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Viscosity Versus Vorticity Stretching: Global Well-posedness for a Family of Navier–Stokes-Alpha-like Models

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We study global well-posedness and regularity of solutions for a family of incompressible three-dimensional Navier–Stokes-alpha-like models that employ fractional Laplacian operators. This family of equations depends on two parameters, θ_1 and θ_2, which affect the strength of nonlinearity (vorticity stretching) and the degree of viscous smoothing. Varying θ_1 and θ_2 interpolates between the incompressible Navier–Stokes equations and the incompressible (Lagrangian averaged) Navier–Stokes-α model. Our main result, which contains previously established results of J.L. Lions and others, provides a relationship between θ_1 and θ_2 that is sufficient to guarantee global existence, uniqueness and regularity of solutions.

1. Introduction

Numerical solution of the Navier–Stokes equations for problems of engineering and geophysical relevance is not possible at present—even on the most powerful computers (see, e.g., [20,49]). Moreover, the mathematical theory for global existence and regularity of solutions to these equations is one of the most challenging open questions of mathematical analysis [17]. In turbulent fluid flows, most of the kinetic energy lies in the large scales, whereas the mathematical and computational difficulties lie in understanding the dynamical interaction between the significantly wide range of relevant scales in this multiscale phenomenon. To overcome this obstacle much effort is being made to produce reliable turbulence models which parameterize the effect of the active small scales in terms of the large scales.

Over the last years, the viscous Camassa–Holm equations have been proposed as a subgrid turbulence model: the (Lagrangian averaged) Navier–Stokes-α or LANS-α model. Physical derivations of the inviscid version of this model were proposed in Holm, Marsden and Ratiu [32], Holm [29] and in [10]. The inviscid version of this model also appears in the context of the non-Newtonian second grade fluids (see, for example, Dunn and Rajagopal [21], Joseph [36] and references therein). Adding an *add-hoc* viscous dissipation term, as proposed in [9,10,11,24], one reaches the viscous Camassa–Holm equations, also known as the LANS-α or Navier–Stokes-α model. This model was used in [9,10,11] as a closure model for the Reynolds averaged Navier–Stokes equations, and it was tested successfully

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against experimental measurements and direct numerical simulations of turbulent channel and pipe flows. Recently, it has been observed that an identical closure for the Reynolds equations may be obtained from the Leray-α model [14, 35], the Clark-α model [8] and the Brandina model [15]. It is worth mentioning that one also obtains a similar set of reduced equations in channels and pipes when applying multipolar viscous fluid models (see, for example, Bellout, Bloom and Nečas [5]). Cheskidov [13] and Holm, Putkaradze, Weidman and Wingate [34] used this model to obtain an extension of the Prandtl equations for the averaged flow in a turbulent boundary layer. Chen, Holm, Margolin and Zhang [12] observed through direct numerical simulations that the energy spectra of the LANS-α model decays faster than $k^{-5/3}$ for wave numbers $k \gg 1/\alpha$. A more refined scaling argument in [23] indicates that the translational kinetic energy spectrum should scale as $\epsilon_\alpha^{2/3} k^{-5/3}(1+\alpha^2k^2)^{-2/3}$. Furthermore, Holm [30] showed that the LANS-α model possesses a Kármán-Howarth theorem consistent with these scalings. In addition, the LANS-α model enjoys a finite dimensional global attractor with an analytical upper bound on its fractal dimension that scales consistently with heuristic arguments for extensive three-dimensional turbulence, that is, as $(L/l_d)^3$. The history of the LANS-α model and information regarding its use as a subgrid scale turbulence model has been summarized by Holm, Jeffery, Kurien, Livescu, Taylor and Wingate in [31]. Similar results have also been obtained for the Leray-α model [14] and the Clark-α model [8], with respective energy spectrum scalings of $(\epsilon_{\mathrm{Leray}})^{2/3}k^{-5/3}(1+\alpha^2k^2)^{-(n-6)/3}$ and $(\epsilon_{\mathrm{Clark}})^{2/3}k^{-5/3}(1+\alpha^2k^2)^{-(n-4)/3}$ where n is an integer between 0 and 2 inclusive that indicates how the average velocity of an eddy of length k^{-1} is computed. The Clark-α model also possesses a finite dimensional global attractor scaling as $(L/l_d)^3$, whereas the Leray-α model possess an attractor with bounds scaling as $(L/l_d)^{12/7}$, somewhere between one and two-dimensional turbulence. Related models may be found in [15, 35]. Note also the numerical work [27,28] of Holm and Guerts concerning the Leray-α model and the approximation of trajectory attractors of the Navier–Stokes equations as $\alpha \to 0$ in [57].

Not only does the LANS-α model have useful computational properties, but, as shown in [24], the mathematical theory of global existence and uniqueness of solutions for this model is complete. In this paper we consider a family of equations which interpolates between the Navier–Stokes equations and the LANS-α model and look for the limiting cases where we can prove global existence and uniqueness of regular solutions.

Let θ_1 and θ_2 be two non-negative parameters. The family of equations we shall consider have the form

$$\begin{cases} \dfrac{\partial v}{\partial t} + \nu(-\Delta)^{\theta_2} v - u \times (\nabla \times v) + \nabla p = f, \\ v = u + (-\alpha^2\Delta)^{\theta_1} u, \\ \nabla \cdot u = 0 \qquad \text{and} \qquad \nabla \cdot v = 0 \end{cases} \tag{1.1}$$

where $u(x,t)$, $v(x,t)$ and $p(x,t)$ are the unknowns, the initial condition $u(x,0) = u_0(x)$ is given and $f(x,t)$ is a known forcing function. For simplicity and lack of natural physical boundary conditions we work on the L-periodic three-dimensional torus $\Omega = [0,L]^3$. We shall assume that the length scale $\alpha > 0$ and the viscosity $\nu \geq 0$. Note, for this equation to make dimensional sense, the dimensional units of ν must be given as $[T]^{-1}[L]^{2\theta_2}$ where $[T]$

represents units of time and $[L]$ units of length. The parameter $\theta_1 \geq 0$ affects the strength of the non-linear term and $\theta_2 \geq 0$ represents the degree of viscous dissipation.

The incompressible Navier–Stokes equations are obtained when $\theta_1 = 0$ and $\theta_2 = 1$. In this case ν is the kinematic viscosity, u is the Eulerian velocity field, and $\frac{1}{2}p - \frac{1}{4}u^2$ is the physical pressure. Note the factor of two in the physical pressure arises from the relationship $v = 2u$ when $\theta_1 = 0$. The LANS-α (NS-α) model is obtained by taking $\theta_1 = \theta_2 = 1$. In this case u is thought to represent a velocity field averaged over a length scale of size α. A simple modification of the scaling arguments in [23] yield that the energy spectrum of u should scale as $\epsilon_{\theta_2}^{2/3} k^{-5/3} \big(1 + (\alpha k)^{2\theta_1}\big)^{-2/3}$ where ϵ_{θ_2} is the average total energy dissipation rate given by

$$\epsilon_{\theta_2} = \left\langle \frac{\nu}{L^3} \int_\Omega u \cdot (-\Delta)^{\theta_2} v \, d^3x \right\rangle .$$

Inviscid versions of equations (1.1) may be obtained by taking $\nu = 0$. Note that given any value of $\theta_1 \geq 0$ the inviscid equations may be derived using variational principles from a Hamiltonian given by a norm equivalent to the H^{θ_1} norm, see, for example, Arnold and Khesin [3], Holm, Marsden, Ratiu [33] and Holm [29]. Thus, solutions to the inviscid equations are geodesic flows with respect to this norm in the group of volume preserving diffeomorphisms. Global existence and regularity of solutions for the three-dimensional LANS-α model, subject to periodic boundary conditions, was shown in [24]. Marsden and Shkoller obtained similar results subject to mathematically convenient Dirichlet-type boundary conditions in [45]. The question of global existence and regularity of solutions to the three-dimensional Navier–Stokes equations remains open [17].

Fractional order Laplace operators have been employed to model diffusion processes in porous media by Meerschaert, Benson and Baeumer [46, 47] and Benson, Schumer, Meerschaert and Wheatcraft [6]. Existence and uniqueness of solutions for other modifications of the Navier–Stokes equations have been studied by Berselli, Galdi, Iliescu and Layton [4], Ju [37], Ladyzhenskaya [38], Lions [41], Sobolevskii [53], Soltanov [54, 55] and references therein. The Smagorinsky model [52] (see also Pope [49]) is another widely studied computational turbulence model. In Ladyzhensakaya [39]an analytical study of the Smagorinsky model and its long-time behavior is reported (see also Ju [37] and Málek and Nečas [44]).

The first section of this paper deals with some mathematical preliminaries while the second part proves our main result. Namely,

> *If $\nu > 0$ and $2\theta_1 + 4\theta_2 \geq 5$, or alternatively, if $\nu = 0$ and $2\theta_1 \geq 5$, then* (1.1) *possesses unique global smooth solutions which depend continuously on the initial data.*

This none surprising result illustrates how a milder non-linearity and/or a stronger viscous dissipation can work together to obtain global existence and regularity of solutions. Note that the parameters $\theta_1 = 0$ and $\theta_2 = 1$ corresponding to the Navier–Stokes equations lie outside the region covered by our main result, whereas the parameters $\theta_1 = \theta_2 = 1$ corresponding to the LANS-α (NS-α) model are within our allowable range of parameters. The case $\theta_1 = 0$ and $\theta_2 = 5/4$ was studied by Lions [41].

The paper is organized as follows. In Section 2 we state a number of identities and facts that will be used in the following sections. Section 3 gives the Galerkin truncation of equations (1.1) and makes an initial estimate on the solution that holds for all values of θ_1 and θ_2. Section 4 provides additional *a-priori* estimates for the viscous case and proves existence and uniqueness of regular solutions when $2\theta_1 + 4\theta_2 \geq 5$ and $\theta_2 > 1/2$. Gevrey regularity is obtained when either $\theta_1 > 0$ or $\theta_2 > 5/2$. Section 5 provides similar existence and uniqueness results for the viscous case when $0 < \theta_2 \leq 1/2$. In this case the viscous effects aren't strong enough to show Gevrey regularity. Section 6 provides existence and uniqueness results for the damped and the inviscid cases when $\theta_2 = 0$. The damping is not essential for our results and hence one can take $\nu = 0$. The final section combines the results of sections 4, 5 and 6 to obtain our main result.

2. Preliminaries

In this section we characterize the fractional Sobolev spaces H^β and V^β and state a number of identities and facts that we shall use later on.

By integrating (1.1) over Ω one can see that if the spatial average of the initial condition u_0 is zero, then the spatial average of the solution remains zero provided the spatial average of the forcing term f is zero. Therefore, in this paper we will consider initial conditions and forcing functions such that $\int_\Omega u_0 = 0$ and $\int_\Omega f = 0$. As a result, solutions to (1.1) have zero spatial averages for all $t \geq 0$.

Let $\mathcal{F}$ be the space of formal Fourier series $\{\, u = \sum_{k\in\mathcal{J}} \hat{u}_k \phi_k : \hat{u}_k \in \mathbf{C}^3 \,\}$ where $\mathcal{J} = \{\, m\alpha_0^{-1} : m \in \mathbf{Z}^3 \setminus \{0\} \,\}$, $\alpha_0 = L/(2\pi)$ and $\phi_k(x) = e^{ik\cdot x}$. Let $\mathcal{V}$ be the space of divergence free trigonometric polynomials consisting of all $u \in \mathcal{F}$ such that $k \cdot \hat{u}_k = 0$ for all $k \in \mathcal{J}$ and $\hat{u}_k = 0$ for all but finitely many values of $k \in \mathcal{J}$. Define H^β and V^β by

$$H^\beta = \Big\{\, u \in \mathcal{F} : \quad \|u\|_\beta < \infty, \quad \hat{u}_k^* = \hat{u}_{-k} \qquad \text{and} \quad \hat{u}_0 = 0 \,\Big\}$$

and

$$V^\beta = \Big\{\, u \in H^\beta : \quad k \cdot \hat{u}_k = 0 \,\Big\} \quad \text{where} \quad \|u\|_\beta^2 = L^3 \sum_{k\in\mathcal{J}} |k|^{2\beta} |\hat{u}_k|^2.$$

When $\beta \geq 0$ the spaces H^β correspond to spaces of periodic functions on Ω taking values in $\mathbf{R}^3$ with zero mean. Furthermore, V^β are those elements in H^β that are divergence free. Note also that V^β is the closure of $\mathcal{V}$ in H^β with respect to the $\|\cdot\|_\beta$ norm.

Following standard notation, we denote the space V^0 by H with norm $|u| = \|u\|_0$ and inner product $(u, v) = L^3 \sum_{k\in\mathcal{J}} (\hat{u}_k \cdot \hat{v}_{-k})$. Thus, H is the Hilbert space of periodic L^2-integrable functions on Ω taking values in $\mathbf{R}^3$ which are divergence free and have zero mean. We identify the continuous dual space of V^β as $V^{-\beta}$ with the pairing given by $\langle u, v\rangle_\beta = L^3 \sum_{k\in\mathcal{J}} (\hat{u}_k \cdot \hat{v}_{-k})$. In particular, we denote V^1 by V with norm $\|u\| = \|u\|_1$ and identify the dual space of V as $V' = V^{-1}$. Note also that $H^{\beta+\varepsilon}$ is compactly embedded in H^β for any $\varepsilon > 0$. Similarly $V^{\beta+\varepsilon}$ is compactly embedded in V^β.

The following Sobolev embedding theorem holds for the fractional spaces H^β. In this result and throughout we use the symbol C to denote a non-dimensional constant.

Theorem 2.1. *If $0 \leq \beta < 3/2$ and $1/p = 1/2 - \beta/3$, then $H^\beta \subset L^p$. Moreover, there exists a non-dimensional constant C depending only on p such that $\|u\|_{L^p} \leq C\|u\|_\beta$ for all $u \in H^\beta$.*

A more general Sobolev embedding theorem from which the Sobolev embedding theorem immediately follows may be found in Adams [1].

We proceed with some definitions and results that will allow us to view (1.1) as an ordinary differential equation with an infinite dimensional phase space. Denote the Leray-Helmholtz projection by P_σ. Note that P_σ is the orthogonal projection from H^β onto V^β. Define A^θ by

$$A^\theta u = \sum_{k\in\mathcal{J}} |k|^{2\theta}\hat{u}_k\phi_k.$$

Note that A^θ maps H^β onto $H^{\beta-2\theta}$. Furthermore, since A^θ preserves the divergence free condition $k\cdot\hat{u}_k = 0$, then $P_\sigma A^\theta = A^\theta P_\sigma$. Therefore A^θ maps V^β to $V^{\beta-2\theta}$. Finally, let us remark that $A^1 u = Au = -\Delta u$ for every $u\in V^\beta$ with $\beta\ge 2$.

Define the bilinear operators B and $\tilde{B}$ from $V\times V$ to V^{-1} as the continuous extensions of the operators given by

$$B(u,v) = P_\sigma\big\{(u\cdot\nabla)v\big\} \quad\text{and}\quad \tilde{B}(u,v) = P_\sigma\big\{u\times(\nabla\times v)\big\}$$

for suitably smooth functions u and v. Details of the construction of B may be found, for example, in [16], [18], [50] or [56]. The construction of $\tilde{B}$ may be found in [24]. We now recall some algebraic properties and bounds that also appear in these references.

Lemma 2.2. *For every $u,v,w\in\mathcal{V}$ the operators B and $\tilde{B}$ satisfy*

$$\big(B(u,v),w\big) = -\big(B(u,w),v\big) \quad\textit{and}\quad \big(\tilde{B}(u,v),w\big) = -\big(\tilde{B}(w,v),u\big). \tag{2.1}$$

Moreover,

$$\big(\tilde{B}(u,v),w\big) = \big(B(u,v),w\big) - \big(B(w,v),u\big). \tag{2.2}$$

Note that the algebraic identities in Lemma 2.2 may be extended to larger spaces by the density of $\mathcal{V}$ in V^β for the appropriate values of β each time that the corresponding trilinear forms are continuous.

