The Orthogonal Projection of Fractal Sets

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To my parents, my wife and daughter:
Carter, Reta, Swatee and Maanasi.

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Contents

Chapter 1

Introduction

The research in this Ph.D. thesis focuses on the problem of embedding a fractal subset of a Hilbert space into a finite dimensional space by means of an orthogonal projection. We first consider subsets with finite fractal dimension and then study what happens in the case of finite Bouligand dimension. Work along these lines was initiated by Mañé [17]. The use of Bouligand dimension in the study of such projections was initiated by Movahedi-Lankarani [19]. Our aim is to obtain more stringent regularity properties for the inverse of the embedding map.

1.1. Overview

Our problem is motivated by the numerical study of the long term behavior of the solutions for many partial differential equations in physics and mechanics. An evolutionary partial differential equation may be viewed in its functional form as the ordinary differential equation

$$\frac{du}{dt} = F(u) \qquad \text{with initial condition} \qquad u(0) = u_0 \tag{1.1.1}$$

where $u(t)$ takes its values in an appropriate Hilbert space. Since the computations a computer is able to perform are finite dimensional, engineers and scientists wishing to study these equations numerically must study finite dimensional approximations. It is of interest to know whether any finite dimensional dynamical system could have the same long term dynamics as the original partial differential equation. Work along these lines includes Eden, Foias, Nicolaenko and Temam [7], Sauer, Yorke and Casdagli [24], and Robinson [21, 22].

The first part of this thesis is dedicated to the geometry of sets with finite fractal dimension. Under general conditions of dissipation, Eden, Foias, Nicolaenko and Temam show in [8] that many physical systems posses an invariant exponentially attracting set with finite fractal dimension. In that work they ask whether the exponential attractors of such infinite dimensional dissipative evolution equations are also the exponential attractors of some finite dimensional ordinary differential equations. An exponential attractor is a set lying inside an infinite dimensional Hilbert space. However, that set does not have natural finite dimensional coordinates. Finite dimensional coordinates can be provided by means of an orthogonal projection which is one-to-one on the attractor and whose inverse when restricted to the image of the attractor has continuity properties suitable for preserving the dynamics of the original system.

A partial solution to this problem was given by Ben-Artzi, Eden, Foias and Nicolaenko in [2]. The complete solution was eventually obtained in this work in collaboration with Foias [11]. In particular, we prove the following:

Theorem 1.1.1. Let $\mathcal{A}$ be a subset of a real infinite dimensional Hilbert space with finite fractal dimension less than $m/2$. Then for any orthogonal projection P of rank m and $\delta > 0$ there is an orthogonal projection $\tilde{P}$ such that $\|\tilde{P} - P\| < \delta$ and $\tilde{P}|_{\mathcal{A}}$ has Hölder inverse.

Hunt and Kaloshin in [15] have recently shown that such projections are prevalent according to the sense of prevalence defined in [16].

The second part of this thesis is concerned with the Bouligand dimension. This dimension was first defined by Bouligand in [3] and is similar to the usual fractal dimension except with an extra scaling condition. Thus, the Bouligand dimension of a compact set is never smaller than its fractal dimension. We observe that the Bouligand dimension of the global attractor must be finite in order for a dynamical system to posses an inertial manifold. Thus, some attractors are already known to have finite Bouligand dimension.

My contribution to the study of this dimension [30] is presented in the second part of this Ph.D. thesis. In this research I have shown that an orthogonal projection can increase the Bouligand dimension of a set. This is surprising, as such a thing cannot happen for the Hausdorff and fractal dimensions, and implies there may be bad projections that make a global attractor appear more complicated than it really is. Along the lines of embedding a fractal set by means of an orthogonal projection, the following result is obtained:

Theorem 1.1.2. Given a subset $\mathcal{A}$ of an arbitrarily large finite dimensional space $\mathbf{R}^n$ such that the Bouligand dimension of the set of differences between elements of $\mathcal{A}$ is less than m, then almost every orthogonal projection P of $\mathcal{A}$ of rank m is injective on $\mathcal{A}$ and $P|_{\mathcal{A}}$ has Lipschitz continuous inverse except for a logarithmic correction term.

Here almost everywhere is understood in terms of the Haar measure invariant with respect to orthogonal transformations on the space of all orthogonal projections of rank m in $\mathbf{R}^n$. Thus, the Bouligand dimension gives coordinates for the attractor with a modulus of continuity that is almost the same as that given by an inertial manifold.

1.2. Motivating Problem

The partial differential equations of physics and mechanics are infinite dimensional dynamical systems. Our motivating problem is to understand and be able to compute the long term dynamics of such infinite dimensional dynamical systems. Such problems include reaction-diffusion equations, Navier-Stokes equations, magneto-hydrodynamics, geophysical flows, thermo-hydraulics, Kuramoto-Sivashinsky equations, Cahn-Hilliard equations, the Ginzburg-Landau equations, damped sine-Gordon equations and the nonlinear wave equations of relativistic quantum mechanics and nonlinear elasticity. The main question motivating this thesis is to what extent can a finite dimensional numerical simulation reflect the long time dynamics of the original infinite dimensional system.

Heuristically, an accurate numerical simulation should be possible. Since dissipation kills small scale features, the original infinite dimensional dynamical system should be essentially finite dimensional. Therefore, it should be possible to capture the long time dynamics of the original system using the necessarily finite resolution of a computer. Our treatment of the underlying dimensionality of a dynamical system focuses on the possibility of constructing finite dimensional dynamical systems with the same long time dynamics as the original system. Following the terminology of Eden, Foias, Nicolaenko and Temam [8] two dynamical systems will be called asymptotically equivalent if they have the same universal attractor, and inertially equivalent if they share an exponential attractor. In both cases we require the dynamics on the shared attractor to be the same.

1.3. The Universal and Exponential Attractors

The universal attractor, sometimes also called the global attractor, is the unique maximal invariant compact set towards which all trajectories and bounded sets are attracted. An exponential attractor is a compact forward invariant set containing the universal attractor towards which all trajectories tend with at least a uniform exponential rate.

Following Temam [28] we now state a precise definition of the universal attractor and a theorem showing its existence. First, assume the solution of (1.1.1) defines a continuous semi-group $S(t)$ such that $u(t) = S(t)u_0$ for $t > 0$.

Definition 1.3.1. A universal attractor for $S(t)$ is a set $\mathcal{A}$ such that $\mathcal{A} = S(t)\mathcal{A}$ for $t > 0$ and $\operatorname{dist}\big(S(t)B, \mathcal{A}\big) \to 0$ as $t \to \infty$ for all bounded B in H.

Here dist denotes the Hausdorff metric defined by

$$\operatorname{dist}(A, B) = \max\big(\rho(A, B), \rho(B, A)\big)$$

where

$$\rho(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|.$$

Existence of a universal attractor generally follows from existence of an absorbing ball B_0 and the asymptotic compactness of $S(t)$. Existence of an absorbing ball is obtained by energy estimates. Compactness comes from regularity and Sobolev embeddings. In particular, Temam [28] proves the following theorem:

Theorem 1.3.2. If for every bounded set B in H there is T such that $\bigcup_{t\geq T} S(t)B$ is compact and contained in B_0, then there exists a unique universal attractor $\mathcal{A} = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)B_0}$.

1.4. Constructing a Finite Dimensional Dynamical System

Inertial manifolds provide one way of constructing a finite dimensional dynamical system from an infinite dimensional one. An inertial manifold is a positively invariant finite dimensional Lipschitz manifold toward which all solutions tend with at least a uniform exponential rate. Thus, an inertial manifold must contain the universal attractor, and moreover the intersection of an inertial manifold with the absorbing ball B_0 is an exponential attractor. Therefore, the finite dimensional dynamical system obtained by means of an inertial manifold is inertially equivalent to the original system. Inertial manifolds have been studied in depth by Constantin in [4] and Constantin, Foias, Nicolaenko and Temam in [5, 6].

At present the construction of an inertial manifold requires a sufficient gap in the spectrum of the linear part of F in (1.1.1). Hence there are dissipative evolution equations for which existence of a finite dimensional universal attractor is known but for which no inertial manifolds have been found. Furthermore, the dimension of an inertial manifold is typically much higher than what other arguments might indicate the dimension of the underlying dynamics to be. Therefore, we would like to find a way of constructing a finite dimensional dynamical system directly from an attractor. What we're interested in finding is a replacement for the inertial manifold that provides an inertially equivalent finite dimensional dynamical system that agrees with the heuristic estimates on the size of the underlying finite dimensionality of the long term dynamics and which can be used when the existence of an inertial manifold can not be shown.

To construct an inertially equivalent finite dimensional dynamical system from an attractor, it is necessary to embed the attractor into a finite dimensional space. This embedding should posses the appropriate continuity properties for preserving the dynamics on the attractor. It is geometrically simplest for that embedding to arise from an orthogonal projection. Before showing the existence of suitable embedding projections, let us first sketch how an inertially equivalent finite dimensional dynamical system may be constructed from such a projection.

The following construction and results are given by Eden, Foias, Nicolaenko and Temam in [8]. Given an attractor $\mathcal{A}$, suppose that P is a suitable projection. Define $s : \mathbf{R}^m \to \mathbf{R}^m$ by

$$s(v) = \begin{cases} v & \text{for } v \in P\mathcal{A} \\ \text{nearest point in } P\mathcal{A} \text{ to } v & \text{otherwise} \end{cases}$$

and $f : \mathbf{R}^m \to \mathbf{R}^m$ by

$$f(v) = PF\big(P^{-1}s(v)\big) + \alpha\big(s(v) - v\big)$$

where $\alpha > 0$ is a fixed constant related to the rate of convergence to the attractor. Then

$$\frac{dv}{dt} = f(v) \tag{1.4.1}$$

is a finite dimensional ordinary differential equation inertially equivalent to the original infinite dimensional dynamical system.

One should note that the maps $s(v)$ and consequently $f(v)$ may not be continuous since the nearest point in $P\mathcal{A}$ to v required for their construction may not be unique. Therefore a choice has to be made and fixed once and for all when defining these functions. It follows that (1.4.1) does not give us a dynamical system in the classical sense. Fortunately, the points of continuity form a dense G_δ subset of $\mathbf{R}^n$ and therefore (1.4.1) can be solved by recasting it in the form of the integral equation

$$v(t) = v_0 + \int_0^t f\big(v(\tau)\big)d\tau.$$

1.5. The Fractal Dimension

To find a projection suitable for the construction in Section 1.4, it turns out that it is enough for the attractor to have finite fractal dimension. By fractal dimension we mean the metric dimension that is sometimes also called the box-counting dimension, the capacity, or the Minkowski dimension. In this section we define the fractal dimension and give an alternative, easier to work with, characterization of it.

Definition 1.5.1. Let A be a totally bounded metric space. The fractal dimension $\dim_f(\mathcal{A})$ is defined as

$$\dim_f(\mathcal{A}) = \limsup_{\rho \to 0} \frac{\log \mathcal{N}_{\mathcal{A}}(\rho)}{\log(1/\rho)} \tag{1.5.1}$$

where $\mathcal{N}_{\mathcal{A}}(\rho) = \min \{ n : a_1, \ldots, a_n \in \mathcal{A} \text{ and } \mathcal{A} \subseteq \bigcup_{i=1}^{n} B_\rho(a_i) \}$ is the minimum number of open balls of radius ρ needed to cover $\mathcal{A}$. If $\mathcal{A}$ is not totally bounded, its fractal dimension $\dim_f(\mathcal{A})$ is, by definition, infinite.

An equivalent definition of fractal dimension in terms of an infimum follows.

Fact 1.5.2. The fractal dimension $\dim_f(\mathcal{A})$ is the infimum over all $d \geq 0$ for which there exists K such that

$$\mathcal{N}_{\mathcal{A}}(\rho) \leq K(1/\rho)^d \qquad \text{for } 0 < \rho < 1. \tag{1.5.2}$$

Remark 1.5.3. The following are simple consequences of the definition:

(i) $\dim_f(M_1 \times M_2) \leq \dim_f(M_1) + \dim_f(M_2)$,

(ii) If $f\colon M_1 \to M_2$ is Lipschitz continuous then $\dim_f(M_2) \leq \dim_f(M_1)$.

