|^{1/2}|Au| \le C\|w\||w||Au|. \tag{A.25}$$

Hence

$$\frac{d}{dt}|w|^2 + \|w\|^2 \le C|w|^2|Au|^2. \tag{A.26}$$

By Poincare's inequality and then Gronwall's inequality

$$|w(t)|^2 \le |w(0)|^2 e^{-t + C\int_0^t |Au(\tau)|^2 d\tau} \le |w(0)|^2 e^{-t} e^{CN_0} = N_1^2 |w(0)|^2 e^{-t}.$$

Thus, we obtain inequality (A.20).

Integrating (A.26) and using estimate (A.20), we have

$$\begin{aligned}\int_0^t \|w(\tau)\|^2 d\tau &\le |w(0)|^2 + C\int_0^t N_1^2 |w(0)|^2 |Au(\tau)|^2 d\tau \\ &\le |w(0)|^2 (1 + CN_1^2 N_0) = N_2 |w(0)|^2.\end{aligned}$$

Letting $t \to \infty$ yields inequality (A.21).

From (A.24), we derive $\frac{d}{dt}R_1 w + R_1 w + R_1 B(w,u) + R_1 B(u,w) = 0$, which yields

$$\frac{1}{2}\frac{d}{dt}|R_1 w|^2 + |R_1 w|^2 \le |\langle B(w,u) + B(u,w), R_1 w\rangle|. \tag{A.27}$$

Since $R_1 H$ is finite dimensional, all the norms in $R_1 H$ are equivalent. Therefore

$$\begin{aligned}|\langle B(u,w), R_1 w\rangle + \langle B(w,u), R_1 w\rangle| &\le |\langle B(u, R_1 w), w\rangle| + |\langle B(w, R_1 w), u\rangle| \\ &\le C|w||u|\|\nabla(R_1 w)\|_{L^\infty(\Omega)} \le C|w||u|\|\nabla(R_1 w)\|_{L^2(\Omega)} \le C|R_1 w||w||u|.\end{aligned}$$

By (5.12), $e^t|u(t)| \le |u^0|$ and $e^t|R_1w(t)| \le |u^0| + |v^0|$. We now have from (A.27)

$$\begin{aligned} e^{2t}|R_1w(t)|^2 - |R_1w(0)|^2 &\le C\int_0^t e^{2\tau}|R_1w(\tau)||u(\tau)||w(\tau)|d\tau \\ &\le C\int_0^t (|u^0| + |v^0|)|u^0|\Big\{N_1|w^0|e^{-\tau/2}\Big\}d\tau \\ &\le CN_1|u^0|(2|u^0| + 1)|w^0| = N_3^2|w^0|. \end{aligned}$$

Hence $e^{2t}|R_1w(t)|^2 \le |R_1w(0)|^2 + N_3^2|w(0)|$, we obtain (A.22). □

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