Lemma 2.3. *Let X be either B or $\tilde{B}$.*

(i) *Let $u\in V^2$, $v\in V$ and $w\in H$. Then $X: V^2\times V\to H$ and*

$$\big|\big(X(u,v),w\big)\big| \le C\|u\|^{1/2}\|u\|_2^{1/2}\|v\||w|. \tag{2.3}$$

(ii) *Let $u\in H$, $v\in V$ and $w\in V^2$. Then $X: H\times V\to V^{-2}$ and*

$$\big|\langle X(u,v),w\rangle_2\big| \le C|u|\|v\|\|w\|^{1/2}\|w\|_2^{1/2}. \tag{2.4}$$

(iii) *Let $u\in H$, $v\in V^3$ and $w\in H$. Then $X: H\times V^3\to H$ and*

$$\big|\big(X(u,v),w\big)\big| \le C|u|\|v\|_2^{1/2}\|v\|_3^{1/2}|w|. \tag{2.5}$$

We conclude this section with the interpolation lemma of Lions–Magenes [42] followed by the Brézis–Gallouet inequality [7], see also Majda and Bertozzi [43].

Lemma 2.4. [Lions–Magenes] *Let* $\varepsilon > 0$ *and suppose that*

$$u \in L^2\big((0,T); H^\varepsilon\big) \qquad \text{and} \qquad du/dt \in L^2\big((0,T); H^{-\varepsilon}\big).$$

(i) *Then* $u \in C\big([0,T], H\big)$ *with*

$$\sup_{t\in[0,T]} \big|u(t)\big| \le C\big\{\|u\|_{L^2((0,T);H^\varepsilon)} + \|du/dt\|_{L^2((0,T);H^{-\varepsilon})}\big\}$$

(ii) *and*

$$\frac{d}{dt}\big|u\big|^2 = 2\Big\langle \frac{du}{dt}, u\Big\rangle_\varepsilon.$$

Theorem 2.5. [Brézis–Gallouet] *Let* $\beta > 3/2$. *Then there exists a constant* $c(\beta)$ *such that*

$$\|u\|_{L^\infty} \le c(\beta)\|u\|_{3/2}\Big\{1 + \log \frac{\alpha_0^{\beta-3/2}\|u\|_\beta}{\|u\|_{3/2}}\Big\}$$

for all $u \in H^\beta \setminus \{0\}$.

We remark that $c(\beta) \to \infty$ as $\beta \to (3/2)+$.

3. Initial Estimates

In this section we state the Galerkin truncation of equations (1.1) and make an initial *a-priori* estimate on the solution.

Let us first cast these equations into a form that will be convenient for mathematical analysis. Our formulation is analogous to the formulation provided in [24] for the LANS-α model and closely related to the functional formulation of the Navier–Stokes equations which may be found, for example, in Constantin and Foias [16], Doering and Gibbon [18], Lions [40], Robinson [50], Sell and You [51], Temam [56] and references therein.

We remove the pressure from further consideration by projecting (1.1) by P_σ and searching for solutions in the space V^β of divergence free functions. Thus, we obtain the infinite dimensional evolution equation

$$\begin{cases} \dfrac{dv}{dt} + \nu A^{\theta_2} v - \tilde{B}(u,v) = f, \\ v = u + \alpha^{2\theta_1} A^{\theta_1} u \end{cases} \tag{3.1}$$

with initial condition $u(0) = u_0$. Note that the pressure may be reconstructed from u and v (up to a constant), if necessary, by solving the elliptic equation $\Delta p = \nabla \cdot \big\{u \times (\nabla \times v)\big\}$ subject to periodic boundary conditions.

We now proceed to define a strong solution to (3.1). Since the relationship between u and v is straightforward, we shall only mention v when we speak of a strong solution. Note that, given $v = u + \alpha^{2\theta_1} A^{\theta_1} u$, the Poincaré inequality immediately leads to

$$\alpha^{2\theta_1}\big\|u\big\|_{2\theta_1} \le \big|v\big| \le \Big\{\alpha_0^{2\theta_1} + \alpha^{2\theta_1}\Big\}\big\|u\big\|_{2\theta_1}. \tag{3.2}$$

Definition 3.1. The definition of a strong solution depends on θ_1 and θ_2. There are two cases:

(i) For $\theta_2 > 1/2$ and $\nu > 0$ we require $v_0 \in H$, $f \in L^2_{\rm loc}\big([0,\infty), V^{-\theta_2}\big)$,

$$v \in L^2_{\rm loc}\big([0,\infty); V^{\theta_2}\big) \qquad \text{and} \qquad \frac{dv}{dt} \in L^2_{\rm loc}\big([0,\infty); V^{-\theta_2}\big).$$

and view (3.1) as an equality in $L^2_{\rm loc}\big([0,\infty); V^{-\theta_2}\big)$. Equivalently,

$$\big\langle dv/dt + \nu A^{\theta_2} v - \tilde{B}(u,v) - f, w\big\rangle_{\theta_2} = 0$$

for every $w \in V^{\theta_2}$ and almost every $t \in (0,\infty)$.

(ii) In the case that either $0 \le \theta_2 \le 1/2$ and $\nu > 0$ or $\theta_2 = 0$ and $\nu \ge 0$, we require that $v_0 \in V$, $f \in L^2_{\rm loc}\big([0,\infty), V^{1-\theta_2}\big)$,

$$v \in L^2_{\rm loc}\big([0,\infty); V^{\theta_2+1}\big) \qquad \text{and} \qquad \frac{dv}{dt} \in L^2_{\rm loc}\big([0,\infty); V^{\theta_2}\big)$$

and view (3.1) as an equality in $L^2_{\rm loc}\big([0,\infty); V^{\theta_2}\big)$. Equivalently,

$$\big\langle \psi, dv/dt + \nu A^{\theta_2} v - \tilde{B}(u,v) - f\big\rangle_{\theta_2} = 0$$

for every $\psi \in V^{-\theta_2}$ and almost every $t \in (0,\infty)$.

We shall show that (3.1) has unique strong solutions in the sense of Definition 3.1 provided $2\theta_1 + 4\theta_2 \ge 5$.

Let H_m be the span of $\{\, \phi_k : |k| \le m \,\}$. Define the orthogonal projector $P_m\colon H^\beta \to H_m$ by

$$P_m u = \sum_{|k|\le m} \hat{u}_k \phi_k. \tag{3.3}$$

The Galerkin approximation of (3.1) is given by

$$\begin{cases} \dfrac{d}{dt} v^m + \nu A^{\theta_2} v^m + P_m \tilde{B}(u^m, v^m) = P_m f \\ v^m = (1 + \alpha^{2\theta_1} A^{\theta_1}) u^m \end{cases} \tag{3.4}$$

with initial condition $u^m(0) = P_m u_0$.

The classical theory of ordinary differential equations implies that (3.4) has a unique solution for a short interval of time. The global existence is guaranteed when we show that the solution of the Galerkin system remains finite for all time. Next, we obtain bounds on u^m that are valid for all values of θ_1 and θ_2.

Lemma 3.2. *Let $\nu > 0$, $f \in L^2_{\rm loc}\big([0,\infty); V^{-\theta_1-\theta_2}\big)$ and $u_0 \in V^{\theta_1}$. Then there exists a non-decreasing function $K_1(t)$ independent of m such that the solution u^m to the Galerkin truncation* (3.4) *satisfies*

$$\big\|u^m(t)\big\|^2_{\theta_1} + \nu \int_0^t \big\|u^m(s)\big\|^2_{\theta_1+\theta_2} ds \le K_1(t) \qquad \textit{for all} \qquad t \ge 0. \tag{3.5}$$

Proof: After taking the inner product of (3.4) with u^m and using (2.1) which implies $\big(P_m\tilde{B}(u^m,v^m),u^m\big)=0$, we have that

$$\frac{1}{2}\frac{d}{dt}\Big\{|u^m|^2+\alpha^{2\theta_1}\big\|u^m\big\|_{\theta_1}^2\Big\}+\nu\Big\{\big\|u^m\big\|_{\theta_2}^2+\alpha^{2\theta_1}\big\|u^m\big\|_{\theta_1+\theta_2}^2\Big\}=(f,u^m). \tag{3.6}$$

The right hand side of equation (3.6) may be estimated using the Cauchy–Schwarz and Young inequalities as

$$\big|(f,u^m)\big|\le\big\|f\big\|_{-\theta_1-\theta_2}\big\|u^m\big\|_{\theta_1+\theta_2}\le\frac{1}{2\nu\alpha^{2\theta_1}}\big\|f\big\|_{-\theta_1-\theta_2}^2+\frac{\nu\alpha^{2\theta_1}}{2}\big\|u^m\big\|_{\theta_1+\theta_2}^2.$$

Substituting this estimate back into (3.6) yields

$$\frac{d}{dt}\Big\{|u^m|^2+\alpha^{2\theta_1}\big\|u^m\big\|_{\theta_1}^2\Big\}+\nu\Big\{\big\|u^m\big\|_{\theta_2}^2+\alpha^{2\theta_1}\big\|u^m\big\|_{\theta_1+\theta_2}^2\Big\}\le\frac{1}{\nu\alpha^{2\theta_1}}\big\|f\big\|_{-\theta_1-\theta_2}^2.$$

Now, integrating and dropping terms obtains the desired inequality (3.5) where

$$K_1(t)=\frac{1}{\alpha^{2\theta_1}}|u_0|^2+\big\|u_0\big\|_{\theta_1}^2+\frac{1}{\nu\alpha^{4\theta_1}}\int_0^t\big\|f(s)\big\|_{-\theta_1-\theta_2}^2ds.$$

This finishes the proof of the lemma. □

Lemma 3.3. *Let $\nu\ge0$, $f\in L^1_{\rm loc}\big([0,\infty);V^{-\theta_1}\big)$ and $u_0\in V^{\theta_1}$. Then there exists a non-decreasing function $K_2(t)$ independent of m such that the solution u^m to the Galerkin truncation* (3.4) *satisfies*

$$\big\|u^m(t)\big\|_{\theta_1}\le K_2(t)\qquad\textit{for all}\qquad t\ge0. \tag{3.7}$$

Proof: Upon dropping the viscous term, equation (3.6) becomes

$$\frac{1}{2}\frac{d}{dt}\Big\{|u^m|^2+\alpha^{2\theta_1}\big\|u^m\big\|_{\theta_1}^2\Big\}\le\big|(f,u^m)\big|. \tag{3.8}$$

The right side of (3.8) may be estimated using the Cauchy–Schwarz and Young inequalities. Thus,

$$\big|(f,u^m)\big|\le\|f\|_{-\theta_1}\|u^m\|_{\theta_1}\le\frac{1}{\alpha^{\theta_1}}\|f\|_{-\theta_1}\Big\{|u^m|^2+\alpha^{2\theta_1}\|u^m\|_{\theta_1}^2\Big\}^{1/2}$$

Substituting this estimate back into (3.8) and setting

$$y_m^2=|u^m|^2+\alpha^{2\theta_1}\|u^m\|_{\theta_1}^2$$

we obtain that

$$\frac{dy_m}{dt}\le\frac{1}{\alpha^{\theta_1}}\|f\|_{-\theta_1}.$$

Now, integrating and dropping terms obtains the desired inequality (3.7) where

$$K_2(t) = \frac{1}{\alpha^{\theta_1}}\Big\{|u_0|^2 + \alpha^{2\theta_1}\|u_0\|_{\theta_1}^2\Big\}^{1/2} + \frac{1}{\alpha^{2\theta_1}}\int_0^t \big\|f(s)\big\|_{-\theta_1} ds.$$

This finishes the proof of the lemma. □

4. The Viscous Case where $\theta_2 > 1/2$

In this section we obtain bounds for the solution of the Galerkin truncation (3.4) and bounds on its time derivative in the viscous case when $\theta_2 > 1/2$ and $\nu > 0$. We prove existence and uniqueness of strong solutions. We end the section by also showing a certain Gevrey class regularity. Note that this Gevrey class regularity is equivalent to analyticity with respect to the spatial variable. Moreover, let us remark that following the tools developed by Foias and Temam [25] (see, for example, Contantin and Foias [16]) one can show analyticity in time with values in the Gevrey class. The latter was done by Foias and Temam in [26]. We begin with some estimates on the non-linear term.

Lemma 4.1. *Suppose $u \in V^{2\theta_1+\theta_2}$ and $v \in V^{\theta_2}$. If $2\theta_1 + 4\theta_2 \geq 5$, then the map $V^{2\theta_1+\theta_2} \times V^{\theta_2} \to \mathbf{R}$ given by $\big\langle \tilde{B}(u,v), v\big\rangle_{\theta_2}$ is well defined and continuous. Moreover,*

$$\big|\big\langle \tilde{B}(u,v), v\big\rangle_{\theta_2}\big| \leq c_6 \big\|u\big\|_{2\theta_1+\theta_2}\big\|v\big\|_{\theta_2}\big|v\big|. \tag{4.1}$$

Proof: We first show this inequality for trigonometric polynomials and then extend to the indicated spaces using the density of the trigonometric polynomials in the appropriate spaces. Let $u, v, w \in \mathcal{V}$. From (2.1) and (2.2) it follows that

$$\big(\tilde{B}(u,v), v\big) = \big(B(u,v), v\big) - \big(B(v,v), u\big) = \big(B(v,u), v\big).$$

Hölder's inequality followed by the Sobolev embedding theorem yields

$$\big|\big(B(v,u), v\big)\big| \leq C\big\|v\big\|_{\beta_1}\big\|u\big\|_{\beta_2+1}\big|v\big|$$

where C is a combination of the dimensionless constants from the Sobolev embedding theorem depending only on β_1 and β_2, $\beta_1 + \beta_2 = 3/2$ and $\beta_1, \beta_2 > 0$. Choose $0 < \beta_1 \leq \theta_2$ and $\beta_2 + 1 \leq 2\theta_1 + \theta_2$. Since $2\theta_1 + \theta_2 - 1 \geq 1/4$ we may choose $\beta_2 > 0$. Moreover, $2\theta_1 + 2\theta_2 - 1 \geq 3/2$ implies there are choices of β_1 and β_2 such that $\beta_1 + \beta_2 = 3/2$. Upon applying Poincaré's inequality and using the density of $\mathcal{V}$ in the appropriate spaces, we obtain (4.1) where where $c_6 = C\alpha_0^{2\theta_1+2\theta_2-5/2}$. □

Lemma 4.2. *Let $u \in V^{2\theta_1+\theta_2}$ and $v, w \in V^{\theta_2}$. Provided $2\theta_1 + 4\theta_2 \geq 5$ and $\theta_2 > 1/2$ then $X: V^{2\theta_1+\theta_2} \times V^{\theta_2} \to V^{-\theta_2}$ and*

$$|\langle X(u,v), w\rangle_{\theta_2}| \leq c_0\big\{\|u\|_{2\theta_1+\theta_2}|w| + \|u\|_{2\theta_1}\|w\|_{\theta_2}\big\}\|v\|_{\theta_2} \tag{4.2}$$

where X can be either B or $\tilde{B}$.