(iii) If $\dim_f(M) < d$ then there is a constant C_M such that $\mathcal{N}_X(\rho) \leq C_M(1/\rho)^d$ holds for all $X \subseteq M$ and $\rho \in (0,1)$.

One way of estimating the fractal dimension of an attractor is by means of the Lyapunov dimension. We begin this task by asking what happens to the shape of a ball under $S(t)$. The answer is that dissipation squashes it and the resulting image may be covered by smaller balls. Frechét derivatives can be used to estimate how the balls are squashed and how many smaller balls are needed to cover the image of each larger ball. The following estimates appear in the work of Constantin, Foias and Temam [9] and also in Temam [28].

For simplicity let us assume $S'(t; u_0)$ to be compact. Let $S'(t; u_0)$ be the Frechét derivative of $\phi \to S(t)\phi$ at u_0 and $U(t; u_0)M(t; u_0)$ its polar decomposition. Define

$$\pi_k(t) = \sup \left\{ \prod_{j=1}^{k} m_j(t; u_0) : u_0 \in \mathcal{A} \right\}$$

where $m_j(t; u_0)$ are the eigenvalues of $M(t; u_0)$ arranged in decreasing order. Note that $\pi_k(t)$ bounds the largest distortion produced by $S(t)$ on any k dimensional slice of the attractor.

The outer Lyapunov exponent μ_k is defined by

$$\mu_1 = l_1 \qquad \text{and} \qquad \mu_k = l_k - l_{k-1}$$

where

$$l_k = \lim_{t \to \infty} \frac{\log \pi_k(t)}{t}.$$

The outer Lyapunov dimension is

$$\dim_l(\mathcal{A}) = (k_0 + 1) \max_{k=1,\ldots,k_0} \left\{ 1 + \frac{l_k}{l_{k_0+1}} \right\}$$

where k_0 is the first integer such that $l_{k_0+1} < 0$. As shown in [9, 28] the following theorem holds:

Theorem 1.5.4. Let $\mathcal{A}$ be the universal attractor, then $\dim_f(\mathcal{A}) \leq \dim_l(\mathcal{A}) + 1$.

Thus the fractal dimension of many universal attractors is finite.

The construction of exponential attractors for a number of dissipative evolution equations is discussed by Eden, Foias, Nicolaenko and Temam in [8]. The strategy presented therein is to enlarge the universal attractor by adding the forward orbits of certain carefully chosen trajectories. This is done so that the fractal dimension of the attractor does not increase. The result is a compact forward invariant set with finite fractal dimension containing the universal attractor towards which all trajectories tend with at least a uniform exponential rate. This exponential fractal attractor then allows the construction of a finite dimensional inertially equivalent dynamical system without having to resort to the theory of inertial manifolds.

1.6. The Hölder Projection Theorem

To find an orthogonal projection suitable for the construction outlined in Section 1.4 we use Theorem 1.1.1 which is proven in Chapter 2. Let $\mathcal{A}$ be an exponential attractor for the dynamical system (1.1.1). In order to ensure that the dynamics on the projected image of $\mathcal{A}$ are the same as the original dynamics, it is enough to ensure that no new steady solutions have been introduced. This can be done by applying our theorem to the set $X = \mathcal{A} + \{\, v + F(v) : v \in \mathcal{A} \,\}$ to obtain a projection $\tilde{P}$ such that the inverse of $\tilde{P}|_X$ is Hölder continuous. As the fractal dimension of X is at most three times the fractal dimension of $\mathcal{A}$, we obtain an inertially equivalent finite dimensional dynamical system for any dimension k such that $k > 6 \dim_f(\mathcal{A})$.

In [2] Ben-Artzi, Eden, Foias and Nicolaenko construct sets that show Hölder continuity for the inverse of $\tilde{P}|_X$ and therefore $\tilde{P}|_{\mathcal{A}}$ is the best one could hope for under the hypothesis of finite fractal dimension. In particular, they conclude that the least upper bound on the Hölder exponent must in general be less than

$$\frac{1}{1+\dim_f(X)/2}. \tag{1.6.1}$$

Hunt and Kaloshin in [15] furthermore provide an estimate that the least upper bound is no more than

$$\frac{k-2\dim_f(X)}{k(1+\dim_f(X)/2)}. \tag{1.6.2}$$

This implies that the most we could know about f in (1.4.1) is that it is Hölder continuous at its points of continuity. It follows that the uniqueness of solutions on $\tilde{P}\mathcal{A}$ must be inferred from their uniqueness in the original evolution equation.

1.7. The Bouligand Dimension

There are two minor flaws with the finite dynamical system (1.4.1): The Hölder regularity of f on the attractor is not enough in itself to guarantee uniqueness of solutions on the attractor and the construction of f outside the attractor introduces additional singularities. In [22] Robinson constructs finite dimensional smooth dynamical systems with approximately the same attractors and long term dynamics as the original system. Chapter 3 of this thesis focuses of obtaining greater regularity for the inverse of the embedding map and consequently f. To do this we consider the Bouligand dimension.

What we shall call Bouligand dimension is the metric dimension defined by Bouligand in [3] and further discussed by Assouad in [1]. The use of Bouligand dimension in studying projections was initiated by Movahedi-Lankarani [19] where he constructs a set $\mathcal{A}$ with finite fractal dimension for which there are no finite rank projections P with $P|_{\mathcal{A}}$ having Lipschitz continuous inverse. Here $\mathcal{A}$ is a subset of a Hilbert space H. To do this, he exhibits a set with finite fractal but infinite Bouligand dimension. He then raises the question: What can happen in the case that the Bouligand dimension of $\mathcal{A}$ is finite? We will attempt to shed some light on this question. Let us first begin with the definition and some basic properties of Bouligand dimension.

Definition 1.7.1. The Bouligand dimension $\dim_b(\mathcal{A})$ is defined as

$$\dim_b(\mathcal{A}) = \lim_{\epsilon\to 0}\lim_{t\to\infty} \Delta_{\epsilon,t}(\mathcal{A})$$

where

$$\Delta_{\epsilon,t}(\mathcal{A}) = \sup\left\{ \frac{\log \mathcal{N}(r,\rho)}{\log(r/\rho)} : 0 < \rho < r < \epsilon \text{ and } r > t\rho \right\}$$

and $\mathcal{N}(r,\rho)$ is the number of ρ-balls required to cover any r-ball in $\mathcal{A}$.

Bouligand dimension may also be characterized as an infimum in a way similar to that for fractal dimension. This shows that Bouligand dimension is essentially the fractal dimension with an extra scaling condition. In particular,

Theorem 1.7.2. The Bouligand dimension $\dim_b(\mathcal{A})$ is the infimum over all $d \geq 0$ for which there exists K such that

$$\mathcal{N}_{\mathcal{A}}(r,\rho) \leq K(r/\rho)^d \qquad \text{for } 0 < \rho < r < 1. \tag{1.7.1}$$

First note that the exact form of the upper bound for r in (1.7.1) is not critical. Equivalently we may require for some $\epsilon > 0$ that there exists K_ϵ such that

$$\mathcal{N}_{\mathcal{A}}(r,\rho) \le K_\epsilon (r/\rho)^d \qquad \text{for } 0 < \rho < r < \epsilon. \tag{1.7.2}$$

It is obvious that if $\epsilon < 1$, then (1.7.1) implies (1.7.2) with $K_\epsilon = K$. Conversely suppose (1.7.2) holds for some $0 < \epsilon < 1$. Since $\mathcal{A}$ is compact, then a finite number N of $\epsilon/4$ balls will cover it. Let r and ρ be such that $0 < \rho < r < 1$. Consider the following cases:

Case $\rho \ge \epsilon/2$. Then

$$\mathcal{N}_{\mathcal{A}}(r,\rho) \le \mathcal{N}_{\mathcal{A}}(r,\epsilon/2) \le \mathcal{N}_{\mathcal{A}}(\epsilon/4) \le N.$$

Case $\rho < \epsilon/2 \le r$. Then

$$\mathcal{N}_{\mathcal{A}}(r,\rho) \le \mathcal{N}_{\mathcal{A}}(r,\epsilon/2)\mathcal{N}_{\mathcal{A}}(\epsilon/2,\rho) \le NK_\epsilon\Big(\frac{\epsilon/2}{\rho}\Big)^d \le NK_\epsilon\Big(\frac{r}{\rho}\Big)^d.$$

Thus taking $K = NK_\epsilon$ we obtain that

$$\mathcal{N}_{\mathcal{A}}(r,\rho) \le K\Big(\frac{r}{\rho}\Big)^d \qquad \text{for } 0 < \rho < r < 1.$$

In the situation where $\epsilon > 1$ the argument is similar. Moreover, if there exists one ϵ and K_ϵ for which (1.7.2) holds, then for each ϵ there is a K_ϵ such that it holds.

Proof of Theorem 1.7.2: Suppose there is d and K such that (1.7.1) holds. Then

$$\frac{\log \mathcal{N}_{\mathcal{A}}(r,\rho)}{\log(r/\rho)} \le d + \frac{\log K}{\log(r/\rho)} \qquad \text{for } 0 < \rho < r < 1.$$

It follows that

$$\Delta_{1,t}(\mathcal{A}) \leq d + \frac{\log K}{\log(t)}$$

and therefore

$$\dim_b(\mathcal{A}) = \lim_{\epsilon\to 0}\lim_{t\to\infty} \Delta_{1,t}(\mathcal{A}) \leq d.$$

Conversely, suppose d is chosen so that $\dim_b(\mathcal{A}) < d$. Then by definition there are values of ϵ and t such that $0 < \epsilon < 1 < t$ and $\Delta_{\epsilon,t}(\mathcal{A}) < d$. It follows that

$$\frac{\log \mathcal{N}_{\mathcal{A}}(r,\rho)}{\log(r/\rho)} < d \qquad \text{for } 0 < t\rho < r < \epsilon$$

and therefore

$$\mathcal{N}_{\mathcal{A}}(r,\rho) < (r/\rho)^d \qquad \text{for } 0 < t\rho < r < \epsilon.$$

Now, suppose $0 < \rho < r < \epsilon$. Then $0 < t(\rho/t) < r < \epsilon$ and

$$\mathcal{N}_{\mathcal{A}}(r,\rho) \leq \mathcal{N}_{\mathcal{A}}(r,\rho/t) \leq \left(\frac{r}{\rho/t}\right)^d = K\left(\frac{r}{\rho}\right)^d \qquad \text{for } 0 < \rho < r < \epsilon$$

where $K = t^d$. It follows that $\dim_b(\mathcal{A})$ is the infimum over all d for which (1.7.1) holds. $\square$

Thus, the Bouligand dimension may be obtained from the usual fractal dimension by subjecting it to an additional scaling condition: Instead of covering the entire set, we cover only a small portion of it and require the number of ρ-balls to scale geometrically with the size of the piece being covered. In particular, (1.7.1) may be obtained from (1.5.2) by the addition of this scaling condition.

The Bouligand dimension satisfies many of the usual properties that a reasonable dimension should satisfy. We now state for reference a few properties of Bouligand dimension as stated by Movahedi-Lankarani [19].

Theorem 1.7.3. The Bouligand dimension has the following properties:

(i) If $\mathcal{A} \subseteq \mathcal{B}$ then $\dim_b(\mathcal{A}) \leq \dim_b(\mathcal{B})$.

(ii) If $\mathcal{A}$ is an open subset of $\mathbf{R}_n$ then $\dim_b(\mathcal{A}) = n$.

(iii) If $\mathcal{A}$ and $\mathcal{B}$ are bi-Lipschitz isomorphic, then $\dim_b(\mathcal{A}) = \dim_b(\mathcal{B})$.

Note that Theorem 1.1.2 is stated with a hypothesis on the Bouligand dimension of the set $\mathcal{A} - \mathcal{A}$ of differences between elements of $\mathcal{A}$ rather than on $\mathcal{A}$ itself. This is necessary because such inequalities as $\dim_f(\mathcal{A} - \mathcal{A}) \leq 2\dim_f(\mathcal{A})$, though true for the fractal dimension, are not true for the Bouligand or Hausdorff dimensions. Corollary 3.2.6 provides an example to show how things are different in the case of Bouligand dimension.