Proof: We first show this inequality for trigonometric polynomials and then extend to the indicated spaces using density. Let $u, v, w \in \mathcal{V}$. Define u so that $\hat{\mathrm{u}}_k = |\hat{u}_k|$. Note that $\|\mathrm{u}\|_\beta = \|u\|_\beta$. Similarly define v and w. Estimate

$$\begin{aligned}
\big|\big(X(u,v),w\big)\big| &\le L^3 \sum_{h+k+l=0} |\hat{u}_h||k||\hat{v}_k||\hat{w}_l| = L^3 \sum_{h+k+l=0} \hat{\mathrm{u}}_h |k| \hat{\mathrm{v}}_k \hat{\mathrm{w}}_l \\
&\le L^3 2^{1/2} \sum_{h+k+l=0} \big\{ |h|^{1/2} \hat{\mathrm{u}}_h |k|^{1/2} \hat{\mathrm{v}}_k \hat{\mathrm{w}}_l + \hat{\mathrm{u}}_h |k|^{1/2} \hat{\mathrm{v}}_k |l|^{1/2} \hat{\mathrm{w}}_l \big\} \\
&= 2^{1/2} \int_\Omega \big\{ (A^{1/4}\mathrm{u})(A^{1/4}\mathrm{v})\mathrm{w} + \mathrm{u}(A^{1/4}\mathrm{v})(A^{1/4}\mathrm{w}) \big\} \\
&\le C\big\{ \|\mathrm{u}\|_{\beta_1+1/2} \|\mathrm{v}\|_{\beta_2+1/2} \|\mathrm{w}\|_{\beta_3} + \|\mathrm{u}\|_{\beta_1'} \|\mathrm{v}\|_{\beta_2'+1/2} \|\mathrm{w}\|_{\beta_3'+1/2} \big\} \\
&= C\big\{ \|u\|_{\beta_1+1/2} \|v\|_{\beta_2+1/2} \|w\|_{\beta_3} + \|u\|_{\beta_1'} \|v\|_{\beta_2'+1/2} \|w\|_{\beta_3'+1/2} \big\}
\end{aligned}$$

where C made up of the non-dimensional constants obtained by applying Theorem 2.1 six times and depends on the parameters β_i and β_i' for $i = 1, 2, 3$, and

$$\textstyle\sum_i \beta_i = \sum_i \beta_i' = 3/2 \qquad \text{and} \qquad 0 \le \beta_i, \beta_i' < 3/2. \tag{4.3}$$

Further choose

$$\begin{aligned}
\beta_1 + 1/2 &\le 2\theta_1 + \theta_2, & \beta_1' &\le 2\theta_1, \\
\beta_2 + 1/2 &\le \theta_2, & \beta_2' + 1/2 &\le \theta_2, \\
\beta_3 &= 0, & \beta_3' + 1/2 &\le \theta_2.
\end{aligned}$$

Since $\theta_2 > 1/2$ and $2\theta_1 + 2\theta_2 - 1 \ge 3/2$ then such a choice can be made while still satisfying (4.3). Setting $c_0 = C\alpha_0^{2\theta_1+2\theta_2-5/2}$ and applying the Poincaré inequality obtains

$$\big|\big(X(u,v),w\big)\big| \le c_0 \big\{ \|u\|_{2\theta_1+\theta_2} |w| + \|u\|_{2\theta_1} \|w\|_{\theta_2} \big\} \|v\|_{\theta_2}$$

for $u, v, w \in \mathcal{V}$. The density of $\mathcal{V}$ in $V^{2\theta_1+\theta_2}$ and V^{θ_2} now yields (4.2). $\square$

Lemma 4.3. *Let $u \in V^{2\theta_1+\theta_2}$ and $v, w \in V^{\theta_2}$. If $2\theta_1 + 4\theta_2 \ge 5$ and $\theta_2 > 1/2$, then $\tilde{B}\colon V^{2\theta_1+\theta_2} \times V^{\theta_2} \to V^{-\theta_2}$ and*

$$|\langle \tilde{B}(u,v), w\rangle_{\theta_2}| \le c_1 \big\{ \|u\|_{2\theta_1+\theta_2} |v| + \|u\|_{2\theta_1} \|v\|_{\theta_2} \big\} \|w\|_{\theta_2}. \tag{4.4}$$

Proof: We first show this inequality for trigonometric polynomials and then extend to the indicated spaces using density. Let $u, v, w \in \mathcal{V}$. ¿From (2.2) followed by (2.1) we have

$$\big|\big(\tilde{B}(u,v),w\big)\big| \le \big|\big(B(u,w),v\big)\big| + \big|\big(B(w,u),v\big)\big|.$$

Estimate $\big|\big(B(u,w),v\big)\big|$ by Lemma 4.2 to obtain

$$\big|\big(B(u,w),v\big)\big| \le c_0 \big\{ \|u\|_{2\theta_1+\theta_2} |v| + \|u\|_{2\theta_1} \|v\|_{\theta_2} \big\} \|w\|_{\theta_2}. \tag{4.5}$$

Estimate $\big|\big(B(w,u),v\big)\big|$ using Hölder's inequality followed by the Sobolev embedding theorem to obtain

$$\big|\big(B(w,u),v\big)\big| \le C\|w\|_{\beta_1} \|u\|_{\beta_2+1} |v|$$

where C is a combination of the dimensionless constants from Theorem 2.1 depending only on β_1 and β_2, $\beta_1+\beta_2=3/2$ and $\beta_1,\beta_2>0$. Choose $\beta_1\le\theta_2$ and $\beta_2+1\le 2\theta_1+\theta_2$. Since $2\theta_1+\theta_2-1\ge 1/4$ we may take $\beta_2>0$. Moreover, $2\theta_1+2\theta_2\ge 3/2$ ensure that we can satisfy $\beta_1+\beta_2=3/2$. Applying the Poincaré inequality gives

$$\big|\big(B(w,u),v\big)\big|\le C\alpha_0^{2\theta_1+2\theta_2-5/2}\|w\|_{\theta_2}\|u\|_{2\theta_1+\theta_2}|v|. \tag{4.6}$$

Combining (4.5) with (4.6) and setting $c_1=c_0+C\alpha_0^{2\theta_1+2\theta_2-5/2}$ we obtain

$$\big|\big(\tilde B(u,v),w\big)\big|\le c_1\big\{\|u\|_{2\theta_1+\theta_2}|v|+\|u\|_{2\theta_1}\|v\|_{\theta_2}\big\}\|w\|_{\theta_2}$$

for every $u,v,w\in\mathcal{V}$. The density of $\mathcal{V}$ in $V^{2\theta_1+\theta_2}$ and V^{θ_2} now yields (4.4). □

Lemma 4.4. *Let $0\le\varepsilon<\theta_2-1/2$ and $4\theta_1+6\theta_2\ge 5+4\varepsilon$. Let $u\in V^{2\theta_1+\theta_2-\varepsilon}$, $v\in V^{\theta_2-\varepsilon}$ and $w\in V^{\theta_2}$. Then $X\colon V^{2\theta_1+\theta_2-\varepsilon}\times V^{\theta_2-\varepsilon}\to V^{-\theta_2}$ and*

$$\big|\big\langle X(u,v),w\big\rangle_{\theta_2}\big|\le c_2\|u\|_{2\theta_1+\theta_2-\varepsilon}\|v\|_{\theta_2-\varepsilon}\|w\|_{\theta_2} \tag{4.7}$$

where X can be either B or $\tilde B$.

Proof: Let $u,v,w\in\mathcal{V}$. Estimate as in Lemma 4.2 to obtain

$$\big|\big(X(u,v),w\big)\big|\le C\big\{\|u\|_{\beta_1+1/2}\|v\|_{\beta_2+1/2}\|w\|_{\beta_3}+\|u\|_{\beta_1'}\|v\|_{\beta_2'+1/2}\|w\|_{\beta_3'+1/2}\big\}$$

where C is a dimensionless constant and β_i,β_i' satisfies (4.3). Choose

$$\begin{aligned}\beta_1+1/2&\le 2\theta_1+\theta_2-\varepsilon, & \beta_1'&\le 2\theta_1+\theta_2-\varepsilon,\\ \beta_2+1/2&\le\theta_2-\varepsilon, & \beta_2'+1/2&\le\theta_2-\varepsilon,\\ \beta_3&\le\theta_2, & \beta_3'+1/2&\le\theta_2.\end{aligned}$$

Since $\theta_2-\varepsilon>1/2$ then $\beta_i,\beta_i'>0$. Moreover, $2\theta_1+3\theta_2-2\varepsilon-1\ge 3/2$ implies all the conditions of (4.3) can be satisfied. Therefore, after applying the Poincaré inequality and using the density of $\mathcal{V}$ in the appropriate spaces we obtain (4.7) where $c_2=C\alpha_0^{2\theta_1+3\theta_2-2\varepsilon-5/2}$. □

Lemma 4.5. *Let $f\in L^2_{\mathrm{loc}}\big([0,\infty);V^{-\theta_2}\big)$ and $v_0\in H$. Suppose $\nu>0$, $2\theta_1+4\theta_2\ge 5$ and $\theta_2>0$. Then there exists a non-decreasing function $K_3(t)$ independent of m such that the solution v^m to the Galerkin truncation (3.4) satisfies*

$$\big|v^m(t)\big|^2+\nu\int_0^t\big\|v^m(s)\big\|_{\theta_2}^2\,ds\le K_3(t)\qquad\text{for all}\qquad t\ge 0. \tag{4.8}$$

Moreover, if $2\theta_1+4\theta_2>5$ and $2\theta_1>1$ then we may take $u_0\in V^{\theta_1}$.

Proof: Take inner product of (3.4) with v^m to obtain

$$\frac12\frac{d}{dt}|v^m|^2+\nu\|v^m\|_{\theta_2}^2\le\big(\tilde B(u^m,v^m),v^m\big)+(f,v^m).$$

Use (2.2), (2.1) followed by Hölder's inequality and the Sobolev embedding theorem to estimate

$$\begin{aligned}\big|\big(\tilde{B}(u^m,v^m),v^m\big)\big| &= \big|\big(B(v^m,v^m),u^m\big)\big| = \big|\big(B(v^m,u^m),v^m\big)\big| \\ &\le C\|v^m\|_{\beta_1}\|u^m\|_{\beta_2+1}|v^m|\end{aligned}$$

where C is a dimensionless constant, $\beta_1+\beta_2=3/2$ and $\beta_1,\beta_2>0$.

To absorb $|v^m|_{\beta_1}$ into the viscosity we must have $\beta_1\le\theta_2$; to control the time integral of $\|u^m\|_{\beta_2+1}$ by Lemma 3.2 we need $\beta_2\le\theta_1+\theta_2-1$. Since $\theta_2>0$, we may take $\beta_1>0$. Since $\theta_1+\theta_2\ge 5/4$, we may take $\beta_2>0$. Moreover, since $\theta_1+2\theta_2-1\ge 3/2$, then β_1 and β_2 may be chosen so that $\beta_1+\beta_2=3/2$.

The Poincaré inequality followed by Young's inequality yields

$$\begin{aligned}\big|\big(\tilde{B}(u^m,v^m),v^m\big)\big| &\le c_1\|v^m\|_{\theta_2}\|u^m\|_{\theta_1+\theta_2}|v^m| \\ &\le \frac{\nu}{4}\|v^m\|_{\theta_2}^2+\frac{c_1^2}{\nu}\|u^m\|_{\theta_1+\theta_2}^2|v^m|^2\end{aligned}$$

where $c_0=C^2\alpha_0^{\theta_1+2\theta_2-5/2}$.

Estimate

$$\big|(f,v^m)\big|\le\frac{1}{\nu}\|f\|_{-\theta_2}^2+\frac{\nu}{4}\|v^m\|_{\theta_2}^2$$

It follows that

$$\frac{d}{dt}|v^m|^2+\nu\|v^m\|_{\theta_2}^2\le\frac{2c_0^2}{\nu}\|u^m\|_{\theta_1+\theta_2}^2|v^m|+\frac{2}{\nu}\|f\|_{-\theta_2}^2. \tag{4.9}$$

Dropping the viscous term of the left, multiplying by the integrating factor

$$\psi_m(t)=\exp\bigg\{-\frac{2c_0^2}{\nu}\int_0^t\|u^m\|_{\theta_1+\theta_2}^2\bigg\}$$

and integrating yields

$$|v^m(t)|^2\le\big\{\psi_m(t)\big\}^{-1}\bigg\{|v^m(0)|^2+\frac{2}{\nu}\int_0^t\|f(s)\|_{-\theta_2}^2\,ds\bigg\}.$$

The magnitude of $\{\psi_m(t)\}^{-1}$ may be estimated using Lemma 3.2 to obtain that $|v^m(t)|^2\le K_3'(t)$ where

$$K_3'(t)=\exp\bigg\{\frac{2c_0^2K_1(t)}{\nu^2}\bigg\}\bigg\{|v^m(0)|^2+\frac{2}{\nu}\int_0^t\|f(s)\|_{-\theta_2}^2\,ds\bigg\}.$$

To finish the proof integrate (4.9), this time without dropping the viscous term, to obtain (4.8) where

$$K_3(t)=|v^m(0)|^2+\frac{2c_0^2}{\nu^2}K_1(t)K_3'(t)+\frac{2}{\nu}\int_0^t\|f(s)\|_{-\theta_2}^2\,ds.$$

This finishes the proof. □

Our next result provides an estimate on the time derivative of v^m.

Lemma 4.6. *Let $f \in L^2_{\rm loc}\big([0,\infty);V^{-\theta_2}\big)$ and $v_0 \in H$. Suppose that $\nu > 0$, $2\theta_1 + 4\theta_2 \geq 5$ and $\theta_2 > 1/2$. Then there exists a non-decreasing function $K_4(t)$ independent of m such that the time derivative of the solution v^m to the Galerkin truncation* (3.4) *satisfies*

$$\int_0^t \Big\| \frac{dv^m}{dt} \Big\|_{-\theta_2}^2 \leq K_4(t) \qquad \textit{for all} \qquad t \geq 0. \tag{4.10}$$

Proof: Applying the $-\theta_2$ norm to (3.4) yields

$$\Big\| \frac{dv^m}{dt} \Big\|_{-\theta_2} \leq \nu \big\| v^m \big\|_{\theta_2} + \big\| \tilde{B}(u^m, v^m) \big\|_{-\theta_2} + \big\| f \big\|_{-\theta_2}.$$

Note that

$$\big\| \tilde{B}(u^m, v^m) \big\|_{-\theta_2} = \sup \Big\{ \big(\tilde{B}(u^m, v^m), w \big) : \| w \|_{\theta_2} \leq 1 \Big\}.$$

By Lemma 4.3 we have that

$$\begin{aligned} \big| \big(\tilde{B}(u^m, v^m), w \big) \big| &\leq c_1 \big\{ \| u^m \|_{2\theta_1 + \theta_2} |v^m| + \| u^m \|_{2\theta_1} \| v^m \|_{\theta_2} \big\} \| w \|_{\theta_2} \\ &\leq c_3 |v^m| \| v^m \|_{\theta_2} \end{aligned}$$

where $c_3 = 2c_1 \alpha^{-2\theta_1}$. It follows from Lemma 4.5 that

$$\begin{aligned} \Big\| \frac{dv^m}{dt} \Big\|_{-\theta_2}^2 &\leq 9 \Big\{ \nu^2 + c_3^2 \big| v^m \big|^2 \Big\} \big\| v^m \big\|_{\theta_2}^2 + 9 \big\| f \big\|_{-\theta_2}^2 \\ &\leq 9 \big\{ \nu^2 + c_3^2 K_3(t) \big\} \big\| v^m \big\|_{\theta_2}^2 + 9 \big\| f \big\|_{-\theta_2}^2. \end{aligned}$$

Integrate to obtain

$$\int_0^t \Big\| \frac{dv^m}{dt} \Big\|_{-\theta_2}^2 \leq 9 \Big\{ \nu^2 + c_3^2 K_3(t) \Big\} \int_0^t \big\| v^m \big\|_{\theta_2}^2 + 9 \int_0^t \| f \|_{-\theta_2}^2.$$

Applying Lemma 4.5 and setting

$$K_4(t) = \frac{9}{\nu} \Big\{ \nu^2 + c_3^2 K_3(t) \Big\} K_3(t) + 9 \int_0^t \| f \|_{-\theta_2}^2.$$

yields (4.10) and finishes the proof. □

We are now ready to take limits along subsequences of v^m to show existence of solutions which satisfy (3.1) viewed as a functional equality in $V^{-\theta_2}$.

Theorem 4.7. *Let $f \in L^2_{\rm loc}\big([0,\infty);V^{-\theta_2}\big)$ and $v_0 \in H$. Suppose that $\nu > 0$, $2\theta_1 + 4\theta_2 \geq 5$ and $\theta_2 > 1/2$. Then for any $T > 0$ equations* (3.1) *viewed as a functional equality in $V^{-\theta_2}$ have a solution that satisfies*

$$v \in L^\infty\big((0,T);H\big) \cap L^2\big((0,T);V^{\theta_2}\big) \qquad \textit{and} \qquad \frac{dv}{dt} \in L^2\big((0,T);V^{-\theta_2}\big).$$

Moreover, v is in $C\big([0,T];H\big)$ and $v(0)=v_0$.

Proof: Lemma 4.5 implies that v^m is uniformly bounded in $L^\infty\big((0,T);H\big)$ and $L^2\big((0,T);V^{\theta_2}\big)$. We may use the Alaoglu compactness theorem to extract a subsequence of v^m that converges weakly to some v in $L^2\big((0,T);V^{\theta_2}\big)$. By Lemma 4.6 we also have a uniform bound on dv^m/dt in $L^2\big((0,T);V^{-\theta_2}\big)$. Therefore, by extracting a subsequence of this subsequence we find a further subsequence of dv^m/dt that also converges weakly to some point in $L^2\big((0,T);V^{-\theta_2}\big)$. Since $C_0^\infty\big((0,T);\mathcal{V}\big)$ is dense in $L^2\big((0,T);V^{\theta_2}\big)$, then integration by parts shows that this subsequence, in fact, converges to dv/dt. Now, Lemma 2.4 implies that this subsequence of v^m converges to v in $C([0,T];H)$ and in particular that $v(0)=v_0$.

Fix ε sufficiently small such that $0<\varepsilon<\theta_2-1/2$ and $4\theta_1+6\theta_2\geq 5+4\varepsilon$. Since $V^{\theta_2-\varepsilon}$ is compactly embedded in V^{θ_2}, then Ehrling's Lemma, see for example [50], implies there exists yet a further subsequence of v^m that converges strongly in $L^2\big((0,T);V^{\theta_2-\varepsilon}\big)$. Denoting this last subsequence by v^{m_i} we obtain

$$\int_0^T \big\langle \psi(t), v^{m_i}(t)\big\rangle_{\theta_2}\,dt \to \int_0^T \big\langle \psi(t), v(t)\big\rangle_{\theta_2}\,dt \quad \text{as} \quad i\to\infty \tag{4.11}$$

for every $\psi\in L^2\big((0,T);V^{-\theta_2}\big)$,

$$\int_0^T \big\langle dv^{m_i}(t)/dt, w(t)\big\rangle_{\theta_2}\,dt \to \int_0^T \big\langle dv(t)/dt, w(t)\big\rangle_{\theta_2}\,dt \quad \text{as} \quad i\to\infty \tag{4.12}$$

for every $w\in L^2\big((0,T);V^{\theta_2}\big)$ and

$$\int_0^T \big\|v^{m_i}(t)-v(t)\big\|_{\theta_2-\varepsilon}^2 dt\to 0 \quad \text{as} \quad i\to\infty. \tag{4.13}$$

It remains to show v is a solution to (3.1) viewed as a functional equality in $V^{-\theta_2}$. To do this it is enough to show that along the subsequence v^{m_i} the equations (3.4) converge weakly in $L^2\big((0,T);V^{-\theta_2}\big)$ to (3.1)

By our choice of subsequences we have already ensured that dv^{m_i}/dt converges weakly to dv/dt in $L^2\big((0,T);V^{-\theta_2}\big)$.

To show $A^{\theta_2}v^{m_i}$ converges weakly to $A^{\theta_2}v$ in $L^2\big((0,T);V^{-\theta_2}\big)$ note that

$$\int_0^T \big\langle A^{\theta_2}v^{m_i}(t), w(t)\big\rangle_{\theta_2}\,dt = \int_0^T \big\langle A^{\theta_2}w(t), v^{m_i}(t)\big\rangle_{\theta_2}\,dt \tag{4.14}$$

for $w\in L^2\big((0,T);V^{\theta_2}\big)$. Now, since $A^{\theta_2}w\in L^2\big((0,T);V^{-\theta_2}\big)$, then (4.11) implies that (4.14) converges to

$$\int_0^T \big\langle A^{\theta_2}w(t), v(t)\big\rangle_{\theta_2}\,dt = \int_0^T \big\langle A^{\theta_2}v(t), w(t)\big\rangle_{\theta_2}\,dt$$

as $i\to\infty$. Thus, $A^{\theta_2}v^{m_i}$ converges weakly to $A^{\theta_2}v$ in $L^2\big((0,T);V^{-\theta_2}\big)$.

We finish by showing that the non-linear term $\tilde{B}(u^{m_i}, v^{m_i})$ converges weakly to $\tilde{B}(u, v)$ in $L^2\big((0,T); V^{-\theta_2}\big)$. By the bilinearity of $\tilde{B}$ we have

$$\tilde{B}(u^{m_i}, v^{m_i}) - \tilde{B}(u, v) = \tilde{B}(u^{m_i}, v^{m_i} - v) + \tilde{B}(u^{m_i} - u, v).$$

Let $w \in L^2\big((0,T); V^{\theta_2}\big)$ and choose $w_j \in C_0^\infty\big((0,T); \mathcal{V}\big)$ such that

$$\int_0^T \|w - w_j\|_{\theta_2}^2 \to 0 \qquad \text{as} \qquad j \to \infty.$$

Let M_j be the bounds such that $\sup\big\{ \|w_j(t)\|_{\theta_2} : t \in (0,T) \big\} \le M_j$. Linearity and Lemma 2.2 implies that

$$\big\langle \tilde{B}(u^{m_i}, v^{m_i} - v), w \big\rangle_{\theta_2} = \big\langle \tilde{B}(u^{m_i}, v^{m_i} - v), w - w_j \big\rangle_{\theta_2} + \big\langle \tilde{B}(u^{m_i}, v^{m_i} - v), w_j \big\rangle_{\theta_2}.$$

By Lemma 4.3 followed by Lemma 4.5 we have

$$\begin{aligned}
&\big|\big\langle \tilde{B}(u^{m_i}, v^{m_i} - v), w - w_j \big\rangle_{\theta_2}\big| \\
&\qquad \le c_1 \big\{ \|u^{m_i}\|_{2\theta_1+\theta_2} |v^{m_i} - v| + \|u^{m_i}\|_{2\theta_1} \|v^{m_i} - v\|_{\theta_2} \big\} \|w - w_j\|_{\theta_2} \\
&\qquad \le c_1 \alpha^{-2\theta_1} \big\{ \|v^{m_i}\|_{\theta_2} |v^{m_i} - v| + |v^{m_i}| \|v^{m_i} - v\|_{\theta_2} \big\} \|w - w_j\|_{\theta_2} \\
&\qquad \le 2 c_1 \alpha^{-2\theta_1} K_3(t)^{1/2} \big\{ \|v^{m_i}\|_{\theta_2} + \|v^{m_i} - v\|_{\theta_2} \big\} \|w - w_j\|_{\theta_2}.
\end{aligned}$$

Applying the Cauchy-Schwartz inequality followed by Lemma 4.5 again we obtain

$$\left| \int_0^T \big\langle \tilde{B}(u^{m_i}, v^{m_i} - v), w - w_j \big\rangle_{\theta_2} \right| \le c_1' K_3(T) \Big\{ \int_0^T \|w - w_j\|_{\theta_2}^2 \Big\}^{1/2} \tag{4.15}$$

where $c_1' = 4 c_1 \alpha^{-2\theta_1} \nu^{-1/2}$.

Estimate, using Lemma 4.4 followed by the Poincaré inequality, to obtain

$$\begin{aligned}
\big|\big\langle \tilde{B}(u^{m_i}, v^{m_i} - v), w_j \big\rangle_{\theta_2}\big| &\le c_2 \|u^{m_i}\|_{2\theta_1+\theta_2-\varepsilon} \|v^{m_i} - v\|_{\theta_2-\varepsilon} \|w_j\|_{\theta_2} \\
&\le c_2 \alpha_0^{\varepsilon} \alpha^{-2\theta_1} \|v^{m_i}\|_{2\theta_1+\theta_2} \|v^{m_i} - v\|_{\theta_2-\varepsilon} \|w_j\|_{\theta_2}.
\end{aligned}$$

The Cauchy-Schwartz inequality followed by Lemma 4.5 yields

$$\left| \int_0^T \big\langle \tilde{B}(u^{m_i}, v^{m_i} - v), w_j \big\rangle_{\theta_2} \right| \le c_2' M_j K_3(T)^{1/2} \Big\{ \int_0^T \|v^{m_i} - v\|_{\theta_2-\varepsilon}^2 \Big\}^{1/2}. \tag{4.16}$$

where $c_2' = c_2 \alpha_0^{\varepsilon} \alpha^{-2\theta_1} \nu^{-1/2}$. Similarly

$$\langle \tilde{B}(u^{m_i} - u, v), w \rangle_{\theta_2} = \langle \tilde{B}(u^{m_i} - u, v), w - w_j \rangle_{\theta_2} + \langle \tilde{B}(u^{m_i} - u, v), w_j \rangle_{\theta_2}.$$

Therefore, by Lemma 4.3, Lemma 4.5 and the Cauchy-Schwartz inequality we obtain

$$\left| \int_0^T \big\langle \tilde{B}(u^{m_i} - u, v), w - w_j \big\rangle_{\theta_2} \right| \le c_2 K_3(T) \Big\{ \int_0^T \|w - w_j\|_{\theta_2}^2 \Big\}^{1/2} \tag{4.17}$$

Estimate, using Lemma 4.4 followed by the Poincaré inequality, to obtain

$$\begin{aligned}\big|\big\langle \tilde B(u^{m_i}-u,v),w_j\big\rangle_{\theta_2}\big| &\le c_2\|u^{m_i}-u\|_{2\theta_1+\theta_2-\varepsilon}\|v\|_{\theta_2-\varepsilon}\|w_j\|_{\theta_2}\\ &\le c_2\alpha_0^{\varepsilon}\alpha^{-2\theta_1}\|v^{m_i}-v\|_{\theta_2-\varepsilon}\|v\|_{\theta_2}\|w_j\|_{\theta_2}\end{aligned}$$

The Cauchy-Schwartz inequality followed by Lemma 4.5 yields

$$\Big|\int_0^T \big\langle \tilde B(u^{m_i}-u,v),w_j\big\rangle_{\theta_2}\Big| \le c_2' M_j K_3(T)^{1/2}\Big\{\int_0^T \|v^{m_i}-v\|^2_{\theta_2-\varepsilon}\Big\}^{1/2}. \tag{4.18}$$

Combining (4.15), (4.16), (4.17) and (4.18) we obtain that

$$\begin{aligned}\int_0^T \big\langle \tilde B(u^{m_i},v^{m_i})-\tilde B(u,v),w\big\rangle_{\theta_2} &\le 2c_1'K_3(T)\Big\{\int_0^T \|w-w_j\|^2_{\theta_2}\Big\}^{1/2}\\ &\quad + 2c_2' M_j K_3(T)^{1/2}\Big\{\int_0^T \|v^{m_i}-v\|^2_{\theta_2-\varepsilon}\Big\}^{1/2}.\end{aligned}$$

First choose j so large that the first term is small, and then choose i so large that the second term is small. Thus

$$\int_0^T \big\langle \tilde B(u^{m_i},v^{m_i})-\tilde B(u,v),w\big\rangle_{\theta_2} \to 0 \qquad \text{as} \qquad i\to\infty.$$

It follows that $\tilde B(u^{m_i},v^{m_i})$ converges to $\tilde B(u,v)$ weakly in $L^2\big((0,T);V^{-\theta_2}\big)$. We have shown that v satisfies (3.1) viewed as a functional equality in $V^{-\theta_2}$. □

We now prove continuous dependence of solutions on initial data and uniqueness.

Theorem 4.8. *Let $v_0\in H$. Suppose that $\nu>0$, $2\theta_1+4\theta_2\ge 5$ and $\theta_2>1/2$. Then there is at most one solution to* (3.1) *that satisfies*

$$v\in L^2_{\rm loc}\big((0,\infty);V^{\theta_2}\big) \qquad \textit{and} \qquad \frac{dv}{dt}\in L^2_{\rm loc}\big((0,\infty);V^{-\theta_2}\big). \tag{4.19}$$

Moreover, this solution depends continuously on the initial data v_0 in the H norm.

Proof: Let v_1 and v_2 be two solutions of (3.1) such that $v_i(0)\in H$ and (4.19) holds for $i=1,2$. Let $\delta u=u_1-u_2$ and $\delta v=v_1-v_2$. Write the evolution equation for δv as an equality in $V^{-\theta_2}$ given by

$$\frac{d}{dt}\delta v+\nu A^{\theta_2}\delta v+\tilde B(\delta u,v_1)+\tilde B(u_2,\delta v)=0. \tag{4.20}$$

Lemma 4.2 implies $\tilde B(\delta u,v_1)$ and $\tilde B(u_2,\delta v)$ are in $V^{-\theta_2}$ for almost every t. Form the $V^{-\theta_2}$ duality pairing of (4.20) with δv to obtain

$$\Big\langle \frac{d}{dt}\delta v,\delta v\Big\rangle_{\theta_2}+\nu\|\delta v\|^2_{\theta_2} \le \big|\big\langle \tilde B(\delta u,v_1),\delta v\big\rangle_{\theta_2}\big| + \big|\big\langle \tilde B(u_2,\delta v),\delta v\big\rangle_{\theta_2}\big|. \tag{4.21}$$

Lemma 2.4 implies that

$$\Big\langle \frac{d}{dt}\delta v, \delta v\Big\rangle_{\theta_2} = \frac{1}{2}\frac{d}{dt}\big|\delta v\big|^2.$$

Therefore (4.21) becomes

$$\frac{1}{2}\frac{d}{dt}\big|\delta v\big|^2 + \nu\|\delta v\|_{\theta_2}^2 \le \big|\big\langle \tilde{B}(\delta u, v_1), \delta v\big\rangle_{\theta_2}\big| + \big|\big\langle \tilde{B}(u_2, \delta v), \delta v\big\rangle_{\theta_2}\big|. \tag{4.22}$$

Estimate $\big\langle \tilde{B}(\delta u, v_1), \delta v\big\rangle_{\theta_2}$ using Lemma 4.2 followed by (3.2) as

$$\begin{aligned} \big|\big\langle \tilde{B}(\delta u, v_1), \delta v\big\rangle_{\theta_2}\big| &\le c_0\big\{\|\delta u\|_{2\theta_1+\theta_2}|\delta v| + \|\delta u\|_{2\theta_1}\|\delta v\|_{\theta_2}\big\}\|v_1\|_{\theta_2} \\ &\le c_2|\delta v|\|\delta v\|_{\theta_2}\|v_1\|_{\theta_2} \end{aligned} \tag{4.23}$$

where $c_2 = 2c_0\alpha^{-2\theta_1}$.