Thus, finiteness of the Bouligand dimension as given by Theorem 1.1.2 allows us to obtain projections with almost Lipschitz inverse. This provides a partial converse to the fact that the attractor of a dynamical system possessing an inertial manifold must have finite Bouligand dimension. In particular, if $\dim_b(\mathcal{A} - \mathcal{A}) < m$, then almost every orthogonal projection of rank m of $\mathcal{A}$ has the same Lipschitz properties except for a logarithmic correction as have the embeddings of an inertial manifold. It should be noted that the theory of constructing a finite dimensional dynamical system based on the Bouligand dimension is not complete. Currently our theorem applies only to the arbitrarily large finite dimensional case. Furthermore, there are no direct proofs that the Bouligand dimension of $\mathcal{A}-\mathcal{A}$ or even $\mathcal{A}$ is finite. Therefore, there is no direct way to use our result to solve the regularity problems of f in (1.1.1). This and additional questions about Bouligand dimension will be discussed in the final chapter of this thesis.

Chapter 2

Fractal Dimension

In this chapter we first extend the Hölder-Mañé Theorem [2] from a finite dimensional space to the general Hilbert space setting. In particular, we prove Theorem 1.1.1 as stated in the introduction. We use this result to prove Theorem 2.8.1 which states that a compact metric space of fractal dimension d can be Hölder-Lipschitz continuously embedded into $\mathbf{R}^n$ for $n > 2d$. This result can be seen as an analytic version of the topological fact that every manifold of dimension d may be continuously embedded into $\mathbf{R}^{2d+1}$.

2.1. Outline

The complete details of the proof of Theorem 1.1.1 will be presented in the rest of this chapter except for the last paragraph which is devoted to the proof of Theorem 2.8.1. For now let us outline the first of these theorems so as to understand the ideas involved.

Let H be a real Hilbert space, X a compact subset of H and $\tilde{P}$ an orthogonal projection. For $\tilde{P}|_{\mathcal{A}}$ to have Hölder continuous inverse $f\colon \tilde{P}(\mathcal{A}) \to H$ means there are constants K and $\theta > 0$ so that $\|f(a_1) - f(a_2)\| \leq K\|a_1 - a_2\|^\theta$ for $a_1, a_2 \in \mathcal{A}$. If $Y = \mathcal{A} - \mathcal{A}$, the set of differences between elements of $\mathcal{A}$, then this is equivalent to saying that $\|\tilde{P}y\| \geq C\|y\|^\eta$ for $y \in Y$ where $\eta = 1/\theta$ and $C = (1/K)^\eta$. Furthermore, since $\dim_f(Y) \le 2\dim_f(\mathcal{A})$, we will focus our attention on obtaining a projection $\tilde{P}$ such that $\|\tilde{P}y\| \ge C\|y\|^\eta$ for $y \in Y$ where the rank of $\tilde{P}$ is greater than the fractal dimension of Y.

As the proof is somewhat complicated, we present the following pictorial representation of the maps involved for reference.

$$\begin{tikzcd}
H \arrow[r, bend right, "Q_N"] \arrow[rrdd, bend right, "P"'] & \mathbf{R}^N \arrow[r, bend right, "P_N"] & \mathbf{R}^m & & \\
& & + & = \mathbf{R}^{2m} \arrow[r, "\tilde{P}_N"] & \mathbf{R}^m \\
& & \mathbf{R}^m & &
\end{tikzcd}$$

Our result in the case of orthogonal projections will be obtained by taking an increasing sequence of finite dimensional subspaces and applying a finite dimensional result on each one thereby obtaining a sequence of projections that converge to one with the desired properties. In fact, given a projection P of rank m, a set Y of finite fractal dimension less than m, and $\delta > 0$, first project H into a large finite dimensional space of dimension N by a suitably choosen orthogonal projection Q_N. We choose Q_N so that the norms of elements in Y do not get too small. Now we consider $Q_N Y$ in the finite dimensional space $Q_N H$ and use [2] ignoring P and δ to construct an orthogonal projection P_0 from $Q_N H$ onto an m dimensional subspace. Then we focus on approximating the original projection.

Let V be a subspace of dimension $2m$ such that $PH \subseteq V$ and $P_0 Q_N H \subseteq V$. Now we use [8] to approximate P in V finding an orthogonal projection $\tilde{P}$ that is δ close to P. Note that, although [2] and [8] each contain complete proofs of the result in the case that H is finite dimensional, the techniques in [8] are better suited for estimating how the result depends on $\dim H$ whereas [2] is better for estimating the dependence on δ. Finally we obtain a sequence of orthogonal projections by letting $N \to \infty$ and $\delta \to 0$ in the above construction. With care this sequence can be made to converge to an orthogonal projection with the desired properties. $\square$

2.2. From Hilbert Space to a Large Finite Dimensional Space

The first step is the construction of the projections Q_N referred to above.

Lemma 2.2.1. Given a compact subset Y of a Hilbert space H such that $\mathcal{N}_Y(\rho) \leq C_Y(1/\rho)^d$ for $\rho \in (0,1)$. Then there are orthogonal projections Q_n of rank less than or equal to n such that $\|y - Q_n y\| \leq (C_Y/n)^{1/d}$ for all $y \in Y$.

Proof: Let $P_0 = I$. We define a sequence of vectors and orthogonal projections inductively as follows: Given P_n let $a_{n+1} \in Y$ be a vector so that $P_n a_{n+1}$ has largest norm in $P_n Y$. Define P_{n+1} to be the orthogonal projection onto $P_n H \ominus \mathbf{R} P_n a_{n+1}$.

Note that if $\|P_n a_{n+1}\| \leq \rho \leq \|P_{n-1} a_n\|$, then, for $j < k \leq n$, one has

$$\|a_k - a_j\| \geq \|P_{k-1} a_k - P_{k-1} a_j\| = \|P_{k-1} a_k\| \geq \|P_{n-1} a_n\| \geq \rho.$$

Thus for $A = \{ a_1, a_2, \dots \}$ we have that $\mathcal{N}_A(\rho) \geq n$. But $\mathcal{N}_A(\rho) \leq C_Y(1/\rho)^d$, since $A \subseteq Y$. It follows that $\rho \leq (C_Y/n)^{1/d}$ and thus $\|P_n a_{n+1}\| \leq (C_Y/n)^{1/d}$.

Let $Q_n = I - P_n$. Then Q_n is an orthogonal projection of rank less than or equal n and $y - Q_n y \in P_n Y$ implies $\|y - Q_n y\| \leq \|P_n a_{n+1}\| \leq (C_Y/n)^{1/d}$. $\square$

2.3. From a Large Finite Dimensional Space to $\mathbf{R}^m$

In this paragraph we present the finite dimension techniques introduced in [2] which we will later apply to $Q_N Y$ in the finite dimensional space $Q_N H$ for N large enough.

Only part of the proof is presented here as we don't approximate the original projection in this step. Careful track is kept of how the estimates depend on the dimension N.

Notation 2.3.1. Let $G_{N,k}$ be the Grassmanian manifold of k dimensional subspaces in $\mathbf{R}^N$, and μ be the invariant measure on $G_{N,k}$ normalized as in [25], equation 12.35, so that

$$\mu(G_{N,k}) = \frac{O_{N-1}O_{N-2}\cdots O_{N-k}}{O_{k-1}O_{k-2}\cdots O_1 O_0}$$

where $O_{N-1} = 2\pi^{N/2}/\Gamma(N/2)$ is the surface area of the $N-1$ dimensional unit sphere. Here Γ represents Euler's Gamma function. Also, let $S(B) = \{\, x \in G_{N,k} : x \cap B \neq \emptyset \,\}$. According to [25], equation 17.52, the measure of $S(B_\rho(a))$ where $a \neq 0$ and $\rho < \|a\|$ is calculated as

$$\mu\big(S(B_\rho(a))\big) = \frac{O_{N-2}\cdots O_{N-k-1}}{O_{k-2}\cdots O_0}\int_0^{\arcsin(\rho/\|a\|)} \cos^{k-1}(s)\sin^{N-k-1}(s)\,ds \qquad \square$$

This result has also been obtained through an explicit computation involving only elementary techniques by Friz and Robinson in [13].

Remark 2.3.2. We obtain for $r \leq \|a\|$ and $\rho < \|a\|$ that

$$\begin{aligned}
\mu\big(S(B_\rho(a))\big) &\leq \frac{1}{N-k}\frac{O_{N-2}\cdots O_{N-k-1}}{O_{k-2}\cdots O_0}\arcsin^{N-k}(\rho/\|a\|) \\
&\leq \frac{\mu(G_{N,k})}{N-k}\frac{O_{k-1}O_{N-k-1}}{O_{N-1}}\Big(\frac{\pi\rho}{2r}\Big)^{N-k} \\
&= \frac{\mu(G_{N,k})O_{N-k-1}}{N-k}\frac{\Gamma(N/2)}{\Gamma(k/2)}\Big(\frac{\rho}{2r}\Big)^{N-k}\pi^{(N-k)/2} \\
&\leq \frac{\mu(G_{N,k})O_{N-k-1}}{N-k}\Big(\frac{N\pi}{2}\Big)^{(N-k+1)/2}\Big(\frac{\rho}{2r}\Big)^{N-k}.
\end{aligned} \qquad \square$$

Lemma 2.3.3. Let $Y \subseteq \mathbf{R}^N$ be such that $\mathcal{N}_Y(\rho) < C_Y(1/\rho)^d$ for all $\rho \in (0,1)$. Then for $m > d$ and $r \in (0,1)$ there exists P_0 of rank m such that

$$\|P_0 y\| \geq C\Big(\frac{r}{\sqrt{N}}\Big)^{(m+1)/(m-d)} \quad \text{for } y \in Y \text{ such that } \|y\| \geq r$$

where $C > 0$ is a constant depending only on m, d and C_Y.

Proof: Define $Z = \{\, y \in Y : \|y\| \geq r \,\}$. Consider a cover of Z by $\mathcal{N}_Z(\rho)$ balls of radius $\rho < r/2$. Let a_j where $j = 1, \ldots, \mathcal{N}_Z(\rho)$ denote the centers of these balls. Estimate the measure α of the projections whose kernels intersect $B_{2\rho}(a_j)$ for some j. Let $k = N - m$ and using Remark 2.3.2 estimate

$$\begin{aligned}
\alpha &= \mu\Big(\bigcup S\big(B_{2\rho}(a_j)\big)\Big) \\
&\leq \mathcal{N}_Z(\rho)\frac{\mu(G_{N,k})O_{N-k-1}}{N-k}\Big(\frac{N\pi}{2}\Big)^{(N-k+1)/2}\Big(\frac{\rho}{r}\Big)^{N-k} \\
&\leq C_Y\frac{\mu(G_{N,k})O_{m-1}}{m}\Big(\frac{N\pi}{2}\Big)^{(m+1)/2}\Big(\frac{1}{r}\Big)^{m}\rho^{m-d} \\
&\leq \frac{\mu(G_{N,k})}{2C^{m-d}}\Big(\frac{\sqrt{N}}{r}\Big)^{m+1}\rho^{m-d}
\end{aligned}$$

where C is an appropriate constant depending only on m, d, and C_Y. It follows that $\alpha < \mu(G_{N,k})$ for $\rho = C(r/\sqrt{N})^{(m+1)/(m-d)}$. Taking $V \in G_{N,k} \setminus \bigcup S\big(B_{2\rho}(a_j)\big)$ and the projection P_0 with kernel V yields

$$\|P_0 y\| \geq \rho \quad \text{for} \quad y \in Z$$

since any $y \in Z$ is at least ρ away from the kernel of P_0. □

2.4. Sharp Conditions on Close Projections

In this section we present some elementary but sharp results from operator theory on orthogonal projections which we need later. For completeness we provide their proofs.

Lemma 2.4.1. Given orthogonal projections P and $\tilde{P}$ of equal and finite rank on a Hilbert space H and $\epsilon \in (0,1)$, if $\tilde{P}x = 0$ implies $\|Px\| \leq \epsilon\|x\|$, then $\|P - \tilde{P}\| \leq \epsilon$.