Estimate $\big\langle \tilde{B}(u_2, \delta v), \delta v)\big\rangle_{\theta_2}$ using Lemma 4.1 followed by (3.2) to obtain

$$\big\langle \tilde{B}(u_2, \delta v), \delta v)\big\rangle_{\theta_2} \le c_6\|u_2\|_{2\theta_1+\theta_2}\|\delta v\|_{\theta_2}|\delta v| \le c_7\|v_2\|_{\theta_2}\|\delta v\|_{\theta_2}|\delta v| \tag{4.24}$$

where $c_7 = c_6\alpha^{-2\theta_1}$. Combining (4.24) with (4.23) yields

$$\frac{1}{2}\frac{d}{dt}|\delta v|^2 + \nu\|\delta v\|_{\theta_2}^2 \le \big\{c_7\|v_1\|_{\theta_2} + c_2\|v_2\|_{\theta_2}\big\}|\delta v|\|\delta v\|_{\theta_2}$$

Apply Young's inequality to obtain

$$\frac{d}{dt}\big|\delta v\big|^2 \le \frac{2}{\nu}\Big\{c_7^2\|v_1\|_{\theta_2}^2 + c_2^2\|v_2\|_{\theta_2}^2\Big\}|\delta v|^2.$$

Multiply by the integrating factor

$$\psi(t) = \exp\bigg\{-\int_0^t \frac{2}{\nu}\Big\{c_7^2\big\|v_1(s)\big\|_{\theta_2}^2 + c_2^2\big\|v_2(s)\big\|_{\theta_2}^2\Big\}ds\bigg\}$$

to obtain that $\big|\delta v(t)\big|^2 \le \big\{\psi(t)\big\}^{-1}\big|\delta v(0)\big|^2$. The magnitude of $\big\{\psi(t)\big\}^{-1}$ may be estimated using (4.19). In particular, let $M(t)$ be the bound such that

$$\int_0^t \big\|v_i(s)\big\|_{\theta_2}^2 ds \le M(t)$$

for $i = 1, 2$ and $t > 0$. It follows that

$$\big|\delta v(t)\big|^2 \le \exp\Big\{2\frac{c_7^2 + c_2^2}{\nu}M(t)\Big\}\big|\delta v(0)\big|^2.$$

If $\delta v(0) = 0$ then $\delta v(t) = 0$ for all $t \in [0, T]$. Therefore solutions are unique. Since $T > 0$ is arbitrary this solution may be uniquely extended for all time. Moreover, this solution depends continuously on the initial data v_0 in the H norm. □

We finish by showing that if $2\theta_1 + 4\theta_2 > 5$ and $\theta_2 > 1/2$ then the initial condition v_0 is smoothed. That is, the solution becomes smoother than the initial data, a characteristic of parabolic evolution equations. We use the Gevrey technique of Foias and Temam [26], see also [19,22,48]. Let $\mathcal{E}(t;x)$ be defined for $x,t \geq 0$ as either

$$\mathcal{E}(t;x) = (1+\eta t x)^{\delta} \qquad \text{or} \qquad \mathcal{E}(t;x) = \exp(\eta t x) \tag{4.25}$$

where $\delta, \eta > 0$. Note that

$$\mathcal{E}(t;x+y) \leq M\mathcal{E}(t;x)\mathcal{E}(t;y) \qquad \text{and} \qquad \frac{\partial \mathcal{E}(t;x)}{\partial t} \leq \eta x M \mathcal{E}(t;x) \tag{4.26}$$

where in the first case $M = \delta + 2^{\delta}$ and in the second $M = 1$.

Theorem 4.9. *Fix η such that $8\eta M \alpha_0^{\theta_2 - 1/2} \leq \nu$ and $T > 0$. Let $\mathcal{E}(A^{1/2})f \in L^2\big([0,T]; V^{-\theta_2}\big)$ and $v_0 \in H$. Suppose that $2\theta_1 + 4\theta_2 \geq 5$, $\theta_2 > 1/2$ and $\theta_1 > 0$. Then there exists T^* such that $0 < T^* \leq T$ and a non-decreasing function $K_5(t)$ such that the unique strong solution v to* (3.1) *satisfies*

$$\big|\mathcal{E}(t;A^{1/2})v(t)\big|^2 + \nu \int_0^t \big\|\mathcal{E}(s;A^{1/2})v(s)\big\|_{\theta_2}^2 ds < K_5(t) \tag{4.27}$$

for all $t \in [0,T^*]$.

Proof: Let us remark that the proof provided here is formal. It can be made rigorous be establishing it first for the Galerkin approximating system and then passing to the limit. For convenience, we make a slight abuse of notation and write

$$\mathcal{E}(A^{1/2}) = \mathcal{E}(t;A^{1/2}) \qquad \text{and} \qquad \dot{\mathcal{E}}(A^{1/2}) = \frac{\partial \mathcal{E}(t;x)}{\partial t}.$$

when the time variable is given by t. Thus, the evolution equation for $\mathcal{E}(A^{1/2})v$ is

$$\frac{d}{dt}\Big\{\mathcal{E}(A^{1/2})v\Big\} = \dot{\mathcal{E}}(A^{1/2})v + \mathcal{E}(A^{1/2})\big\{f - \nu A^{\theta_2}v - \tilde{B}(u,v)\big\}.$$

Take inner product with $\mathcal{E}(A^{1/2})v$ and apply (4.26) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big|\mathcal{E}(A^{1/2})u\big|^2 + \nu\big\|\mathcal{E}(A^{1/2})u\big\|_{\theta_2}^2 \\
&\qquad \leq \eta M \big\|\mathcal{E}(A^{1/2})v\big\|_{1/2} + \big(\mathcal{E}(A^{1/2})f, \mathcal{E}(A^{1/2})v\big) - \big(\mathcal{E}(A^{1/2})\tilde{B}(u,v), \mathcal{E}(A^{1/2})u\big)
\end{aligned}$$

Since $8\eta M \alpha_0^{\theta_2 - 1/2} \leq \nu$ then the Poincaré inequality implies

$$\eta M \big\|\mathcal{E}(A^{1/2})v\big\|_{1/2} \leq \frac{\nu}{8}\big\|\mathcal{E}(A^{1/2})v\big\|_{\theta_2}.$$

The forcing term may be estimated as

$$\left|\left\langle \mathcal{E}(A^{1/2})f, \mathcal{E}(A^{1/2})v\right\rangle\right| \le \frac{2}{\nu}\left\|\mathcal{E}(A^{1/2})f\right\|_{-\theta_2}^2 + \frac{\nu}{8}\left\|\mathcal{E}(A^{1/2})v\right\|_{\theta_2}^2.$$

To estimate the trilinear term first define the function u so that $\hat{\mathrm{u}}_k = |\hat{u}_k|$. Note that $\|\mathrm{u}\|_\beta = \|u\|_\beta$. Similarly define v. Thus

$$\begin{aligned}
&\left|\left\langle \mathcal{E}(A^{1/2})\tilde{B}(u,v), \mathcal{E}(A^{1/2})v\right\rangle\right| \\
&\quad\le L^3 \sum_{h+k+l=0} \mathcal{E}\big(|h+k|\big)|\hat{u}_h||k||\hat{v}_k|\mathcal{E}\big(|l|\big)|\hat{v}_l| \\
&\quad\le L^3 M \sum_{h+k+l=0} \mathcal{E}\big(|h|\big)\hat{\mathrm{u}}_h\mathcal{E}\big(|k|\big)|k|\hat{\mathrm{v}}_k\mathcal{E}\big(|l|\big)\hat{\mathrm{v}}_l \\
&\quad\le L^3 M 2^{1/2} \sum_{h+k+l=0} \Big\{\mathcal{E}\big(|h|\big)\hat{\mathrm{u}}_h\mathcal{E}\big(|k|\big)|k|^{1/2}\hat{\mathrm{v}}_k\mathcal{E}\big(|l|\big)|l|^{1/2}\hat{\mathrm{v}}_l \\
&\qquad\qquad + \mathcal{E}\big(|h|\big)|h|^{1/2}\hat{\mathrm{u}}_h\mathcal{E}\big(|k|\big)|k|^{1/2}\hat{\mathrm{v}}_k\mathcal{E}\big(|l|\big)\hat{\mathrm{v}}_l\Big\} \\
&\quad= M2^{1/2}\int_\Omega \Big\{\big(\mathcal{E}(A^{1/2})\mathrm{u}\big)\big(\mathcal{E}(A^{1/2})A^{1/2}\mathrm{v}\big)\big(\mathcal{E}(A^{1/2})A^{1/2}\mathrm{v}\big) \\
&\qquad\qquad + \big(\mathcal{E}(A^{1/2})A^{1/2}\mathrm{u}\big)\big(\mathcal{E}(A^{1/2})A^{1/2}\mathrm{v}\big)\big(\mathcal{E}(A^{1/2})\mathrm{v}\big)\Big\}.
\end{aligned}$$

The first integral may be estimated by Hölder's inequality followed by the Sobolev embedding theorem as

$$\begin{aligned}
I_1 &= M2^{1/2}\int_\Omega \big(\mathcal{E}(A^{1/2})\mathrm{u}\big)\big(\mathcal{E}(A^{1/2})A^{1/2}\mathrm{v}\big)\big(\mathcal{E}(A^{1/2})A^{1/2}\mathrm{v}\big) \\
&\le M2^{1/2}C\big\|\mathcal{E}(A^{1/2})u\big\|_{\beta_1}\big\|\mathcal{E}(A^{1/2})v\big\|_{\beta_2+1/2}\big\|\mathcal{E}(A^{1/2})v\big\|_{\beta_3+1/2}
\end{aligned}$$

where C is a dimensionless constant depending on the β_i and

$$\sum_{i=1}^3 \beta_i = 3/2 \qquad \text{and} \qquad 0 \le \beta_i < 3/2 \qquad \text{for} \qquad i = 1,2,3.$$

Since $\theta_1 > 0$ then $2\theta_1 + 2\theta_2 > \theta_1 + 2\theta_2 \ge 5/2$. Therefore, there exists γ such that $1/2 < \gamma < \theta_2$ and $2\theta_1 + 2\gamma > 5/2$. It follows that we may further choose β_i so that $\beta_1 \le 2\theta_1$, $\beta_2 + 1/2 \le \gamma$ and $\beta_3 + 1/2 \le \gamma$. Therefore after applying the Poincaré inequality we obtain

$$I_1 \le c_1\big|\mathcal{E}(A^{1/2})u\big|\big\|\mathcal{E}(A^{1/2})v\big\|_\gamma^2$$

where $c_1 = M2^{1/2}C\alpha^{-2\theta_1}\alpha_0^{2\theta_1+2\gamma-5/2}$. Similarly

$$\begin{aligned}
I_2 &= M2^{1/2}\int_\Omega \big(\mathcal{E}(A^{1/2})A^{1/2}\mathrm{u}\big)\big(\mathcal{E}(A^{1/2})A^{1/2}\mathrm{v}\big)\big(\mathcal{E}(A^{1/2})\mathrm{v}\big) \\
&\le M2^{1/2}C\big\|\mathcal{E}(A^{1/2})u\big\|_{\beta_1'+1/2}\big\|\mathcal{E}(A^{1/2})v\big\|_{\beta_2'+1/2}\big\|\mathcal{E}(A^{1/2})v\big\|_{\beta_3'} \\
&\le c_1\big|\mathcal{E}(A^{1/2})v\big|\big\|\mathcal{E}(A^{1/2})v\big\|_\gamma^2
\end{aligned}$$

where $\beta_1' + 1/2 \le 2\theta_1 + \gamma$, $\beta_2' + 1/2 \le \gamma$ and $\beta_3' = 0$. Use the interpolating inequality

$$\|w\|_{(1-\varepsilon)\beta_1+\varepsilon\beta_2} \le \|w\|_{\beta_1}^{(1-\varepsilon)} \|w\|_{\beta_2}^{\varepsilon}$$

with $w = \mathcal{E}(A^{1/2})v$, $(1-\varepsilon)\beta_1 + \varepsilon\beta_2 = \gamma$, $\beta_1 = 0$ and $\beta_2 = \theta_2$. Solving for ε yields $\varepsilon = \gamma/\theta_2$. It follows that

$$\begin{aligned}
I_1 + I_2 &\le 2c_1 \big|\mathcal{E}(A^{1/2})v\big| \big\|\mathcal{E}(A^{1/2})v\big\|_\gamma^2 \\
&\le 2c_1 \big|\mathcal{E}(A^{1/2})v\big|^{3-2\gamma/\theta_2} \big\|\mathcal{E}(A^{1/2})v\big\|_{\theta_2}^{2\gamma/\theta_2} \\
&= 2c_1 \Big\{\frac{4\gamma}{\nu\theta_2}\Big\}^{\gamma/\theta_2} \big|\mathcal{E}(A^{1/2})v\big|^{3-2\gamma/\theta_2} \Big\{\frac{\nu\theta_2}{4\gamma}\Big\}^{\gamma/\theta_2} \big\|\mathcal{E}(A^{1/2})v\big\|_{\theta_2}^{2\gamma/\theta_2} \\
&\le \frac{\nu}{4} \big\|\mathcal{E}(A^{1/2})v\big\|_{\theta_2}^2 + c_2 \big|\mathcal{E}(A^{1/2})v\big|^k
\end{aligned}$$

where

$$c_2 = \frac{\theta_2 - \gamma}{\theta_2} \big\{2c_1\big\}^{\theta_2/(\theta_2-\gamma)} \Big\{\frac{4\gamma}{\nu\theta_2}\Big\}^{\gamma/(\theta_2-\gamma)} \qquad \text{and} \qquad k = 2 + \frac{\theta_2}{\theta_2 - \gamma}.$$

After combining the above estimates, we obtain the differential inequality

$$\frac{d}{dt}\big|\mathcal{E}(A^{1/2})v\big|^2 + \nu\big\|\mathcal{E}(A^{1/2})v\big\|_{\theta_2}^2 \le \frac{4}{\nu}\big\|\mathcal{E}(A^{1/2})f\big\|_{-\theta_2}^2 + c_2\big|\mathcal{E}(A^{1/2})v\big|^k. \tag{4.28}$$

Let R be a constant with the same dimensions as $|\mathcal{E}(A^{1/2})v|^2$. Dropping the viscous term and setting $y = |\mathcal{E}(A^{1/2})v|^2 + R$ and $F = (4/\nu)\big\|\mathcal{E}(A^{1/2})f\big\|_{-\theta_2}^2$ yields

$$\frac{dy}{dt} \le (c_2 + R^{-k}F)y^k.$$

Integrating yields $y(t) \le g_0(t)$ where

$$g_0(t) = \frac{|v_0|^2 + R}{\Big\{1 - (k-1)\big\{|v_0|^2 + R\big\}^{k-1}\big(c_2 t + R^{-k}\int_0^t F\big)\Big\}^{1/(k-1)}}.$$

Choose $T^* > 0$ so small that

$$(k-1)\big\{|v_0|^2 + R\big\}^{k-1}\big(c_2 T^* + R^{-k}\textstyle\int_0^{T^*} F\big) < 1.$$

Integrate (4.28) once more to obtain (4.27) with

$$K_5(t) = |v_0|^2 + \int_0^t \Big\{c_2\big\{g_0(s)\big\}^k + F(s)\Big\}ds.$$

This finishes the proof of the lemma. □

A similar proof yields Gevrey regularity when $\theta_2 > 5/2$ and $\theta_1 = 0$.

5. The Viscous Case where $0 < \theta_2 \le 1/2$

In the case $0 < \theta_2 \le 1/2$ we shall employ a stronger estimate with more strict requirements on the regularity of the initial condition v_0 and of the forcing term f. This is

Lemma 5.1. *Let $f \in L^2_{\mathrm{loc}}\big([0,\infty); V^{1-\theta_2}\big)$ and $v_0 \in V$. Suppose that $\nu > 0$, $2\theta_1 + 4\theta_2 \ge 5$ and $\theta_2 > 0$. Then there exists a non-decreasing function $K_6(t)$ independent of m such that the solution v^m to the Galerkin truncation (3.4) satisfies*

$$\big\|v^m(t)\big\|^2 + \nu \int_0^t \big\|v^m(s)\big\|^2_{\theta_2+1} ds \le K_6(t) \qquad \textit{for all} \qquad t \ge 0. \tag{5.1}$$

Moreover, if $2\theta_1 + 4\theta_2 > 5$ and $2\theta_1 > 1$ then we may take $u_0 \in V^{\theta_1}$.