Proof: Define $T = \tilde{P}|_{PH}: PH \to \tilde{P}H$. If $Tx = 0$ for some $x \in PH$ then $x = Px$ and $\tilde{P}x = 0$. Therefore $\|x\| = \|Px\| \leq \epsilon\|x\|$, which is possible only if $x = 0$. Thus T is one-to-one. But $\dim TPH = \dim PH = \dim \tilde{P}H$ implies $TPH = \tilde{P}H$, and so T is also onto. Define

$$S = (I - \tilde{P})\big|_{(I-P)H}: (I - P)H \to (I - \tilde{P})H.$$

By [29], Chapter IX, Lemma 1.1, we have that S is one-to-one and onto and therefore so is its adjoint S^*. Furthermore,

$$\left\|S^*(I - \tilde{P})x\right\|^2 = \left\|(I - \tilde{P})x\right\|^2 - \left\|P(I - \tilde{P})x\right\|^2 \geq (1 - \epsilon^2)\left\|(I - \tilde{P})x\right\|^2.$$

We can again apply [29], Chapter IX, Lemma 1.1, to obtain that $T^*: \tilde{P}H \to PH$ is one-to-one and onto with

$$\|T^*\tilde{P}x\|^2 \geq (1 - \epsilon^2)\|\tilde{P}x\|^2.$$

It follows that

$$\begin{aligned}
\left\|(\tilde{P} - P)x\right\|^2 &= \left\|(I - P)\tilde{P}x - P(I - \tilde{P})x\right\|^2 \\
&= \left\|(I - P)\tilde{P}x\right\|^2 + \left\|P(I - \tilde{P})x\right\|^2 \\
&= \left\|\tilde{P}x\right\|^2 - \left\|T^*\tilde{P}x\right\|^2 + \left\|P(I - \tilde{P})x\right\|^2 \\
&\leq \epsilon^2\left\|\tilde{P}x\right\|^2 + \epsilon^2\left\|(I - \tilde{P})x\right\|^2
\end{aligned}$$

thus $\|\tilde{P} - P\| \leq \epsilon$. $\square$

Lemma 2.4.2. Let P and $\tilde{P}$ be orthogonal projections of finite rank such that $\|P - \tilde{P}\| < \delta$ for some $\delta \in (0,1)$. Then there is a unitary operator U such that $\tilde{P} = UPU^{-1}$ and $\|U - I\|^2 \leq 2(1 - \sqrt{1-\delta^2})$.

Proof: First suppose $\|P - \tilde{P}\| < \delta$. Define $T = \tilde{P}|_{PH}$. Since $\delta < 1$ then $T: PH \to \tilde{P}H$ is one-to-one and onto, and thus in the polar decomposition VA of T where $A: PH \to PH$ is a positive symmetric transformation we have that $V: PH \to \tilde{P}H$ is a unitary transformation (see [20], sections 105 and 110). Let N be the rank of P. As A is positive symmetric, there must be N orthogonal eigenvectors e_i of A with corresponding eigenvalues α_i. Thus we have $Ae_i = \alpha_i e_i$ where $i = 1, \ldots, N$. Define $\tilde{e}_n = Ve_n$. Clearly $\tilde{e}_n$ is an orthogonal basis for $\tilde{P}H$. Furthermore,

$$\begin{aligned}(e_i, \tilde{e}_j) &= (e_i, Ve_j) = (\tilde{P}e_i, Ve_j) = (Te_i, Ve_j) \\ &= (VAe_i, Ve_j) = \alpha_i(Ve_i, Ve_j) = \alpha_i(e_i, e_j).\end{aligned}$$

Therefore $\|\tilde{P}e_i\| = \|Te_i\| = \|VAe_i\| = \alpha_i$ implies

$$1 = \|Pe_i\|^2 = \|\tilde{P}e_i + Pe_i - \tilde{P}e_i\|^2 = \|\tilde{P}e_i\|^2 + \|Pe_i - \tilde{P}e_i\|^2 \leq (\tilde{e}_i, e_i)^2 + \delta^2$$

so that $(e_i, \tilde{e}_i) \geq \sqrt{1-\delta^2}$.

For $i = 1, 2, \ldots, N$ define $Z_i = \mathbf{R}e_i + \mathbf{R}\tilde{e}_i$ and write $H = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_N \oplus Z$. Let U_i be the rotation in Z_i taking e_i into $\tilde{e}_i$ and define $U = U_1 \oplus U_2 \oplus \cdots \oplus U_N \oplus I|_Z$. Clearly U is unitary and $\tilde{P} = UPU^{-1}$. Furthermore, denoting by I_i the identity operator on Z_i, we have

$$\|U - I\|^2 = \max\left\{\, \|U_i - I_i\|^2 : i = 1, 2, \ldots, N \,\right\}$$

$$= \max \left\{ \|\tilde{e}_i - e_i\|^2 : i = 1, 2, \ldots, N \right\}$$

$$= \max \left\{ 2\big(1 - (\tilde{e}_i, e_i)\big) : i = 1, 2, \ldots, N \right\}$$

$$\leq 2(1 - \sqrt{1 - \delta^2}). \qquad \square$$

Lemma 2.4.2 has the following strong reciprocal:

Lemma 2.4.3. Let P be a finite dimensional orthogonal projection and U a unitary operator. If $\|U - I\|^2 \leq 2(1 - \sqrt{1 - \delta^2})$ then $\|P - \tilde{P}\| \leq \delta$ where $\tilde{P} = UPU^{-1}$.

Proof: Let $\tilde{P}x = 0$ and $\|x\| = 1$. Define $y = Ux$. Then $\|y\| = 1$ and

$$\rho^2 = \|x - y\|^2 \leq \|I - U\|^2 \leq 2(1 - \sqrt{1 - \delta^2})$$

imply $(x, y) = 1 - \rho^2/2 \geq \sqrt{1 - \delta^2}$. Since $\tilde{P}x = 0$ then $\tilde{P} \leq I - x^* \otimes x$ and therefore

$$\|Px\| = \|UPx\| = \|\tilde{P}y\| \leq \|y - (x, y)x\| = \sqrt{1 - (x, y)^2} \leq \delta.$$

Applying Lemma 2.4.1 gives $\|P - \tilde{P}\| \leq \delta$. $\square$

2.5. Finite Dimensional Hölder Continuity

We now need the one step at a time constructive proof of the finite dimensional case of Theorem 1.1.1 in [8], Proposition A.1. Using the results from the previous section we give a simplified presentation carefully keeping track of how the constants depend on the closeness δ of the approximating projection.

Lemma 2.5.1. Let P be a projection of rank $N - 1$ in $\mathbf{R}^N$ and $A = \{\ker \tilde{P} : \|P - \tilde{P}\| < \delta\}$. Then $\mu(A) \geq C\delta^{N-1}$ where $C > 0$ is a constant that depends only on N.

Proof: Write $P = I - h^* \otimes h$. It follows from Lemma 2.4.1 that

$$A = \{\ker \tilde{P} : \|\tilde{P}h\| < \delta\} = \{\mathbf{R}\tilde{h} : \|\tilde{h}\| = 1 \text{ and } \|h - \tilde{h}(\tilde{h}, h)\| < \delta\}$$

$$= \{x \in G_{N,1} : x \cap B_\delta(h) \neq \emptyset\} = S\big(B_\delta(h)\big).$$

Therefore

$$\mu(A) = O_{N-2} \int_0^{\arcsin(\delta)} \sin^{N-2}(s)\, ds \geq C\delta^{N-1}$$

where C is a constant that depends only on N. □

Lemma 2.5.2. Let $Y \subseteq B_1(0) \subseteq \mathbf{R}^N$ be such that $\mathcal{N}_Y(\rho) \leq C_Y(1/\rho)^d$ for $\rho \in (0,1)$ where $d < N-1$. There are constants $C > 0$ and $\eta \geq 1$ depending only on N, d, and C_Y such that for any projection P of rank $N-1$ and $\delta \in (0,1)$ there is a projection $\tilde{P}$ such that

$$\|P - \tilde{P}\| < \delta \quad \text{and} \quad \|\tilde{P}y\| \geq C\big(\delta\|y\|\big)^\eta \quad \text{for all } y \in Y.$$

Proof: Let $\eta = 2(N-1)/(N-1-d)$. Since P is of rank $N-1$ there is h with $\|h\| = 1$ such that $P = I - h^* \otimes h$. Define $Z_n = \{y \in Y : r_n \leq \|y\| \leq r_{n-1}\}$ where $r_n = 1/2^n$. Consider a cover of Z_n by $\mathcal{N}_{Z_n}(\rho_n)$ balls of radius $\rho_n = \epsilon r_n^\eta$. Let a_j^n where $j = 1, \ldots, \mathcal{N}_{Z_n}(\rho_n)$ denote the centers of these balls.

Estimate the measure α of projections whose kernels intersect the balls $B_{2\rho_n}(a_j^n)$ in $G_{N,1}$:

$$\begin{aligned}
\alpha &= \mu\Big(\bigcup_{n,j} S\big(B_{2\rho_n}(a_j^n)\big)\Big) \\
&\leq \sum_{n=1}^\infty \mathcal{N}_{Z_n}(\rho_n) \frac{\mu(G_{N,1})O_{N-2}}{N-1}\Big(\frac{N\pi}{2}\Big)^{N/2}\Big(\frac{\rho_n}{r_n}\Big)^{N-1} \\
&\leq C_Y \epsilon^{N-1-d} \frac{\mu(G_{N,1})O_{N-2}}{N-1}\Big(\frac{N\pi}{2}\Big)^{N/2} \sum_{n=1}^\infty r_n^{(N-1-d)\eta - N+1} \\
&\leq C_1 \epsilon^{N-1-d}
\end{aligned}$$

where C_1 is an appropiate constant depending only on N, d and C_Y.

From Lemma 2.5.1 it follows that $\alpha < \mu\big(\{\ker \tilde{P} : \|P - \tilde{P}\| < \delta\}\big)$ by setting $\epsilon = C_2\delta^{(N-1)/(N-1-d)}$ where C_2 is some suitable constant depending only on N, d and C_Y. Take

$$\tilde{V} \in \{\ker \tilde{P} : \|P - \tilde{P}\| < \delta\} \setminus \Big(\bigcup_{j,n} S\big(B_{2\rho_n}(a_j^n)\big)\Big).$$

and the projection $\tilde{P}$ with kernel $\tilde{V}$. If $y \in Y$ then $r_n \le \|y\| \le r_{n-1}$ for some n and $y \in B_{\rho_n}(a_j^n)$ for some j. Since $\tilde{V} \notin S\big(B_{2\rho_n}(a_j^n)\big)$ then y is at least ρ_n away from the kernel of $\tilde{P}$. Therefore

$$\|\tilde{P}y\| \ge \rho_n = \epsilon r_n^\eta \ge \epsilon 2^{-\eta}\|y\|^\eta = C_2 2^{-\eta}\delta^{\eta/2}\|y\|^\eta \ge C_2 2^{-\eta}\big(\delta\|y\|\big)^\eta$$

because $\delta \in (0,1)$. The desired conclusion follows with $C = C_2 2^{-\eta}$. □

Theorem 2.5.3.[2, 8] Given $Y \subseteq B_1(0) \subseteq \mathbf{R}^N$ such that $\mathcal{N}_Y(\rho) \le C_Y(1/\rho)^d$ for $\rho \in (0,1)$ where $d < m \le N-1$. There are constants $C > 0$ and $\eta \ge 1$ depending only on N, m, d and C_Y such that for any projection P on $\mathbf{R}^N$ of rank m and $\delta \in (0,1)$ there is a projection $\tilde{P}$ of rank m such that

$$\|P - \tilde{P}\| < \delta \quad \text{and} \quad \|\tilde{P}y\| \ge C\big(\delta\|y\|\big)^\eta \quad \text{for every } y \in Y.$$

Proof: If $N - 1 = m$ then this is simply Lemma 2.5.2, otherwise assume for induction on N that the result holds in $\mathbf{R}^{N-1}$.

Let $Q : \mathbf{R}^N \to \mathbf{R}^{N-1}$ be chosen so $P \le Q$. By Lemma 2.5.2 there is a $\tilde{Q}$ such that $\|Q - \tilde{Q}\| < \delta/2$ and $\|\tilde{Q}y\| \ge C_1\big(\delta\|y\|\big)^{\eta_1}$ for $y \in Y$. Let U be as in Lemma 2.4.2

so that $\tilde{Q} = UQU^{-1}$ and $\|U - I\|^2 \leq 2(1 - \sqrt{1 - \delta^2/4})$. From Lemma 2.4.3 it follows that $\|P - UPU^{-1}\| < \delta/2$.

Now consider the $N-1$ dimensional space $\tilde{Q}\mathbf{R}^N$. By the induction hypothesis there is $\tilde{P} \leq \tilde{Q}$ such that $\|\tilde{P} - UPU^{-1}\| < \delta/2$ and $\|\tilde{P}\tilde{Q}y\| \geq C_2\big(\delta\|\tilde{Q}y\|\big)^{\eta_2}$ for $y \in Y$. Thus

$$\|\tilde{P} - P\| \leq \|\tilde{P} - UPU^{-1}\| + \|P - UPU^{-1}\| < \delta$$

and

$$\|\tilde{P}y\| \geq C_2\big(\delta\|\tilde{Q}y\|\big)^{\eta_2} \geq C_2\big(\delta C_1(\delta\|y\|)^{\eta_1}\big)^{\eta_2} \geq C\big(\delta\|y\|\big)^{\eta} \quad \text{for } y \in Y$$

where C and η depend only on N, m, d, and C_Y. $\square$

2.6. Approximation of P in the $2m$-Dimensional Space

In this section the previous results are put together in the way outlined in Section 2.1 to obtain an orthogonal projection $\tilde{P}$ on H that is close to the original projection and behaves reasonably well on Y.

Lemma 2.6.1. Let H be a Hilbert space and Y be a compact subset of $B_1(0)$ such that $\dim_f(Y) < m$. Then there exists constants $\eta \geq 1$, $\epsilon > 0$ and $C > 0$ such that for any orthogonal projection P of rank m and $r \in (0, \epsilon)$ there is an orthogonal projection $\tilde{P}$ with

$$\|\tilde{P} - P\| < r \quad \text{and} \quad \|\tilde{P}y\| \geq Cr^{\eta} \quad \text{for } y \in Y \text{ such that } \|y\| \geq r.$$

Proof: Choose d so $\dim_f(Y) < d < m$, let C_Y be as in Remark 1.5.3 c, and take $\epsilon = C_Y^{1/d}$. Thus there is $N \geq 2$ so that

$$\left(\frac{C_Y}{N}\right)^{1/d} \leq \frac{r}{2} < \left(\frac{C_Y}{N-1}\right)^{1/d}.$$

Defining $Q = Q_N$ from Lemma 2.2.1 yields

$$\|Qy\| \geq \|y\| - (C_Y/N)^{1/d} \geq r/2 \quad \text{for } y \in Y \text{ such that } \|y\| \geq r.$$

Since $C_{QY} \leq C_Y$ one may use Lemma 2.3.3 to find an orthogonal projection P_0 of rank m in QH such that

$$\|P_0 Qy\| \geq C_1 \left(\frac{r/2}{\sqrt{N}}\right)^{(m+1)/(m-d)} \quad \text{for } y \in Y \text{ such that } \|Qy\| \geq r/2$$

where C_1 depends only on m, d and C_Y. Since $N \geq 2$ we have

$$\frac{1}{N} > \frac{1}{2C_Y}\left(\frac{r}{2}\right)^d$$

and therefore also

$$\|P_0 Qy\| \geq C_2 r^{(1+d/2)(m+1)/(m-d)} \quad \text{for } y \in Y \text{ such that } \|y\| \geq r$$

where C_2 depends only on m, d and C_Y.

Choose a subspace $V \subseteq H$ of dimension $2m$ such that $P_0 QH \subseteq V$ and $PH \subseteq V$ and let $\tilde{Q}$ be the orthogonal projection from H onto V. By Theorem 2.5.3 applied to $\tilde{Q}Y$ in V there is an orthogonal projection $\tilde{P}_1$ in V such that

$$\|\tilde{P}_1 - P\| < r \quad \text{and} \quad \|\tilde{P}_1 \tilde{Q}y\| \geq C_3 \big(r\|\tilde{Q}y\|\big)^{\eta_1} \quad \text{for all } y \in Y$$

where C_3 and $\eta_1 \geq 1$ are constants depending only on m, d and C_Y.

Defining $\tilde{P} = \tilde{P}_1 \tilde{Q}$, we obtain

$$\|\tilde{P}y\| \geq C_3\big(r\|P_0 Qy\|\big)^{\eta_1} \geq C_3\big(rC_2 r^{(1+d/2)(m+1)/(m-d)}\big)^{\eta_1} = Cr^{\eta}$$

for $y \in Y$ such that $\|y\| \geq r$ with $\eta = \big(1 + (1 + d/2)(m+1)/(m-d)\big)\eta_1$ and C depending only on $C_2^{\eta_1}$ and C_3, that is, only on m, d, and C_Y. $\square$

2.7. The Hölder-Mañé Projection in Hilbert Space

In this section we conclude the proof of Theorem 1.1.1 by creating a Cauchy sequence of orthogonal projections using the previous theorem and letting the distance between successive projections decrease so quickly that a projection with Hölder inverse is obtained in the limit.

Lemma 2.7.1. Given a set $Y \subseteq B_1(0)$ suppose there are constants $\eta \geq 1$, $\epsilon > 0$ and $C > 0$ such that for every orthogonal projection P of rank m and $r \in (0, \epsilon)$ there is an orthogonal projection $\tilde{P}$ satisfying

$$\|\tilde{P} - P\| < r \quad \text{and} \quad \|\tilde{P}y\| \geq Cr^{\eta} \quad \text{for} \quad y \in Y \text{ such that } \|y\| > r.$$

Then for every $\gamma > \eta + 1$ and $\delta \in (0, 1)$ there is constant K such that for every orthogonal projection P of rank m there is an orthogonal projection $\tilde{P}$ of rank m satisfying

$$\|\tilde{P} - P\| < \delta \quad \text{and} \quad \|\tilde{P}y\| \geq K\|y\|^{\gamma^2} \quad \text{for} \quad y \in Y.$$

Proof: Let $\alpha = \min(\delta, \epsilon)$. Choose $\rho < 1/2$ so that $K = C(\alpha\rho)^\eta - 2\alpha\rho^\gamma > 0$. Define $r_n = \alpha\rho^{\gamma^n}$ for $n = 0, 1, 2, \ldots$ and choose a sequence of orthogonal projections so that

$$\|P_n - P_{n-1}\| < r_n \quad \text{and} \quad \|P_n y\| \geq C r_n^\eta \quad \text{for} \quad \|y\| \geq r_n.$$

for $n = 0, 1, 2, \ldots$ where $P_{-1} = P$. Since $r_{n+1} \leq r_n/2$ then $\sum_{k=n+1}^\infty r_k \leq 2r_{n+1}$ implies P_n is a Cauchy sequence of projections. Let $\tilde{P} = \lim P_n$. Clearly $\|\tilde{P} - P\| < \sum_{n=0}^\infty r_n \leq 2r_0 = 2\alpha\rho < \delta$. Fix $y \in Y$ with $y \neq 0$. If $r_0 \leq \|y\| \leq 1$ then

$$\|\tilde{P}y\| \geq \|P_0 y\| - \|\tilde{P} - P_0\|\|y\| \geq C r_0^\eta - 2r_1$$

$$= C(\alpha\rho)^\eta - 2\alpha\rho^\gamma = K \geq K\|y\|^{\gamma^2}.$$

Otherwise there is n so that $r_n \leq \|y\| \leq r_{n-1}$ and

$$\|\tilde{P}y\| \geq \|P_n y\| - \|\tilde{P} - P_n\|\|y\| \geq C r_n^\eta - 2r_{n+1}r_{n-1}$$

$$= C(\alpha\rho^{\gamma^n})^\eta - 2\alpha\rho^{\gamma^{n+1}}\alpha\rho^{\gamma^{n-1}} \geq (C\alpha^\eta - 2\alpha^2\rho^{\gamma^n(\gamma-\eta)})\rho^{\gamma^n\eta}$$

$$\geq \rho^{-\eta}K\rho^{\gamma^n\eta} \geq K\rho^{\gamma^n\eta} \geq K r_{n-1}^{\gamma\eta} \geq K\|y\|^{\gamma^2}. \qquad \square$$

We are now able to conclude with the proof of Theorem 1.1.1.

Proof of Theorem 1.1.1: Since X must be totally bounded, its closure $\bar{X}$ in H is compact. It is easy to see that $\dim_f(\bar{X}) = \dim_f(X)$ and therefore writing $Y = \bar{X} - \bar{X}$ yields $\dim_f(Y) \leq 2\dim_f(X) < m$ and Y compact. Clearly Theorem 1.1.1 is now a consequence of Lemma 2.6.1 and Lemma 2.7.1. $\square$

Theorem 1.1.1 is still true for general projections, that is, operators P on the Hilbert space satisfying only $P^2 = P$.

Theorem 2.7.2. Let H be a Hilbert space and $X \subseteq H$ be such that $\dim_f(X) < m/2$. Then for any projection P of rank m and $\delta > 0$ there is a projection $\tilde{P}$ such that $\|\tilde{P} - P\| < \delta$ and $\tilde{P}|_X$ has Hölder inverse.

Proof: Let P_0 be an orthogonal projection such that $P_0 H = PH$ and let U be a unitary operator from $(I - P_0)H$ onto $(I - P)H$. Define $V = P_0 + U(I - P_0)$. Then

$$VP = \big(P_0 + U(I - P_0)\big)P = P_0 P = P$$

and

$$VU^{-1}(I - P) = \big(P_0 + U(I - P_0)\big)U^{-1}(I - P) = I - P$$

imply $V^{-1} = P + U^{-1}(I - P)$ and thus

$$VP_0V^{-1} = VP_0\big(P + U^{-1}(I - P)\big) = VP_0P = VP = P.$$

By Theorem 1.1.1 applied to $V^{-1}X$ and P_0 there is an orthogonal projection $\tilde{P}_0$ and constants K and η such that

$$\|\tilde{P}_0 - P_0\| \le \delta/\big(\|V\|\|V^{-1}\|\big) \quad \text{and} \quad \|\tilde{P}_0 V^{-1} y\| \ge K\|V^{-1}y\|^{\eta} \quad \text{for} \quad y \in Y$$

where $Y = \bar{X} - \bar{X}$. Define $\tilde{P} = V\tilde{P}_0V^{-1}$. It follows that

$$\|\tilde{P}y\| = \|V\tilde{P}_0V^{-1}y\| \ge \frac{\|\tilde{P}_0V^{-1}y\|}{\|V^{-1}\|} \ge \frac{K\|V^{-1}y\|^{\eta}}{\|V^{-1}\|} \ge \frac{K\|y\|^{\eta}}{\|V^{-1}\|\|V\|^{\eta}}$$

and $\|\tilde{P} - P\| \le \big\|V(\tilde{P}_0 - P_0)V^{-1}\big\| \le \|V\|\|\tilde{P}_0 - P_0\|\|V^{-1}\| \le \delta.$ $\square$

2.8. Consequences

The Hölder-Lipschitz parametrization of a compact metric space M of finite fractal dimension presented in Theorem 2.8.1 is a natural application of Theorem 1.1.1.

In this way any metric space of finite fractal dimension is the range of a Hölder expanding map defined on a bounded set in a finite dimensional Euclidean space.