Proof: We establish our estimates via the vorticity $q^m = \nabla \times v^m$. Taking the curl of equations (3.4) we obtain the equations governing the evolution of q^m given by

$$\frac{dq^m}{dt} + \nu A^{\theta_1} q^m - \nabla \times \big\{P_m(u^m \times q^m)\big\} = \nabla \times P_m f. \tag{5.2}$$

Using the vector identity

$$\nabla \times (F \times G) = F(\nabla \cdot G) - G(\nabla \cdot F) + (G \cdot \nabla)F - (F \cdot \nabla)G$$

and the fact that u^m and q^m are divergence free we obtain

$$\nabla \times \big\{P_m(u^m \times q^m)\big\} = P_m\big\{\nabla \times (u^m \times q^m)\big\} = P_m\big\{(q^m \cdot \nabla)u^m - (u^m \cdot \nabla)q^m\big\}.$$

It follows taking inner product of (5.2) with q^m that

$$\frac{1}{2}\frac{d}{dt}|q^m|^2 + \nu\big\|q^m\big\|^2_{\theta_2} + \big((q^m \cdot \nabla)u^m, q^m\big) = (\nabla \times f, q^m). \tag{5.3}$$

The right side of equation (5.3) may be estimated using the Cauchy–Schwarz and Young inequalities as

$$\big|(\nabla \times f, q^m)\big| \le \big\|\nabla \times f\big\|_{-\theta_2}\big\|q^m\big\|_{\theta_2} \le \frac{1}{\nu}\big\|f\big\|^2_{1-\theta_2} + \frac{\nu}{4}\big\|q^m\big\|^2_{\theta_2}. \tag{5.4}$$

Next, let us estimate the term $\big((q^m \cdot \nabla)u^m, q^m\big)$. Use Hölder's inequality followed by the Sobolev embedding theorem to obtain

$$\big|\big((q^m \cdot \nabla)u^m, q^m\big)\big| \le C\big\|q^m\big\|_{\beta_1}\big\|u^m\big\|_{\beta_2+1}|q^m|$$

where C is a dimensionless constant, $\beta_1 + \beta_2 = 3/2$ and $\beta_1, \beta_2 > 0$. Further choose $\beta_1 \le \theta_2$ and $\beta_2 + 1 \le \theta_1 + \theta_2$. Since $\theta_1 + \theta_2 - 1 > 0$, then $\beta_2 > 0$. Moreover, since $\theta_1 + 2\theta_2 - 1 \ge 3/2$ then β_1 and β_2 may be chosen so that $\beta_1 + \beta_2 = 3/2$.

Now, applying the Poincaré and Young's inequalities yields

$$\begin{aligned}\big|\big((q^m\cdot\nabla)u^m,q^m\big)\big| &\le c_0\big\|q^m\big\|_{\theta_2}\big\|u^m\big\|_{\theta_1+\theta_2}\big|q^m\big| \\ &\le \frac{\nu}{4}\big\|q^m\big\|_{\theta_2}^2+\frac{c_0^2}{\nu}\big\|u^m\big\|_{\theta_1+\theta_2}^2\big|q^m\big|^2,\end{aligned}$$

where $c_0=C\alpha_0^{\theta_1+2\theta_2-5/2}$. This we substitute along with (5.4) into (5.3) to obtain

$$\frac{d}{dt}|q^m|^2+\nu\big\|q^m\big\|_{\theta_2}^2\le\frac{2}{\nu}\Big\{c_0^2\big\|u^m\big\|_{\theta_1+\theta_2}^2\big|q^m\big|^2+\big\|f\big\|_{1-\theta_2}^2\Big\}. \tag{5.5}$$

Dropping the viscous term on the left, multiplying by the integrating factor

$$\psi_m(t)=\exp\Big\{-\frac{2c_0^2}{\nu}\int_0^t\|u^m\|_{\theta_1+\theta_2}^2\Big\}$$

and integrating yields

$$\big|q^m(t)\big|^2\le\big\{\psi_m(t)\big\}^{-1}\Big\{\big|q^m(0)\big|^2+\frac{2}{\nu}\int_0^t\psi(s)\big\|f(s)\big\|_{1-\theta_2}^2ds\Big\}.$$

The magnitude of $\big\{\psi_m(t)\big\}^{-1}$ may be estimated using Lemma 3.2 while observing that $|q^m(0)|\le|q(0)|$. We obtain $\big|q^m(t)\big|^2\le K_6'(t)$ where

$$K_6'(t)=\exp\Big\{\frac{2c_0^2K_1(t)}{\nu^2}\Big\}\Big\{\big|q(0)\big|^2+\frac{2}{\nu}\int_0^t\big\|f(s)\big\|_{1-\theta_2}^2ds\Big\}.$$

To finish the proof integrate (5.5), this time without dropping terms, to obtain

$$\big|q^m(t)\big|^2+\nu\int_0^t\big\|q^m(s)\big\|_{\theta_2}^2ds\le K_6(t)$$

where

$$K_6(t)=\big|q(0)\big|^2+\frac{2}{\nu^2}K_1(t)K_6'(t)+\frac{2}{\nu}\int_0^t\big\|f(s)\big\|_{1-\theta_2}^2ds.$$

Noting that $|q^m|=\|v^m\|$ finishes the proof. $\square$

Lemma 5.2. *Let $u\in V^{2\theta_1}$, $v\in V$ and $w\in V^{\theta_2}$. If $0<\theta_2\le 1/2$ and $2\theta_1+4\theta_2\ge 5$, then $\tilde{B}:V^{2\theta_1}\times V\to V^{-\theta_2}$ and*

$$\big|\big\langle\tilde{B}(u,v),w\big\rangle_{\theta_2}\big|\le c_8|u|_{2\theta_1}\|v\|\|w\|_{\theta_2} \tag{5.6}$$

Proof: We first show this inequality for trigonometric polynomials and then extend to the indicated spaces using density. Let $u,v,w\in\mathcal{V}$. Estimate using Hölder's inequality followed by the Sobolev embedding theorem as

$$\big|\big(\tilde{B}(u,v),w\big)\big|\le C\|u\|_{\beta_1}\|v\|\|w\|_{\beta_2}$$

where C is a non-dimensional constant, $\beta_1 + \beta_2 = 3/2$ and $\beta_1, \beta_2 > 0$. Choose $\beta_1 \le 2\theta_1$ and $\beta_2 \le \theta_2$. Since $\theta_2 \le 1/2$ implies $\theta_1 \ge 3/2$, then $2\theta_1 + \theta_2 \ge 3$. Thus, we may satisfy $\beta_1 + \beta_2 = 3/2$. The Poincaré inequality then implies

$$\left|\left(\tilde{B}(u,v),w\right)\right| \le c_8 |u|_{2\theta_1} \|v\| \|w\|_{\theta_2}.$$

where $c_8 = C\alpha_0^{2\theta_1+\theta_2-3/2}$. Inequality (5.6) now follows from the density of $\mathcal{V}$ in the appropriate spaces. $\square$

Lemma 5.3. *Let* $0 < \theta_2 \le 1/2$ *and* $2\theta_1 + 4\theta_2 \ge 5$. *Let* $u \in V^{2\theta_1}$, $v \in V^{\theta_2+1}$ *and* $\psi \in V^{-\theta_2}$. *Then* $X\colon V^{2\theta_1} \times V^{\theta_2+1} \to V^{\theta_2}$ *and*

$$\left|\left\langle \psi, X(u,v)\right\rangle_{\theta_2}\right| \le c_3 \|u\|_{2\theta_1} \|v\|_{\theta_2+1} \|\psi\|_{-\theta_2} \tag{5.7}$$

where X *can be either* B *or* $\tilde{B}$.

Proof: We first show this inequality for trigonometric polynomials and then extend to the indicated spaces using density. Let $u, v, \psi \in \mathcal{V}$. Define u so that $\hat{\mathrm{u}}_k = |\hat{u}_k|$. Similarly define v and Ψ. Using the Fourier series estimate

$$\begin{aligned}
\left|\left(\psi, X(u,v)\right)\right| &\le L^3 \sum_{h+k+l=0} |\hat{u}_h||k||\hat{v}_k||\hat{\psi}_l| \\
&\le L^3 2^{\theta_2} \sum_{h+k+l=0} \left\{ |h|^{\theta_2} \hat{\mathrm{u}}_h |k| \hat{\mathrm{v}}_k |l|^{-\theta_2} \hat{\Psi}_l + \hat{\mathrm{u}}_h |k|^{1+\theta_2} \hat{\mathrm{v}}_k |l|^{-\theta_2} \hat{\Psi}_l \right\} \\
&= 2^{\theta_2} \int_\Omega \left\{ (A^{\theta_2/2}\mathrm{u})(A^{1/2}\mathrm{v})(A^{-\theta_2/2}\Psi) + \mathrm{u}(A^{(1+\theta_2)/2}\mathrm{v})(A^{-\theta_2/2}\Psi) \right\}.
\end{aligned}$$

Estimate the first term in the integral using Hölder's inequality followed by the Sobolev embedding theorem as

$$\begin{aligned}
I_1 &= 2^{\theta_2} \int_\Omega (A^{\theta_2/2}\mathrm{u})(A^{1/2}\mathrm{v})(A^{-\theta_2/2}\Psi) \\
&\le C \|u\|_{\beta_1+\theta_2} \|v\|_{\beta_2+1} \|\psi\|_{-\theta_2}
\end{aligned}$$

where C is a dimensionless constant, $\beta_1 + \beta_2 = 3/2$ and $\beta_1, \beta_2 > 0$. Since $\theta_2 \le 1/2$ then $\theta_1 \ge 3/2$. Therefore we may further choose $\beta_1 + \theta_2 \le 2\theta_1$ and $\beta_2 + 1 \le \theta_2 + 1$. The Poincaré inequality now yields

$$I_1 \le c_1 \|u\|_{2\theta_1} \|v\|_{\theta_2+1} \|\psi\|_{-\theta_2}$$

where $c_1 = C\alpha_0^{2\theta_1-3/2}$. Estimate the second term using Hölder's inequality and the inequality $\|u\|_{L^\infty} \le |u|^{1/2}\|u\|^{1/2}$ of Agmon [2] (see also [16]) followed by the Poincaré inequality as

$$\begin{aligned}
I_2 &= \int_\Omega \mathrm{u}(A^{(1+\theta_2)/2}\mathrm{v})(A^{-\theta_2/2}\Psi) \\
&\le C \|u\|^{1/2} \|u\|_2^{1/2} \|v\|_{\theta_2+1} \|\psi\|_{-\theta_2} \\
&\le c_2 \|u\|_{2\theta_1} \|v\|_{\theta_2+1} \|\psi\|_{-\theta_2}
\end{aligned}$$

where $c_2 = C\alpha_0^{2\theta_1-3/2}$. By taking $c_3 = c_1 + c_2$, the conclusion (5.7) now follows from the density of $\mathcal{V}$ in the appropriate spaces. $\square$

Lemma 5.4. *Let $f \in L^2_{\rm loc}\big([0,\infty); V^{1-\theta_2}\big)$ and $v_0 \in V$. Suppose that $\nu > 0$, $2\theta_1 + 4\theta_2 \geq 5$ and $0 < \theta_2 \leq 1/2$. Then there exists a non-decreasing function $K_4(t)$ independent of m such that the time derivative of the solution v^m to the Galerkin truncation (3.4) satisfies*

$$\int_0^t \Big\|\frac{dv^m}{dt}\Big\|_{\theta_2}^2 \leq K_7(t) \qquad \textit{for all} \qquad t \geq 0. \tag{5.8}$$

Proof: Applying the θ_2 norm to (3.4) yields

$$\Big\|\frac{dv^m}{dt}\Big\|_{\theta_2} \leq \nu\big\|v^m\big\|_{2\theta_2} + \big\|\tilde{B}(u^m,v^m)\big\|_{\theta_2} + \|f\|_{\theta_2}.$$

Since $\theta_2 \leq 1/2$ then $1-\theta_2 \leq \theta_2$. Applying the Poincaré inequality we obtain

$$\Big\|\frac{dv^m}{dt}\Big\|_{\theta_2} \leq \nu\alpha_0^{1-2\theta_2}\big\|v^m\big\| + \big\|\tilde{B}(u^m,v^m)\big\|_{\theta_2} + \alpha_0^{1-2\theta_2}\|f\|_{1-\theta_2}.$$

Let $\|\psi\|_{-\theta_2} \leq 1$. Estimate, using Lemma 5.3 followed by the Poincaré inequality, to obtain

$$\begin{aligned}\big|\big\langle\psi, \tilde{B}(u^m,v^m)\big\rangle_{\theta_2}\big| &\leq c_3\|u^m\|_{2\theta_1}\|v^m\|_{\theta_2+1}\|\psi\|_{-\theta_2}\\ &\leq c_4\|v^m\|\|v^m\|_{\theta_2+1}\end{aligned}$$

where $c_4 = c_3\alpha^{-2\theta_1}\alpha_0$. It follows from Lemma 5.1 that

$$\begin{aligned}\Big\|\frac{dv^m}{dt}\Big\|_{\theta_2}^2 &\leq 9\Big\{\nu^2\alpha_0^{2-4\theta_2} + c_4^2\big\|v^m\big\|^2\Big\}\big\|v^m\big\|_{\theta_2+1}^2 + 9\alpha_0^{2-4\theta_2}\big\|f\big\|_{1-\theta_2}^2\\ &\leq 9\big\{\nu^2\alpha_0^{2-4\theta_2} + c_4^2K_6(t)\big\}\big\|v^m\big\|_{\theta_2+1}^2 + 9\alpha_0^{2-4\theta_2}\big\|f\big\|_{1-\theta_2}^2.\end{aligned}$$

Integrate and apply Lemma 5.1 once more to obtain (5.8) where

$$K_7(t) = \frac{9}{\nu}\Big\{\nu^2\alpha_0^{2-4\theta_2} + c_4^2K_6(t)\Big\}K_6(t) + 9\alpha_0^{2-4\theta_2}\int_0^t \|f\|_{1-\theta_2}^2.$$

This finishes the proof. $\square$

Theorem 5.5. *Let $f \in L^2_{\rm loc}\big([0,\infty); V^{1-\theta_2}\big)$ and $v_0 \in V$. Suppose that $\nu > 0$, $2\theta_1 + 4\theta_2 \geq 5$ and $0 < \theta_2 \leq 1/2$. Then for any $T > 0$ equations (3.1) viewed as a functional equality in V^{θ_2} have a solution that satisfies*

$$v \in L^\infty\big((0,T);V\big) \cap L^2\big((0,T);V^{\theta_2+1}\big) \qquad \textit{and} \qquad \frac{dv}{dt} \in L^2\big((0,T);V^{\theta_2}\big).$$

Moreover, v is in $C\big([0,T];V^{\theta_2+1/2}\big)$ and $v(0) = v_0$.

Proof: Let $T > 0$ be given. The proof begins like the proof of Theorem 4.7. We choose ε so that $0 < \varepsilon < \theta_2$ and in place of Lemma 4.5 and Lemma 4.6 we employ Lemma 5.1 and Lemma 5.4. We know based on the above that the Galerkin system has a solution on the interval $[0, T]$. Applying Lemma 2.4 to the subsequence $A^{\theta_2/2+1/4}v^{m_i}$ we obtain that v^{m_i} converges to v in $C\big([0,T]; V^{\theta_2+1/2}\big)$ and consequently $v(0) = v_0$. Moreover,

$$\int_0^T \big\langle \phi(t), v^{m_i}(t) \big\rangle_{\theta_1+1}\, dt \to \int_0^T \big\langle \phi(t), v(t) \big\rangle_{\theta_1+1}\, dt \quad \text{as} \quad i \to \infty \tag{5.9}$$

for every $\phi \in L^2\big((0,T); V^{-\theta_2-1}\big)$,

$$\int_0^T \big\langle \psi(t), dv^{m_i}(t)/dt \big\rangle_{\theta_2}\, dt \to \int_0^T \big\langle \psi(t), dv(t)/dt \big\rangle_{\theta_2}\, dt \quad \text{as} \quad i \to \infty \tag{5.10}$$

for every $\psi \in L^2\big((0,T); V^{-\theta_2}\big)$ and

$$\int_0^T \big\| v^{m_i}(t) - v(t) \big\|^2_{\theta_2+1-\varepsilon} dt \to 0 \quad \text{as} \quad i \to \infty. \tag{5.11}$$

We now show v is a solution to (3.1) viewed as a functional equality in V^{θ_2}. To do this it is enough to show that along the subsequence v^{m_i} the equations (3.4) converge weakly in $L^2\big((0,T); V^{\theta_2}\big)$ to (3.1).