Theorem 2.8.1. Given a metric space M of fractal dimension less than $m/2$, there is a function $g: M \to \mathbf{R}^m$ and constants $L > 0$ and $\gamma \geq 1$ such that

$$Ld(a,b)^\gamma \leq \big\|g(a) - g(b)\big\| \leq d(a,b) \quad \text{for all } a, b \in M.$$

For the proof of this theorem we need the following Hölder-Lipschitz embedding of M into a Hilbert space:

Lemma 2.8.2. Given a metric space M of finite fractal dimension and a Hilbert space H, there is function $f: M \to H$ and constants $K_1 > 0$, $K_2 > 0$, and $\eta \geq 1$ such that

$$K_2 d(a,b)^\eta \leq \big\|f(a) - f(b)\big\| \leq K_1 d(a,b) \quad \text{for all } a, b \in M.$$

Proof: Fix $d > \max\big(1, \dim_f(M)\big)$. By Remark 1.5.3 c there is C_M so that $\mathcal{N}_M(\rho) \leq C_M(1/\rho)^d$. For each n cover M with $N_n = \mathcal{N}_M(\rho_n)$ balls of radius $\rho_n = 1/2^n$ and denote the centers of the balls by a_j^n where $j = 1, \ldots, N_n$. Let $I = \bigcup_{n=1}^\infty \big\{\, (n,j) : j = 1, \ldots, N_n \,\big\}$ and choose a bijection $\sigma: I \to \mathbf{N}$. Define $f: M \to H$ by

$$f(a) = \sum_{n=1}^{\infty} \sum_{j=1}^{N_n} e_{\sigma(n,j)} \rho_n^d d(a, a_j^n)$$

where $\{ e_k : k \in \mathbf{N} \}$ is an orthogonal basis for H. The existence of the constant K_1 is given by the following computation:

$$\begin{aligned}\big\|f(a)-f(b)\big\|^2 &= \sum_{n=1}^{\infty}\sum_{j=1}^{N_n} \rho_n^{2d}\big|d(a,a_j^n)-d(b,a_j^n)\big|^2\\ &\le \sum_{n=1}^{\infty} C_M(1/\rho_n)^d\rho_n^{2d}\big|d(a,b)\big|^2\\ &\le C_M\big|d(a,b)\big|^2\sum_{n=1}^{\infty}\rho_n^d\\ &= K_1^2 d(a,b)^2 \quad \text{for all } a,b\in M\end{aligned}$$

with $K_1 = \big(C_M \sum_{n=1}^{\infty} \rho_n^d\big)^{1/2}$. To obtain the constants K_2 and η observe that $\big\|f(a)-f(b)\big\|^2 \ge \rho_n^{2d}\big|d(a,a_j^n)-d(b,a_j^n)\big|^2$ and therefore $d(a,a_j^n) \le \epsilon/\rho_n^d + d(b,a_j^n)$ where $\epsilon = \big\|f(a)-f(b)\big\|$. Choosing j so $d(b,a_j^n) < \rho_n$ yields

$$d(a,b) \le d(a,a_j^n) + d(b,a_j^n) \le \epsilon/\rho_n^d + 2\rho_n.$$

Since M has finite fractal dimension, it is bounded. Let D be the diameter of M. Thus $\epsilon \le DK_1$. Choose n so that $4^{d(n-1)} \le DK_1/\epsilon < 4^{dn}$. Then

$$d(a,b) \le 2^d\big(DK_1\epsilon\big)^{1/2} + 2\big(\epsilon/(DK_1)\big)^{1/2d}$$

and thus

$$K_2 d(a,b)^\eta \le \|f(a)-f(b)\| \quad \text{for all } a,b\in M$$

provided that $\eta = 2d$ and $K_2 = DK_1(2^dDK_1+2)^{-2d}$. □

We are now able to conclude with a proof of Theorem 2.8.1.

Proof of Theorem 2.8.1. Let f be the function given by Lemma 2.8.2 and P be an arbitrary orthogonal projection of rank m. Since f is Lipschitz, Remark 1.5.3 b gives that $\dim_f(X) < m/2$ where $X = f(M)$. Apply Theorem 1.1.1 to X and P to obtain an orthogonal projection $\tilde{P}$ and constants C and γ_1 such that $\|\tilde{P}x - \tilde{P}y\| \geq C\|x-y\|^{\gamma_1}$ for $x, y \in X$. Define $g(a) = K_1^{-1}\tilde{P}f(a)$. Thus for all $a, b \in M$ we have

$$\left\|g(a) - g(b)\right\| \leq \frac{1}{K_1}\left\|f(a) - f(b)\right\| \leq d(a,b)$$

and

$$\left\|g(a) - g(b)\right\| \geq \frac{C}{K_1}\left\|f(a) - f(b)\right\|^{\gamma_1} \geq \frac{C}{K_1}K_2^{\gamma_1}d(a,b)^{\gamma_1\eta}.$$

To conclude we let $L = CK_2^{\gamma_1}/K_1$, and $\gamma = \gamma_1\eta$. $\square$

Chapter 3

Bouligand Dimension

This chapter discusses the Bouligand dimension. In Section 3.1 we show that Bouligand dimension behaves well under cartesion products and calculate the Bouligand dimension for a self-similar fractal. In Section 3.2 we provide a set $\mathcal{A}$ for which $\dim_b(\mathcal{A} - \mathcal{A}) > 2\dim_b(\mathcal{A})$ and find orthogonal projections that increase the Bouligand dimension of certain sets. Finally, in Section 3.3 we prove Theorem 1.1.2.

3.1. Self-similar Fractals

In this section we prove Bouligand dimension is well behaved with respect to Cartesian products and calculate the Bouligand dimension for a self-similar fractal. In particular, we show the Bouligand dimension agrees with the similarity dimension for such sets, and therefore the fractal and Hausdorff dimensions as well.

We first discuss how Bouligand dimension behaves with respect to Cartesian products. We would like

$$\dim_b(\mathcal{A} \times \mathcal{B}) = \dim_b(\mathcal{A}) + \dim_b(\mathcal{B}).$$

However, as with any metric dimension, the Bouligand dimension depends on what metric is used. Therefore, care must be taken in choosing a metric. Let $\mathcal{A}$ and

$\mathcal{B}$ be compact metric spaces with metrics $d_{\mathcal{A}}$ and $d_{\mathcal{B}}$. Define the metric spaces $X_p = \mathcal{A} \times \mathcal{B}$ for $1 \leq p \leq \infty$ by the metrics d_p given by

$$d_p\big((a_1, b_1), (a_2, b_2)\big) = \begin{cases} \left(d_{\mathcal{A}}(a_1, a_2)^p + d_{\mathcal{B}}(b_1, b_2)^p\right)^{1/p} & \text{for } 1 \leq p < \infty \\ \max\left(d_{\mathcal{A}}(a_1, a_2), d_{\mathcal{B}}(b_1, b_2)\right) & \text{for } p = \infty. \end{cases}$$

Theorem 3.1.1. $\dim_b(X_p) = \dim_b(\mathcal{A}) + \dim_b(\mathcal{B})$.

Proof: Since d_∞ is Lipschitz equivalent to d_p for $1 \leq p < \infty$, it follows from property (iii) of Theorem 1.7.3 that

$$\dim_b(X_\infty) = \dim_b(X_p) \qquad \text{for} \qquad 1 \leq p < \infty.$$

Hence, we may prove the theorem for $X = X_\infty$ without loss of generality.

For every α and β such that $\dim_b(\mathcal{A}) < \alpha$ and $\dim_b(\mathcal{B}) < \beta$ there exists $K_{\mathcal{A}}$ and $K_{\mathcal{B}}$ such that

$$\mathcal{N}_{\mathcal{A}}(r, \rho) < K_{\mathcal{A}}\left(\frac{r}{\rho}\right)^{\alpha} \qquad \text{and} \qquad \mathcal{N}_{\mathcal{B}}(r, \rho) < K_{\mathcal{B}}\left(\frac{r}{\rho}\right)^{\beta}$$

for $0 < \rho < r < 1$. Let B be a ball of radius r in X. Then by definition $B = U \times V$ where U and V are balls of radius r. Cover U by balls U_i of radius ρ in $\mathcal{A}$ and V by balls V_j of radius ρ in $\mathcal{B}$. Since $U_i \times V_j$ form a cover of B, it follows that

$$\mathcal{N}_X(r, \rho) \leq \mathcal{N}_{\mathcal{A}}(r, \rho)\mathcal{N}_{\mathcal{B}}(r, \rho) \leq K_{\mathcal{A}} K_{\mathcal{B}}\left(\frac{r}{\rho}\right)^{\alpha+\beta}.$$

Hence $\dim_b(X) \leq \alpha + \beta$ and so $\dim_b(X) \leq \dim_b(\mathcal{A}) + \dim_b(\mathcal{B})$.

Now, let n be the maximum number of disjoint $\rho/2$-radius balls with centers u_i in U. Let U_i be balls of radius ρ with the same centers. Thus the points u_i are at least a distance $\rho/2$ apart from each other and U_i covers U. Do the same for V to obtain m balls V_j of radius ρ whose centers v_j are at least a distance $\rho/2$ apart from each other which cover V. Let $z_k \in B$ be an enumeration of the points (u_i, v_j) and define $Z = \{ z_k : k = 1 \dots nm \}$. It follows that

$$\mathcal{N}_X(r, \rho/4) \geq \mathcal{N}_Z(\rho/2) \geq nm \geq \mathcal{N}_U(\rho)\mathcal{N}_V(\rho).$$

Choose U and V such that $\mathcal{N}_U(\rho)$ and $\mathcal{N}_V(\rho)$ attain their maximum values. Given d such that $\dim_b(X) < d$, there is K such that

$$\mathcal{N}_X(r, \rho) \leq K\left(\frac{r}{\rho}\right)^d \qquad \text{for } 0 < \rho < r < 1.$$

Thus

$$\mathcal{N}_{\mathcal{A}}(r, \rho)\mathcal{N}_{\mathcal{B}}(r, \rho) \leq \mathcal{N}_X(r, \rho/4) \leq 4^d K\left(\frac{r}{\rho}\right)^d,$$

and therefore $\dim_b(\mathcal{A}) + \dim_b(\mathcal{B}) = \dim_b(X)$. □

It is interesting to note that the Hausdorff dimension $\dim_h$ does not behave well under Cartesian products. For example, Falconer [10] exhibits sets $\mathcal{A}$ and $\mathcal{B}$ such that $\dim_h(\mathcal{A}) + \dim_h(\mathcal{B}) < \dim_h(\mathcal{A} \times \mathcal{B})$.

To calculate the Bouligand dimension of a self similar set, note that Hausdorff dimension is bounded by the fractal dimension which is in turn bounded by the Bouligand dimension. Thus given a compact metric space $\mathcal{A}$ we have that

$$\dim_h(\mathcal{A}) \leq \dim_f(\mathcal{A}) \leq \dim_b(\mathcal{A}). \tag{3.1.1}$$

Let $f_i\colon \mathbf{R}^n \to \mathbf{R}^n$ where $i = 1, \ldots, N$ be similarities with ratios $c_i \in (0,1)$ and define $F(M) = \cup f_i(M)$ for $M \subseteq \mathbf{R}^n$. Let $\mathcal{A}$ be the compact invariant set such that $F(\mathcal{A}) = \mathcal{A}$.

Definition 3.1.2. The similarity dimension of $\mathcal{A}$ is defined as $\dim_s(\mathcal{A}) = s$ where $\Sigma c_i^s = 1$.

Theorem 3.1.3. Let $\mathcal{A}$ be defined as above. Then $\dim_b(\mathcal{A}) = \dim_s(\mathcal{A})$.

Proof: Since $\mathcal{A}$ is compact there is ϵ such that $0 < \epsilon < 1$ and

$$\operatorname{dist}\big(f_i(\mathcal{A}), f_j(\mathcal{A})\big) > \epsilon \quad \text{for} \quad i \neq j.$$

Let $c = \min\{\, c_i : i = 1, \ldots, N \,\}$. Let $R > \epsilon/c$ be chosen so large that $\mathcal{A}$ fits inside an R ball. Let r and ρ be chosen so that $0 < \rho < r < \epsilon/2$. Let I be the set of finite sequences $(i_1, \ldots, i_n)$ such that

$$c_{i_1} \cdots c_{i_n} \epsilon \leq 2r < c_{i_1} \cdots c_{i_{n-1}} \epsilon.$$

Since $2r < \epsilon$, then I is non-empty. Furthermore, $\mathcal{A} = \cup\{ f_{i_1} \circ \cdots \circ f_{i_n}(\mathcal{A}) : (i_1, \ldots, i_n) \in I \,\}$. Let $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_m)$ be distinct elements of I. Suppose for definiteness that $n \leq m$. Consider the sets $f_{i_1} \circ \cdots \circ f_{i_n}(\mathcal{A})$ and $f_{j_1} \circ \cdots \circ f_{j_m}(\mathcal{A})$. Let k be the largest index such that $i_l = j_l$ for all $l \leq k$. Obviously $k < n$ for otherwise $k = n$ would imply $n < m$ leading to the contradiction

$$c_{i_1} \cdots c_{i_n} \epsilon \leq 2r < c_{j_1} \cdots c_{j_{m-1}} \epsilon \leq c_{j_1} \cdots c_{j_n} \epsilon = c_{i_1} \cdots c_{i_n} \epsilon.$$

Define $M = f_{i_k+1} \circ \cdots \circ f_{i_n}(\mathcal{A})$ and $N = f_{j_k+1} \circ \cdots \circ f_{j_m}(\mathcal{A})$. Then $i_{k+1} \neq j_{k+1}$ implies $\mathrm{dist}(M, N) > \epsilon$. It follows that

$$\mathrm{dist}\big(f_{i_1} \circ \cdots \circ f_{i_n}(\mathcal{A}), f_{j_1} \circ \cdots \circ f_{j_m}(\mathcal{A})\big) > c_{i_1} \cdots c_{i_k}\epsilon \geq c_{i_1} \cdots c_{i_{n-1}}\epsilon > 2r.$$

Now let B be a ball of radius r. If follows that $B \cap f_{i_1} \circ \cdots \circ f_{i_n}(\mathcal{A}) \neq \emptyset$ for at most one $(i_1, \ldots, i_n) \in I$. Let $(i_1, \ldots, i_n)$ be that one and fix it.