By our choice of subsequences we have already ensured that dv^{m_i}/dt converges weakly to dv/dt in $L^2\big((0,T); V^{\theta_2}\big)$.

Thanks to (5.9) v^{m_i} converges to v weakly in $L^2\big((0,T); V^{1+\theta_2}\big)$. As a result $A^{\theta_2} v^{m_i}$ converges to $A^{\theta_2} v$ weakly in $L^2\big((0,T); V^{1-\theta_2}\big)$. Since $\theta_2 \le 1/2$ we have $L^2\big((0,T); V^{1-\theta_2}\big) \subseteq L^2\big((0,T); V^{\theta_2}\big)$ and consequently $A^{\theta_2} v^m$ converges weakly to $A^{\theta_2} v$ in $L^2\big((0,T); V^{\theta_2}\big)$.

We finish by showing that the non-linear term $\tilde{B}(u^{m_i}, v^{m_i})$ converges weakly to $\tilde{B}(u, v)$ in $L^2\big((0,T); V^{\theta_2}\big)$. By the bilinearity of $\tilde{B}$ we have

$$\tilde{B}(u^{m_i}, v^{m_i}) - \tilde{B}(u, v) = \tilde{B}(u^{m_i}, v^{m_i} - v) + \tilde{B}(u^{m_i} - u, v).$$

Let $\psi \in L^2\big((0,T); V^{-\theta_2}\big)$ and choose $\psi_j \in C_0^\infty\big((0,T); \mathcal{V}\big)$ such that

$$\int_0^T \|\psi - \psi_j\|^2_{-\theta_2} \to 0 \qquad \text{as} \qquad j \to \infty.$$

Let M_j be the bounds such that $\sup\{\, |\psi_j(t)| : t \in (0,T) \,\} \le M_j$. Linearity implies that

$$\big\langle \psi, \tilde{B}(u^{m_i}, v^{m_i} - v) \big\rangle_{\theta_2} = \big\langle \psi - \psi_j, \tilde{B}(u^{m_i}, v^{m_i} - v) \big\rangle_{\theta_2} + \big\langle \psi_j, \tilde{B}(u^{m_i}, v^{m_i} - v) \big\rangle_{\theta_2}.$$

Lemma 5.3 followed by Lemma 5.1 yields

$$\begin{aligned} &\big|\big\langle \psi - \psi_j, \tilde{B}(u^{m_i}, v^{m_i} - v) \big\rangle_{\theta_2}\big| \\ &\qquad \le c_3 \|u^{m_i}\|_{2\theta_1} \|v^{m_i} - v\|_{\theta_2+1} \|\psi - \psi_j\|_{-\theta_2} \\ &\qquad \le c_3 K_6(t)^{1/2} \|v^{m_i} - v\|_{\theta_2+1} \|\psi - \psi_j\|_{-\theta_2}. \end{aligned}$$

Applying the Cauchy–Schwartz inequality followed by Lemma 5.1 again we obtain

$$\begin{aligned}
&\Big|\int_0^T \big\langle \psi-\psi_j, \tilde{B}(u^{m_i}, v^{m_i}-v)\big\rangle_{\theta_2}\Big| \\
&\qquad\qquad \le c_3 K_6(T)^{1/2}\Big\{\int_0^T \|v^{m_i}-v\|^2_{\theta_2+1}\Big\}^{1/2}\Big\{\int_0^T \|\psi-\psi_j\|^2_{-\theta_2}\Big\}^{1/2} \\
&\qquad\qquad \le c_3' K_6(T)\Big\{\int_0^T \|\psi-\psi_j\|^2_{-\theta_2}\Big\}^{1/2}. \qquad (5.12)
\end{aligned}$$

where $c_3'=c_3\nu^{-1/2}$. Estimate using (2.3) and the Poincaré inequality as

$$\begin{aligned}
\big|\big\langle \psi_j, \tilde{B}(u^{m_i}, v^{m_i}-v)\big\rangle_{\theta_2}\big| &= \big|\big(\tilde{B}(u^{m_i}, v^{m_i}-v), \psi_j\big)\big| \\
&\le C\|u^{m_i}\|^{1/2}\|u^{m_i}\|_2^{1/2}\|v^{m_i}-v\||\psi_j| \\
&\le c_4 M_j \|v^{m_i}\|\|v^{m_i}-v\|_{\theta_2+1-\varepsilon}
\end{aligned}$$

where $c_4 = C\alpha_0^{2\theta_1+\theta_2+1-\varepsilon-3/2}\alpha^{-2\theta_1}$. Cauchy-Schwartz followed by Lemma 5.1 yields

$$\Big|\int_0^T \big|\big\langle \psi_j, \tilde{B}(u^{m_i}, v^{m_i}-v)\big\rangle_{\theta_2}\big|\Big| \le c_4' M_j K_6(T)^{1/2}\Big\{\int_0^T \|v^{m_i}-v\|^2_{\theta_2+1-\varepsilon}\Big\}^{1/2} \qquad (5.13)$$

where $c_4'=c_4\nu^{-1/2}$. Similarly

$$\big\langle \psi, \tilde{B}(u^{m_i}-u, v)\big\rangle_{\theta_2} = \big\langle \psi-\psi_j, \tilde{B}(u^{m_i}-u, v)\big\rangle_{\theta_2} + \big\langle \psi_j, \tilde{B}(u^{m_i}-u, v)\big\rangle_{\theta_2}.$$

By Lemma 5.3, Lemma 5.1 and the Cauchy-Schwartz inequality we obtain

$$\Big|\int_0^T \big\langle \psi_j-\psi, \tilde{B}(u^{m_i}-u, v)\big\rangle_{\theta_2}\Big| \le c_3' K_6(T)\Big\{\int_0^T \|\psi-\psi_j\|^2_{-\theta_2}\Big\}^{1/2}. \qquad (5.14)$$

Similarly (2.3) followed by the Poincaré inequality yields

$$\begin{aligned}
\big|\langle \psi_j, \tilde{B}(u^{m_i}-u, v)\rangle_{\theta_2}\big| &= \big|(\tilde{B}(u^{m_i}-u, v), \psi_j)\big| \\
&\le C\|u^{m_i}-u\|^{1/2}\|u^{m_i}-u\|_2^{1/2}\|v\||\psi_j| \\
&\le c_4\|v^{m_i}-v\|_{\theta_2+1-\varepsilon}\|v\||\psi_j|.
\end{aligned}$$

The Cauchy-Schwartz inequality followed by Lemma 5.1 yields

$$\Big|\int_0^T \langle \psi_j, \tilde{B}(u^{m_i}-u, v)\rangle_{\theta_2}\Big| \le c_4' M_j K_6(T)^{1/2}\Big\{\int_0^T \|v^{m_i}-v\|^2_{\theta_2+1-\varepsilon}\Big\}^{1/2}. \qquad (5.15)$$

Combining (5.12), (5.13), (5.14) and (5.15) we obtain that

$$\begin{aligned}
\Big|\int_0^T \big\langle \psi, \tilde{B}(u^{m_i}, v^{m_i}) - \tilde{B}(u, v)\big\rangle_{\theta_2}\Big| &\le 2c_4' K_6(T)\Big\{\int_0^T \|\psi-\psi_j\|^2_{\theta_2}\Big\}^{1/2} \\
&\quad + 2c_5' M_j K_6(T)^{1/2}\Big\{\int_0^T \|v^{m_i}-v\|^2_{\theta_2+1-\varepsilon}\Big\}^{1/2}.
\end{aligned}$$

First choose j so large that the first term is small, and then choose i so large that the second term is small. Thus

$$\Big| \int_0^T \langle \psi, \tilde{B}(u^{m_i}, v^{m_i}) - \tilde{B}(u,v) \rangle \Big| \to 0 \qquad \text{as} \qquad i \to \infty.$$

It follows that $\tilde{B}(u^{m_i}, v^{m_i})$ converges to $\tilde{B}(u,v)$ weakly in $L^2\big((0,T); V^{\theta_2}\big)$. We have shown that v satisfies (3.1) viewed as a functional equality in V^{θ_2}. □

Theorem 5.6. *Let $v_0 \in H$. Suppose $\nu > 0$, $2\theta_1 + 4\theta_2 \geq 5$ and $0 < \theta_2 \leq 1/2$. Then there is at most one solution to* (3.1) *that satisfies*

$$v \in L^2_{\rm loc}\big([0,\infty); V\big) \qquad \textit{and} \qquad \frac{dv}{dt} \in L^2_{\rm loc}\big([0,\infty); V^{-1}\big). \tag{5.16}$$

Moreover, this solution depends continuously on the initial data v_0 in the H norm.

Proof: The proof proceeds exactly as the proof of Theorem 4.8 until (4.24) at which point we use Poincaré equality once more to obtain

$$\big|\big\langle B(\delta v, u_2)\delta v\big\rangle\big| \leq c_1' |\delta v| \|v_2\| \|\delta v\|_{\theta_2} \tag{5.17}$$

where $c_1' = c_1 \alpha_0^{1-\theta_2}$. Estimate $\big\langle \tilde{B}(\delta u, v_1), \delta v\big\rangle$ using Lemma 5.2 followed by (3.2). Thus

$$\big|\big\langle \tilde{B}(\delta u, v_1), \delta v\big\rangle_{\theta_2}\big| \leq c_2' |\delta v| \|v_1\| \|\delta v\|_{\theta_2} \tag{5.18}$$

where $c_2' = c_8 \alpha^{-2\theta_1}$. Let $M'(t)$ be the bound obtained from (5.16) such that

$$\int_0^t \big\|v_i(s)\big\|^2 ds \leq M'(t).$$

The proof now proceeds exactly as in Theorem 4.8. □

6. The Damped and Inviscid Cases where $\theta_2 = 0$ and $\nu \geq 0$

We now prove existence and uniqueness results analogous to Theorem 5.5, Theorem 4.8 and Theorem 5.6 for the damped inviscid case where $\theta_2 = 0$ and $\nu > 0$ and for the undamped inviscid case where $\theta_2 = 0$ and $\nu = 0$. In these cases (5.3) becomes

$$\frac{1}{2}\frac{d}{dt}|q^m|^2 + \nu\big|q^m\big|^2 + \big((q^m \cdot \nabla)u^m, q^m\big) = (\nabla \times f, q^m). \tag{6.1}$$

Let us again remark that the short time existence of the Galerkin system is based on the existence and uniqueness theory for ordinary differential equations. Global existence is guaranteed when we show that these solutions remain finite for all time.

Lemma 6.1. *Let $f \in L^1_{\rm loc}\big([0,\infty); V\big)$ and $v_0 \in V$. Suppose that $\nu \geq 0$, $2\theta_1 \geq 5$ and $\theta_2 = 0$. Then there exists a non-decreasing function $K_8(t)$ independent of m such that the solution v^m to the Galerkin truncation* (3.4) *satisfies*

$$\big\|v^m(t)\big\| \leq K_8(t) \qquad \textit{for all} \qquad t \geq 0. \tag{6.2}$$

Proof: Let $\epsilon > 0$ be arbitrarily small and define $z_{m,\epsilon} = \big(|q^m|^2 + \epsilon\big)^{1/2}$. Thus, $|q^m| < z_{m,\epsilon}$. Note for consistency ϵ must have the same dimensions as $|q^m|^2$.

Estimate $(\nabla \times f, q^m)$ using the Cauchy–Schwarz and Young inequalities as

$$\big|(\nabla \times f, q^m)\big| \le \|f\||q^m| \le \|f\| z_{m,\epsilon} \tag{6.3}$$

In the case $u^m \neq 0$, estimate $\big((q^m \cdot \nabla)u^m, q^m\big)$ using the Brézis–Gallouet inequality Theorem 2.5 followed by the Poincaré inequality as

$$\begin{aligned}\big|\big((q^m \cdot \nabla)u^m, q^m\big)\big| &\le |q^m|^2 \|\nabla u^m\|_{L^\infty} \\ &\le c(2\theta_1)|q^m|^2\|u^m\|_{5/2}\Big\{1 + \log \frac{\alpha_0^{2\theta_1 - 3/2}\|u^m\|_{2\theta_1+1}}{\|u^m\|_{5/2}}\Big\}.\end{aligned}$$

By Lemma 3.3 we have $\|u^m(t)\|_{5/2} \le K_2(t)$ where $K_2(t) = \alpha_0^{\theta_1 - 5/2} K_1(t)$. Define $\delta(t)$ so that $\|u^m(t)\|_{5/2} = K_2(t)\delta(t)$. Thus $0 < \delta(t) \le 1$. Let $c_0 = e^{-1}$ be the maximum of $-x\log x$ on the interval $(0, 1]$. It follows for $u^m \neq 0$ that

$$\begin{aligned}\big|\big((q^m \cdot \nabla)u^m, q^m\big)\big| &\le K_2(t)\delta(t)|q^m|^2\Big\{1 + \log \frac{c_1|q^m|}{K_2(t)\delta(t)}\Big\} \\ &\le K_2(t)|q^m|^2\Big\{1 + c_0 + \log \frac{c_1|q^m|}{K_2(0)}\Big\} \\ &\le K_2(t)z_{m,\epsilon}^2\Big\{1 + c_0 + \log \frac{c_1 z_{m,\epsilon}}{K_2(0)}\Big\}\end{aligned} \tag{6.4}$$

where $c_1 = \alpha_0^{2\theta_1 - 3/2}\alpha^{-2\theta_1}$. When $u^m = 0$ the final result in (6.4) trivially holds.

Since

$$\frac{d}{dt} z_{m,\epsilon} = \frac{1}{2z_{m,\epsilon}} \frac{d}{dt}|q^m|^2,$$

then (6.1) along with (6.3) and (6.4) yields

$$\frac{d}{dt} z_{m,\epsilon} \le \|f\| + K_2(t) z_{m,\epsilon}\Big\{1 + c_0 + \log \frac{c_1 z_{m,\epsilon}}{K_2(0)}\Big\}.$$

Setting

$$y_{m,\epsilon} = 1 + \frac{c_1 z_{m,\epsilon}}{K_2(0)} \qquad \text{and} \qquad g_2(t) = \frac{c_1\|f\|}{K_2(0)} + (1 + c_0)K_2(t)$$

yields $dy_{m,\epsilon}/dt \le c_2(t) y_{m,\epsilon}\{1 + \log y_{m,\epsilon}\}$. Therefore $y_{m,\epsilon}(t) \le K_8'(t)$ where

$$K_8'(t) = \exp\Big\{\Big\{1 + \log\Big(1 + \frac{c_1\big(|q(0)|^2 + \epsilon\big)^{1/2}}{K_2(0)}\Big)\Big\} \exp\big\{\textstyle\int_0^t g_2(t)dt\big\} - 1\Big\}.$$

Letting $\epsilon \to 0$ and setting

$$K_8(t) = K_2(0)c_1^{-1} \exp\left\{\left\{1 + \log\left(1 + \frac{c_1|q(0)|}{K_2(0)}\right)\right\} \exp\left\{\, \int_0^t g_2(t)dt\right\} - 1\right\}$$

yields (6.2) and finishes the proof. ☐

Lemma 6.2. *Let $f \in L^2_{\text{loc}}\big([0,\infty); V\big)$ and $v_0 \in V$. Suppose that $\nu \geq 0$, $2\theta_1 \geq 5$ and $\theta_2 = 0$. Then there exists a non-decreasing function $K_9(t)$ independent of m such that the time derivative of the solution v^m to the Galerkin truncation* (3.4) *satisfies*

$$\int_0^t \left|\frac{dv^m}{dt}\right|^2 \leq K_9(t) \qquad \textit{for all} \qquad t \geq 0. \tag{6.5}$$

Proof: Applying the H norm to (3.4) yields

$$\left|\frac{dv^m}{dt}\right| \leq \nu\big|v^m\big| + \big|\tilde{B}(u^m, v^m)\big| + \big|f\big|.$$

Now, using (2.3) followed by the Poincaré inequality and (3.2), we obtain

$$\begin{aligned}\big|\tilde{B}(u^m, v^m)\big| &= \sup\left\{\, \big(\tilde{B}(u^m, v^m), w\big) : |w| \leq 1\,\right\} \\ &\leq C\|u^m\|_2^{1/2}\|u^m\|^{1/2}\|v^m\| \leq c_1\|v^m\|^2\end{aligned}$$

where $c_1 = C\alpha_0^{2\theta_1 - 1/2}\alpha^{-2\theta_1}$. It follows from Lemma 6.1 that

$$\left|\frac{dv^m}{dt}\right| \leq \big\{\nu + c_1 K_8(t)\big\}K_8(t) + \big|f\big|.$$

Therefore

$$\left|\frac{dv^m}{dt}\right|^2 \leq g_2(t) + 4\big|f\big|^2$$

where $g_2(t) = 4\big\{\nu + c_1 K_8(t)\big\}^2 K_8(t)^2$ is a non-decreasing function. Integrating yields (6.5) where

$$K_9(t) = t g_2(t) + 4\int_0^t |f(s)|^2 ds.$$

This finishes the proof. ☐

Theorem 6.3. *Let $f \in L^2_{\text{loc}}\big([0,\infty); V\big)$ and $v_0 \in V$. Suppose $\nu \geq 0$, $2\theta_1 \geq 5$ and $\theta_2 = 0$. Then for any $T > 0$ equations* (3.1) *viewed as a functional equality in H have a solution that satisfies*

$$v \in L^\infty\big((0,T); V\big) \qquad \textit{and} \qquad \frac{dv}{dt} \in L^2\big((0,T); H\big).$$

Moreover, v is in $C\big([0,T]; V^{1/2}\big)$ and $v(0) = v_0$.