Hence $B \cap \mathcal{A} = B \cap f_{i_1} \circ \cdots \circ f_{i_n}(\mathcal{A})$. Define $\gamma = c_{i_1} \cdots c_{i_n}$ and let J be the set of $(j_1, \ldots, j_m)$ such that

$$R\gamma c_{j_1} \cdots c_{j_m} \leq \rho < R\gamma c_{j_1} \cdots c_{j_{m-1}}.$$

Since $\rho < r \leq \gamma\epsilon/c \leq R\gamma$, then J is non-empty. Since $\mathcal{A} = \cup\{f_{j_1} \circ \cdots \circ f_{j_m}(\mathcal{A}) : (j_1, \ldots, j_n) \in J\}$ it follows that

$$B \cap \mathcal{A} \subseteq \cup\{\, f_{i_1} \circ \cdots \circ f_{i_n} \circ f_{j_1} \circ \cdots \circ f_{j_m}(\mathcal{A}) : (j_1, \ldots, j_m) \in J\,\}.$$

Furthermore, since $R\gamma c_{j_1} \cdots c_{j_m} \leq \rho$, it follows that

$$f_{i_1} \circ \cdots \circ f_{i_n} \circ f_{j_1} \circ \cdots \circ f_{j_m}(\mathcal{A}) \subseteq B_\rho\big(f_{i_1} \circ \cdots \circ f_{i_n} \circ f_{j_1} \circ \cdots \circ f_{j_m}(\alpha)\big).$$

We now estimate the number of elements in J. Since $\Sigma c_i^s = 1$, it follows by induction that $\sum_J (c_{j_1} \cdots c_{j_m})^s = 1$. Therefore $\sum_J \big(c\rho/(R\gamma)\big)^s \leq 1$ and so J has no more than $\big(R\gamma/(c\rho)\big)^s$ elements. Hence, $\mathcal{N}_{B\cap\mathcal{A}}(\rho) \leq \big(R\gamma/(c\rho)\big)^s$. Since $\gamma = c_{i_1} \cdots c_{i_n} \leq 2r/\epsilon$, it follows that $\mathcal{N}_{\mathcal{A}}(r, \rho) \leq K(r/\rho)^s$ where $K = \big(2R/(c\epsilon)\big)^s$. As this holds for all $0 < \rho < r < \epsilon/2$, it further follows that $\dim_b(\mathcal{A}) \leq s$. By [10], Theorem 9.3, the Hausdorff dimension $\dim_h(\mathcal{A}) = s$; therefore, in light of (3.1.1) we obtain that $\dim_b(\mathcal{A}) = s$. $\square$

Thus, for self-similar sets, the Bouligand dimension agrees with the similarity, fractal, and Hausdorff dimensions. In the case that $c_i = c$ for all i, the similarity dimension has an easy to calculate form and we obtain the following corollary:

Corollary 3.1.4. Let $f_i: \mathbf{R}^n \to \mathbf{R}^n$ where $i = 1, \ldots, N$ be similarities with ratio $c \in (0,1)$ and define $F(M) = \cup f_i(M)$ for $M \subseteq \mathbf{R}^n$. Take $\mathcal{A}$ to be the compact invariant set such that $F(\mathcal{A}) = \mathcal{A}$. If the images under the f_i are disjoint then $\dim_b(\mathcal{A}) = -\log(N)/\log(c)$.

3.2. Dimension Increasing Projections

The results in this section are motivated by examples involving orthogonal sequences in a Hilbert space. The fractal dimension of such sequences has been already studied by Ben-Artzi, Eden, Foias and Nicolaenko in [2] and by Ladislav Mišík Jr. and Tibor Žáčik in [18]. In some sense, an orthogonal sequence is the farthest thing possible from the regular self-similar fractals discussed in the previous section. Not surprisingly, it is in the treatment of these sets that the Bouligand dimension differs most dramatically from the fractal dimension. In particular, we exhibit a set $\mathcal{A}$ such that $\dim_b(\mathcal{A}) = 0$ and $\dim_b(\mathcal{A} - \mathcal{A}) = \infty$. This section closes with a proof of Theorem 3.2.9 concerning the existence orthogonal projections that increase Bouligand dimension.

Recall, Bouligand dimension is characterized by a scaling condition between r and ρ. In particular, the number of ρ balls needed to cover a r sized piece should be no different than the number of $\lambda\rho$ balls needed to cover a λr sized piece. This homogeneous scaling makes the Bouligand dimension sensitive to inhomogeneities in the set $\mathcal{A}$. We will consider orthogonal sequences in a Hilbert space converging to zero. Let H be a Hilbert space and e_i an orthonormal sequence.

Before proceeding, it will be usefull to have a simple test to know whether a set has finite Bouligand dimension or not. The following lemma provides a simple test for determining this.

Lemma 3.2.1. If there is some K such that $\mathcal{N}_{\mathcal{A}}(r, r/2) < K$ holds for all $r < 1$ then $\mathcal{A}$ has finite Bouligand dimension. Moreover, $\dim_b(\mathcal{A}) \leq \log_2 K$.

Proof: Given r and ρ such that $0 < \rho < r < 1$ choose n so that $r/2^n \leq \rho < r/2^{n-1}$. Then

$$\mathcal{N}_{\mathcal{A}}(r,\rho) \leq \mathcal{N}_{\mathcal{A}}(r, r/2)\mathcal{N}_{\mathcal{A}}(r/2, r/2^2)\cdots\mathcal{N}_{\mathcal{A}}(r/2^{n-1}, r/2^n) \leq K^n$$

and since $n - 1 \leq \log_2(r/\rho)$ it follows that

$$\mathcal{N}_{\mathcal{A}}(r,\rho) \leq K(K^{n-1}) \leq K(r/\rho)^{\log_2 K}.$$

Therefore $\dim_b(\mathcal{A}) \leq \log_2 K < \infty$. $\square$

Lemma 3.2.2. Let $\mathcal{A} = \{\, 0 \,\} \cup \{\, e_n/n^\alpha : n \in \mathbf{N} \,\}$. Then $\dim_f(\mathcal{A}) = 1/\alpha$ and $\dim_b(\mathcal{A}) = \infty$.

Proof: The first fact appears in [2]. For the second, consider the ball of radius $r = 1/m^\alpha$ centered at the origin

$$B = \{\, a \in \mathcal{A} : \|a\| < r \,\} = \{\, 0 \,\} \cup \{\, e_n/n^\alpha : n > m \,\}.$$

Cover B by $r/2$ balls. Each point a distance more that $r/2$ from the origin will require a separate ball. Since

$$1/n^\alpha > r/2 \qquad \text{implies} \qquad n < (2/r)^{1/\alpha} = m2^{1/\alpha}$$

then

$$\mathcal{N}_{\mathcal{A}}(r, r/2) \geq \mathcal{N}_B(r/2) > m2^{1/\alpha} - m - 1.$$

This is unbounded; therefore, by Lemma 3.2.1 it follows that $\dim_b(\mathcal{A}) = \infty$. ☐

Fact 3.2.3. If there is K and α such that

$$a_n \leq K/n^\alpha$$

then $\dim_f \big(\{\, 0 \,\} \cup \{\, a_n e_n : n \in \mathbf{N} \,\}\big) \leq 1/\alpha$.

The Bouligand dimension of a geometric sequence is finite because geometric sequences have the scaling property needed for $\mathcal{N}_{\mathcal{A}}(r, r/2)$ to be bounded.

Fact 3.2.4. If there is K and α such that $0 < \alpha < 1$ and

$$(1/K)\alpha^n \leq a_n \leq K\alpha^n \tag{3.2.1}$$

then $\dim_b \big(\{\, 0 \,\} \cup \{\, a_n e_n : n \in \mathbf{N} \,\}\big) = 0$.

The lower bound in the inequality (3.2.1) is required. This follows immediately from

Theorem 3.2.5. There exist sequences converging arbitrarily fast to zero that have infinite Bouligand dimension.

Proof: Let b_j be a sequence converging to zero. Consider the sequence $a_n e_n$ where

$$a_n = b_j \qquad \text{for } n = 2^{j-1}, \ldots, 2^j - 1.$$

Let B be the ball of radius $r = b_j + \epsilon$ centered at the origin

$$B = \{\, a_n e_n : n \geq 2^{j-1} \,\}.$$

Cover B by $r/2$ balls. Then

$$\mathcal{N}_{\mathcal{A}}(r, r/2) \geq \mathcal{N}_B(r/2) \geq 2^j - 2^{j-1} = 2^{j-1}$$

is unbounded; therefore, $\dim_b(\mathcal{A}) = \infty$. Since no conditions on b_j were imposed, then a_n may converge arbitrarily fast to zero. □

Corollary 3.2.6. There exist a set $\mathcal{A}$ such that $\dim_b(\mathcal{A}) = 0$ and $\dim_b(\mathcal{A}-\mathcal{A}) = \infty$.

Proof: Let x_j be a sequence of the type given in Theorem 3.2.5 such that $\|x_j\| \leq 4^{-j}$. We assume the complement of the closed linear span of the x_j to be infinite dimensional and define y_j to be an orthogonal sequence in that complement such that $\|y_j\| = 4^{-j}$. Let $\mathcal{A}$ be the closure of the set of all a_j where $a_{2j} = y_j$ and $a_{2j+1} = x_j + y_j$. Clearly the set of differences $\mathcal{A} - \mathcal{A}$ contains the set x_j and therefore $\dim_b(\mathcal{A} - \mathcal{A}) = \infty$.

Claim $\dim_b(\mathcal{A}) = 0$. If $k = 2j$ then

$$\|a_k\| = \|y_j\| = 4^{-j} = 2^{-k}$$

and so condition (3.2.1) is satisfied with $\alpha = 1/2$. If $k = 2j+1$, then

$$\|a_k\| \leq \|x_j\| + \|y_j\| \leq 4^{-j} + 4^{-j} = 2(4^{-(k-1)/2}) \leq 4(2^{-k})$$

and

$$\|a_k\| \geq \|y_j\| = 2(2^{-k}).$$

Therefore, taking $K = 4$ and $\alpha = 1/2$ satisfies (3.2.1) for all terms of the sequence a_j. Applying Fact 3.2.4 now yields that $\dim_b(\mathcal{A}) = 0$. □

We will now use the same scaling properties exploited in the above examples to construct orthogonal projections that increase Bouligand dimension. Note that for fractal and Hausdorff dimensions, the dimension of the projected image is always less than or equal the dimension of the original set. Thus, Bouligand dimension gives us further insight over what can happen to the geometry of a set under orthogonal projection. In particular, there may be bad projections that make a global attractor appear more complicated than it really is. First, we need the following definitions and results.

Definition 3.2.7. Let $\vee\mathcal{A}$ denote the closed linear span of $\mathcal{A}$.

Lemma 3.2.8. There exists a projection Q such that $Q\mathcal{A}$ contains an orthogonal sequence if and only if there exists $V \subseteq \mathcal{A}$ such that $\dim(\vee V) = \infty$ and

$$\vee\big(V \setminus \{v\}\big) \neq \vee V \qquad \text{for all } v \in V.$$

Proof: Suppose there is such a projection Q. Choose $v_n \in \mathcal{A}$ such that the Qv_n form an orthogonal sequence. Define $V = \{ v_n : n \in \mathbf{N} \}$. It follows that Qv is orthogonal to $\vee\big(QV \setminus \{Qv\}\big)$ for any $v \in V$. Therefore $\vee\big(QV \setminus \{Qv\}\big) \neq \vee\big(QV\big)$. Since orthogonal projections are continuous and linear, then Q commutes with $\vee$ and we obtain $Q\big(\vee\big(V \setminus \{v\}\big)\big) \neq Q(\vee V)$, which implies $\vee\big(V \setminus \{v\}\big) \neq \vee V$.