Proof: Let $T > 0$ be given. The proof begins like the proof of Theorem 4.7. We choose ε so that $0 < \varepsilon < 1/2$. In place of Lemma 4.5 and Lemma 4.6 we employ Lemma 6.1 and Lemma 6.2. Note that Lemma 6.1 implies that v^m is also uniformly bounded in $L^2\big((0,T);V\big)$. We know based on the above that the Galerkin system has a solution on the interval $[0,T]$. Applying Lemma 2.4 to the subsequence $A^{1/4}v^{m_i}$ we obtain that v^{m_i} converges to v in $C\big([0,T];V^{1/2}\big)$ and consequently $v(0) = v_0$. Moreover,

$$\int_0^T \big\langle \psi(t), v^{m_i}(t)\big\rangle_1 \, dt \to \int_0^T \big\langle \psi(t), v(t)\big\rangle_1 \, dt \quad \text{as} \quad i \to \infty \tag{6.6}$$

for every $\psi \in L^2\big((0,T);V^{-1}\big)$,

$$\int_0^T \big(dv^{m_i}(t)/dt, w(t)\big), dt \to \int_0^T \big(dv(t)/dt, w(t)\big)\, dt \quad \text{as} \quad i \to \infty \tag{6.7}$$

for every $w \in L^2\big((0,T);H\big)$ and

$$\int_0^T \big\|v^{m_i}(t) - v(t)\big\|_{1-\varepsilon}^2 dt \to 0 \quad \text{as} \quad i \to \infty. \tag{6.8}$$

It remains to show v is a solution to (3.1) viewed as a functional equality in H. To do this it is enough to show that along the subsequence v^{m_i} the equations (3.4) converge weakly in $L^2\big((0,T);H\big)$ to (3.1).

By our choice of subsequences we have already ensured that dv^{m_i}/dt converges weakly to dv/dt in $L^2\big((0,T);H\big)$. Similarly the damping term νv^{m_i}, if $\nu > 0$, converges weakly to νv in $L^2\big((0,T);V\big)$ and hence strongly in $L^2\big((0,T);H\big)$.

We finish by showing that the non-linear term $\tilde{B}(u^{m_i}, v^{m_i})$ converges weakly to $\tilde{B}(u,v)$ in $L^2\big((0,T);H\big)$. By the bilinearity of $\tilde{B}$ we have

$$\tilde{B}(u^{m_i}, v^{m_i}) - \tilde{B}(u,v) = \tilde{B}(u^{m_i}, v^{m_i} - v) + \tilde{B}(u^{m_i} - u, v).$$

Let $w \in L^2\big((0,T);H\big)$ and choose $w_j \in C_0^\infty\big((0,T);\mathcal{V}\big)$ such that

$$\int_0^T |w - w_j|^2 dt \to 0 \qquad \text{as} \qquad j \to \infty.$$

Let M_j be the bounds such that $\sup\big\{\, \|w_j(t)\| : t \in (0,T)\,\big\} \le M_j$. Equation (2.2) along with (2.5) implies

$$\big(\tilde{B}(u^{m_i}, v^{m_i} - v), w\big) = \big(B(u^{m_i}, v^{m_i} - v), w\big) + \big(B(w, u^{m_i}), v^{m_i} - v\big).$$

Linearity and Lemma 2.2 implies that

$$\big(B(u^{m_i}, v^{m_i} - v), w\big) = \big(B(u^{m_i}, v^{m_i} - v), w - w_j\big) - \big(B(u^{m_i}, w_j), v^{m_i} - v\big).$$

Now (2.3) and the Poincaré inequality followed by Lemma 6.1 yields

$$\begin{aligned}\big|\big(B(u^{m_i}, v^{m_i} - v), w - w_j\big)\big| &\le C\|u^{m_i}\|^{1/2}\|u^{m_i}\|_2^{1/2}\|v^{m_i} - v\||w - w_j| \\ &\le C\alpha_0^{2\theta_1 - 1/2}\alpha^{-2\theta_1}\|v^{m_i}\|\|v^{m_i} - v\|\big|w - w_j\big| \\ &\le g_1(t)|w - w_j|\end{aligned}$$

where $g_1(t) = 2C\alpha_0^{2\theta_1 - 1/2}\alpha^{-2\theta_1}\big\{K_8(t)\big\}^2$. Thus, by the Cauchy–Schwarz inequality

$$\left|\int_0^T \big(\tilde{B}(u^{m_i}, v^{m_i} - v), w - w_j\big)\right| \le g_1(T)T^{1/2}\Big\{\int_0^T |w - w_j|^2\Big\}^{1/2}. \tag{6.9}$$

Similarly, (2.3) followed by the Poincaré inequality and then Lemma 6.1 yields

$$\begin{aligned}\big|\big(B(u^{m_i}, w_j), v^{m_i} - v\big)\big| &\le C\|u^{m_i}\|^{1/2}\|u^{m_i}\|_2^{1/2}\|w_j\||v^{m_i} - v| \\ &\le g_2(t)\|v^{m_i} - v\|_{1-\varepsilon}\end{aligned}$$

where $g_2(t) = CM_j\alpha_0^{2\theta_1 - \varepsilon}K_8(t)$ is a non-decreasing function. Thus,

$$\left|\int_0^T \big(B(u^{m_i}, w_j), v^{m_i} - v\big)\right| \le g_2(T)T^{1/2}\Big\{\int_0^T \|v^{m_i} - v\|_{1-\varepsilon}^2 dt\Big\}^{1/2}. \tag{6.10}$$

Estimate using (2.5) followed by the Poincaré inequality and Lemma 6.1 as

$$\begin{aligned}\big|\big(B(w, u^{m_i}), v^{m_i} - v\big)\big| &\le C|w|\|u^{m_i}\|_2^{1/2}\|u^{m_i}\|_3^{1/2}|v^{m_i} - v| \\ &\le C\alpha_0^{2\theta_1 - 1/2 - \varepsilon}\alpha^{-2\theta_1}|w|\|v^{m_i}\|\|v^{m_i} - v\|_{1-\varepsilon} \\ &\le g_3(t)|w|\|v^{m_i} - v\|_{1-\varepsilon}\end{aligned}$$

where $g_3(t) = C\alpha_0^{2\theta_1 - 1/2 - \varepsilon}\alpha^{-2\theta_1}K_8(t)$ is a non-decreasing function. Thus,

$$\left|\int_0^T \big\langle B(w, u^{m_i}), v^{m_i} - v\big\rangle\right| \le g_3(T)M\Big\{\int_0^T \|v^{m_i} - v\|_{1-\varepsilon}^2\Big\}^{1/2} \tag{6.11}$$

where $M = \big\{\int_0^T |w(t)|^2\big\}^{1/2}dt$. Now use (2.3) followed by the Poincaré inequality and Lemma 6.1 to obtain

$$\begin{aligned}\big|\big(\tilde{B}(u^{m_i} - u, v), w\big)\big| &\le C\big\|u^{m_i} - u\big\|^{1/2}\big\|u^{m_i} - u\big\|_2^{1/2}\|v\||w| \\ &\le g_3(t)\big\|v^{m_i} - v\big\|_{1-\varepsilon}\big|w\big|.\end{aligned}$$

Thus,

$$\left|\int_0^T \big(\tilde{B}(u^{m_i} - u, v), w\big)\right| \le g_3(T)M\Big\{\int_0^T \big\|u^{m_i} - u\big\|_{2\theta_1 + 1 - \varepsilon}^2\Big\}^{1/2}. \tag{6.12}$$

We now combine the estimates given in (6.9), (6.10), (6.11) and (6.12) to show that $\tilde{B}(u^{m_i}, v^{m_i})$ converges to $\tilde{B}(u,v)$ weakly in $L^2\big((0,T);H\big)$. We obtain

$$\begin{aligned}\left|\int_0^T \big(\tilde{B}(u^{m_i}, v^{m_i}) - \tilde{B}(u,v), w\big)\right| &\le g_1(T)T^{1/2}\Big\{\int_0^T |w-w_j|^2\Big\}^{1/2}\\ &+ \Big(g_2(T)M_j + 2g_3(T)M\Big)\Big\{\int_0^T \|v^{m_i} - v\|_{1-\varepsilon}^2\Big\}^{1/2}.\end{aligned}$$

The first term can be made small by choosing j large enough. The second term can then be made small by further choosing i large enough. It follows that $\tilde{B}(u^{m_i}, v^{m_i})$ converges to $\tilde{B}(u,v)$ weakly in $L^2\big((0,T);H\big)$. □

Lemma 6.4. *Let $v_0 \in H$. Suppose that $\nu \ge 0$, $2\theta_1 \ge 5$ and $\theta_2 = 0$. Then there is at most one solution that satisfies*

$$v \in L^2\big((0,\infty);V\big) \qquad \textit{and} \qquad \frac{dv}{dt} \in L^2_{\rm loc}\big([0,\infty);V^{-1}\big). \tag{6.13}$$

Moreover, this solution depends continuously on the initial data in the H norm.

Proof: Let v_1 and v_2 be solutions of (3.1) such that $v_i(0) \in H$ and (6.13) holds for $i = 1,2$. Let $\delta u = u_1 - u_2$ and $\delta v = v_1 - v_2$. Let $\epsilon > 0$ be arbitrary and define $\delta_\epsilon = \big(|\delta u|^2 + \epsilon\big)^{1/2}$. It follows, after adding and subtracting $\tilde{B}(u_2, v_1)$, that the evolution equation for δv may be written as

$$\frac{d}{dt}\delta v + \nu \delta v + \tilde{B}(\delta u, v_1) + \tilde{B}(u_2, \delta v) = 0. \tag{6.14}$$

Form the V^{-1} duality paring of (6.14) with δv and apply Lemma 2.4 to obtain

$$\frac{1}{2}\frac{d}{dt}\big|\delta v\big|^2 + \nu|\delta v|^2 \le \big|\big\langle \tilde{B}(\delta u, v_1), \delta v\big\rangle_1\big| + \big|\big\langle \tilde{B}(u_2, \delta v), \delta v\big\rangle_1\big|. \tag{6.15}$$

Now (2.3) followed by the Poincaré inequality yields

$$\begin{aligned}\big|\big\langle \tilde{B}(\delta u, v_1), \delta v\big\rangle_1\big| = \big|\big(\tilde{B}(\delta u, v_1), \delta v\big)\big| &\le C\|\delta u\|^{1/2}\|\delta u\|_2^{1/2}\|v_1\||\delta v|\\ &\le c_0\|v_1\||\delta v|^2 \le c_0\|v_1\|\delta_\epsilon^2\end{aligned} \tag{6.16}$$

where $c_0 = \alpha_0^{2\theta_1 - 3/2}\alpha^{-2\theta_1}$. By Lemma 2.2, inequality (2.5) and techniques similar to the proof of Lemma 4.1 it follows that

$$\begin{aligned}\big|\big(\tilde{B}(u_2, \delta v), \delta v\big)\big| = \big|\big(B(\delta v, u_2), \delta v\big)\big| &\le C\|u_2\|_2^{1/2}\|u_2\|_3^{1/2}|\delta v|^2\\ &\le c_1\|v_2\||\delta v|^2 \le c_1\|v_2\|\delta_\epsilon^2\end{aligned} \tag{6.17}$$

Since

$$\frac{d}{dt}\delta_\epsilon = \frac{1}{2\delta_\epsilon}\frac{d}{dt}|\delta v|^2,$$

then (6.15) along with (6.16) and (6.17) yields

$$\frac{d}{dt}\delta_\epsilon \leq \big\{c_0\|v_1\| + c_1\|v_2\| - \nu\big\}\delta_\epsilon.$$

Multiply by the integrating factor

$$\psi(t) = \exp\Big\{ - \int_0^t \{c_0\|v_1(s)\| + c_1\|v_2(s)\| - \nu\}ds\Big\}$$

to obtain that

$$|\delta_\epsilon(t)| \leq \big\{\psi(t)\big\}^{-1}|\delta_\epsilon(0)|.$$

Now, letting $\epsilon \to 0$ yields

$$|\delta v(t)| \leq \big\{\psi(t)\big\}^{-1}|\delta v(0)|.$$

If $\delta v(0) = 0$ then $\delta v(t) = 0$ for all time. Therefore solutions are unique. ¿From the above we observe that the solution depends continuously on the initial data v_0 in the H norm. $\square$

7. Conclusions

In this section we summarize our main result: If $2\theta_1 + 4\theta_2 \geq 5$, then equations (3.1) have unique strong solutions in the sense of Definition 3.1. This follows immediately from Theorem 4.7, Theorem 4.8, Theorem 5.5, Theorem 5.6, Theorem 6.3 and Lemma 6.4. In addition, if $\theta_2 > 1/2$, then Theorem 4.9 implies that the initial condition v_0 is instantaneous smoothed to become as regular as the forcing permits. Thus we obtain

Theorem 7.1. *Suppose* $2\theta_1 + 4\theta_2 \geq 5$.

(i) *Let* $f \in L^2_{\text{loc}}\big([0,\infty); V^{-\theta_2}\big)$ *and* $v_0 \in H$. *Suppose* $\theta_2 > 1/2$ *and* $\nu > 0$. *Then, equations* (3.1) *have unique strong solutions*

$$v \in L^2_{\text{loc}}\big([0,\infty); V^{\theta_2}\big) \cap C\big([0,\infty); H\big)$$

that depend continuously on the initial data v_0 *in the* H *norm. Moreover, this solution is instantaneously smoothed to analyticity if the forcing is analytic.*

(ii) *Let* $f \in L^2_{\text{loc}}\big([0,\infty); V^{1-\theta_2}\big)$ *and* $v_0 \in V$. *Suppose either* $0 \leq \theta_2 \leq 1/2$ *and* $\nu > 0$ *or* $\theta_2 = 0$ *and* $\nu \geq 0$. *Then, equations* (3.1) *have unique strong solutions*

$$v \in L^2_{\text{loc}}\big([0,\infty); V^{\theta_2+1}\big) \cap C\big([0,\infty); H\big)$$

that depend continuously on the initial data v_0 *in the* H *norm.*

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