Conversely, by omitting some elements of V if necessary, we write $V = \{ v_n : n \in \mathbf{N} \}$ and suppose the orthogonal complement of $\vee V$ to be infinite dimensional. Let f_i be an orthonormal sequence contained in that complement.

Define $V_1 = V$ and $W_1 = V_1 \setminus \{v_1\}$. For induction suppose $\vee W_n \neq \vee V_n$. Then there exists $e_n \in \vee V_n$ such that $\|e_n\| = 1$ and $e_n \perp \vee W_n$. Let $g_n = (e_n + f_n)/\sqrt{2}$. Define $V_{n+1} = V_n \cup \{g_n\}$ and $W_{n+1} = V_{n+1} \setminus \{v_{n+1}\}$. Clearly $\vee W_{n+1} \neq \vee V_{n+1}$.

The above construction yields an orthonormal sequence g_n such that $(g_i, v_i) \neq 0$ and $(g_i, v_j) = 0$ for $i \neq j$. Let Q be the projection onto the space spanned by the g_n. It follows that $Pv_n = \alpha_n g_n$ where $\alpha_n \neq 0$ and so $QV \subseteq Q\mathcal{A}$ contains an orthogonal sequence. $\square$

Theorem 3.2.9. If $\mathcal{A}$ satisfies Lemma 3.2.8, then there exists an orthogonal projection P such that the Bouligand dimension of $P\mathcal{A}$ is infinite.

Note that Fact 3.2.4 provides an example of a set with finite Bouligand dimension that satisfies Lemma 3.2.8. However, a slight modification of the solution given by Halmos for Problem 11 in [14] yields an infinite dimensional compact set which does not satisfy these hypotheses. It is unknown whether the attractors of naturally occurring physical systems satisfy these hypothesis or not.

Proof of Theorem 3.2.9: By Lemma 3.2.8 there exists an orthogonal projection Q such that $Q\mathcal{A}$ contains the orthogonal sequence $\alpha_n g_n$. Let

$$m_j = \min\{\, |\alpha_n| : n = 2^{j-1}, \ldots, 2^j - 1 \,\}$$

and choose a monotone sequence β_j strictly decreasing to zero such that $\beta_j \leq m_j$. We may assume that the orthogonal complement of $\vee\{\, \alpha_n g_n : n \in \mathbf{N} \,\}$ in QH is infinite. Let h_n be an orthonormal sequence in that complement and define

$$\tilde{g}_n = \sin(\theta_n) g_n + \cos(\theta_n) h_n$$

where θ_n has been chosen in such a way that

$$\alpha_n \sin(\theta_n) = \beta_j \qquad \text{for} \qquad n = 2^{j-1}, \ldots, 2^j - 1.$$

This can be done since $\beta_j < \alpha_n$. Let P be the projection onto the space spanned by the $\tilde{g}_n$. It follows that $P\mathcal{A} = PQ\mathcal{A}$ contains a sequence similar to the one found in Theorem 3.2.5 and therefore

$$\dim_b(P\mathcal{A}) \geq \dim_b\{\, \alpha_n \sin(\theta_n) g_n : n \in \mathbf{N} \,\} = \infty. \qquad \square$$

3.3. The Projection Theorem

In this section we prove Theorem 1.1.2 restated more explicitly in the form of Theorem 3.3.2 below. We want to find rank m orthogonal projections such that

$$\|Px - Py\| \geq \epsilon f\big(\|x - y\|\big) \qquad \text{for } x, y \in \mathcal{A} \tag{3.3.1}$$

where f is linear with a logarithmic correction. Moreover, we show that the measure of all the projections which don't satisfy this condition for any constant $\epsilon > 0$ is zero. Writing $Y = \mathcal{A} - \mathcal{A}$ simplifies (3.3.1) to the requirement that

$$\|Py\| \geq \epsilon f\big(\|y\|\big) \qquad \text{for } y \in Y.$$

Let G be the space of orthogonal projections in $\mathbf{R}^N$ of rank m and μ be the invariant measure on G with respect to orthogonal transformations. Define the shadow of a set B in $\mathbf{R}^N$ to be

$$S(B) = \{\, P \in G : 0 \in PB \,\}.$$

Showing that $\mu\big(S(Y)\big) = 0$ would prove almost every rank m orthogonal projection is injective. To obtain continuity of the inverse we construct a slightly larger set U containing Y and show that $\mu\big(S(U)\big) = 0$. We shall use the following simplified version of Remark 2.3.2 in our computations:

Remark 3.3.1. The measure of the shadow of a ρ-ball B centered a distance r from the origin is bounded by $\mu\big(S(B)\big) \leq C(\rho/r)^m$.

Theorem 3.3.2. Given $\mathcal{A} \subseteq \mathbf{R}^N$ such that $\mathcal{A} \subseteq B_R(0)$ and $\dim_b(\mathcal{A} - \mathcal{A}) < d < m$. Let $\eta > 1$ and define

$$f(x) = x\Big(\frac{1}{\log_2(2R/x)}\Big)^{\eta/(m-d)}$$

where $R = 2\sup\big\{\, \|a\| : a \in \mathcal{A} \,\big\}$. Then for almost every projection P of rank m there is a constant C such that (3.3.1) is satisfied.

Proof: Define $r_n = R/2^n$ and $\rho_n = f(r_{n-1})$. Divide Y into shells

$$Z_n = \{\, y \in Y : r_n \le \|y\| \le r_{n-1} \,\}$$

and cover Z_n by balls of radius ρ_n.

Let $Y = \mathcal{A} - \mathcal{A}$ and choose K such that $\mathcal{N}_{\mathcal{B}}(r,\rho) \le K(r/\rho)^d$ for all $\mathcal{B} \subseteq Y$. Further choose $c > 2$ so large that $c^{m-d} > K$. Cover each ρ_n-ball by ρ_n/c-balls and each of those by ρ_n/c^2-balls and so on. Label the centers of the ρ_n/c^i-balls that cover Z_n by a_{nij}. Thus

$$Z_n \subseteq \textstyle\bigcup_j B_{\rho_n/c^i}(a_{nij})$$

where the index j ranges over at most $K^{i+1}c^{id}(r_{n-1}/\rho_n)^d$ balls.

Since $c > 2$, the covers formed by doubling the radius of the balls in each generation will be nested. Let

$$U_i = \textstyle\bigcup_{n,j} B_{2\rho_n/c^i}(a_{nij})$$

be the union of all the radius doubled balls in the i-th generation. For each $y \in Y$ there is a ball $B_\rho(a)$ of radius $\rho = \rho_n/c^i$ centered at a such that $y \in B_\rho(a)$. If $P \notin S(U_i)$ then $0 \notin PB_{2\rho}(a)$ and hence

$$\|Py\| \ge \rho = \rho_n/c^i = f(r_{n-1})/c^i \ge f\big(\|y\|\big)/c^i$$

shows that (3.3.1) holds for the i-th generation with $\epsilon = 1/c^i$.

Let $U = \limsup_i U_i$. It remains to show the measure of $S(U)$ is zero. Estimate

$$\begin{aligned}
\mu\big(S(U)\big) \le \mu\big(S(U_i)\big) &\le \sum_{n,j} \mu\Big(S\big(B_{2\rho_n/c^i}(a_{nij})\big)\Big) \le \sum_{n,j} C\Big(\frac{2\rho_n}{c^i r_n}\Big)^m \\
&\le \sum_n CK^{i+1}c^{id}\Big(\frac{r_{n-1}}{\rho_n}\Big)^d\Big(\frac{2\rho_n}{c^i r_n}\Big)^m = 4^m CK\Big(\frac{K}{c^{m-d}}\Big)^i \sum_n \Big(\frac{f(r_{n-1})}{r_{n-1}}\Big)^{m-d} \\
&= 4^m CK\Big(\frac{K}{c^{m-d}}\Big)^i \sum_n \frac{1}{n^\eta} = L\Big(\frac{K}{c^{m-d}}\Big)^i
\end{aligned}$$

where L is a constant. Letting i become large, we obtain $\mu\big(S(U)\big) = 0$. Thus, almost every orthogonal projection of rank m satisfies (3.3.1) for some $\epsilon > 0$. $\square$

Chapter 4

Concluding Remarks

This thesis has focused on some theoretical aspects of finding finite dimensional dynamical systems sharing an exponential attractor with a given dissipative partial differential equation. In Chapter 2 we proved in collaboration with Foias [11] the following theorem:

> **Theorem 1.1.1.** Let H be a real Hilbert space and $\mathcal{A} \subseteq H$ be such that $\dim_f(\mathcal{A}) < m/2$. Then for any orthogonal projection P of rank m and $\delta > 0$ there is an orthogonal projection $\tilde{P}$ such that $\|\tilde{P} - P\| < \delta$ and $\tilde{P}|_{\mathcal{A}}$ has Hölder inverse.

This result extends the work of Ben-Artzi, Eden and Foias [2] to infinite dimensional spaces and in light of the work of Eden, Foias, Nicolaenko, and Temam [8] shows that the exponential attractors of infinite dimensional dissipative evolution equations are also the exponential attractors of some finite dimensional ordinary differential equations. Thus, we have given a theoretical justification for the numerical study of dissipative partial differential equations. Using this result in the case that $m = 3$, Robinson [23] has obtained arbitrarily large finite time approximations of all trajectories on the attractor by a smooth three dimensional dynamical system. Ben-Artzi, Eden, Foias and Nicolaenko show in [2] that Hölder continuity for the inverse of $P|_{\mathcal{A}}$ is the best one could hope for under the hypothesis that $\dim_f(\mathcal{A})$ is finite in general. Thus our work appears to complete the theory when all that is known about $\mathcal{A}$ is that it has finite fractal dimension.

In Chapter 3 we studied a stronger notion of dimension defined by Bouligand [3]. Our main goal was to obtain greater regularity for the inverse of the projections appearing in Theorem 1.1.1. Our main result along these lines was a strengthing of the finite dimensional version of Theorem 1.1.1 proved earlier by Ben-Artzi, Eden, Foias and Nicolaenko in [2].

> **Theorem 1.1.2**. Given any compact subset $\mathcal{A} \subseteq \mathbf{R}^n$ such that $\dim_b(\mathcal{A} - \mathcal{A}) < m$, then almost every orthogonal projection P of $\mathcal{A}$ of rank m is injective on $\mathcal{A}$ and $P|_{\mathcal{A}}$ has Lipschitz continuous inverse except for a logarithmic correction term.

At present there are many questions left to answer concerning the application of Bouligand dimension to the study of attractors. I am working on extending Theorem 1.1.2 to cover the case where $\mathcal{A}$ is a subset of an infinite dimensional Hilbert space. It may be possible to do this in a way similar to the way Chapter 2 extends the finite dimensional projection theorem of [2] to produce Theorem 1.1.1. One difficulty in following the exact procedure used there is the fact that, as shown in Theorem 3.2.9, the Bouligand dimension of $\mathcal{A}$ may increase under orthogonal projections. This is probably not the typical case, and I am working on proving a result along these lines. Alternatively, it may be possible to extend Theorem 1.1.2 using the notions of prevalence defined by Hunt, Sauer and Yorke in [16] and recently used by Hunt and Kaloshin [15] to show that the projections of Theorem 1.1.1 are prevalent.

It was shown in [2] using examples of orthogonal sequences that the least upper bound for the Hölder exponent in Theorem 1.1.1 was less than (1.6.1). I am working on obtaining a similar upper bound on the exponent of the logarithmic term for any possible infinite dimensional extension of Theorem 1.1.2.

Finally, I would like to to prove directly that an attractor $\mathcal{A}$ has finite Bouligand dimension in cases where existence of an inertial manifold has not been established. In particular, I think it may be possible to prove that the universal attractor of the Navier-Stokes equation on the sphere has finite Bouligand dimension. In light of Corollary 3.2.6 it would be of further interest to show the Bouligand dimension of $\mathcal{A} - \mathcal{A}$ is also finite. In addition, I would like to find a way of constructing an exponentially attracting forward invariant set from the universal attractor preserving the Bouligand dimension. This may be possible using a method similar to the method used by Eden, Foias, Nicolaenko and Temam in [8] for fractal dimension.

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