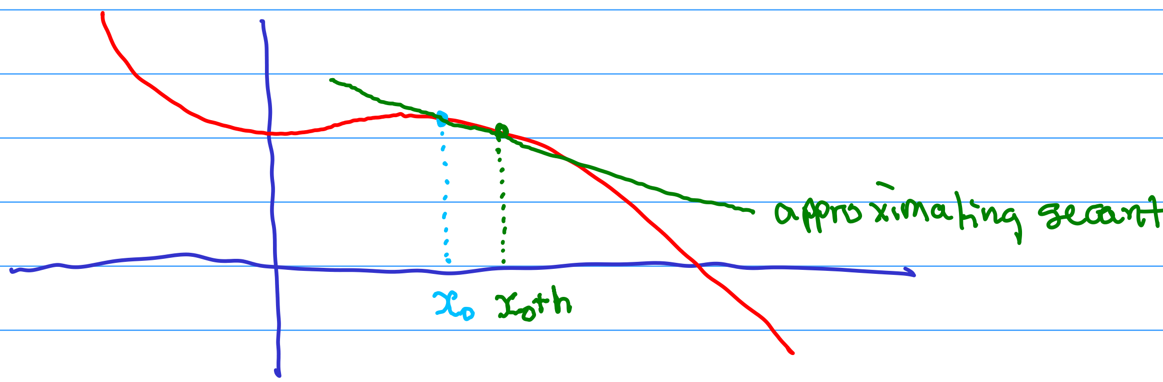
Idea of a limit: Define something through better and better approximations.

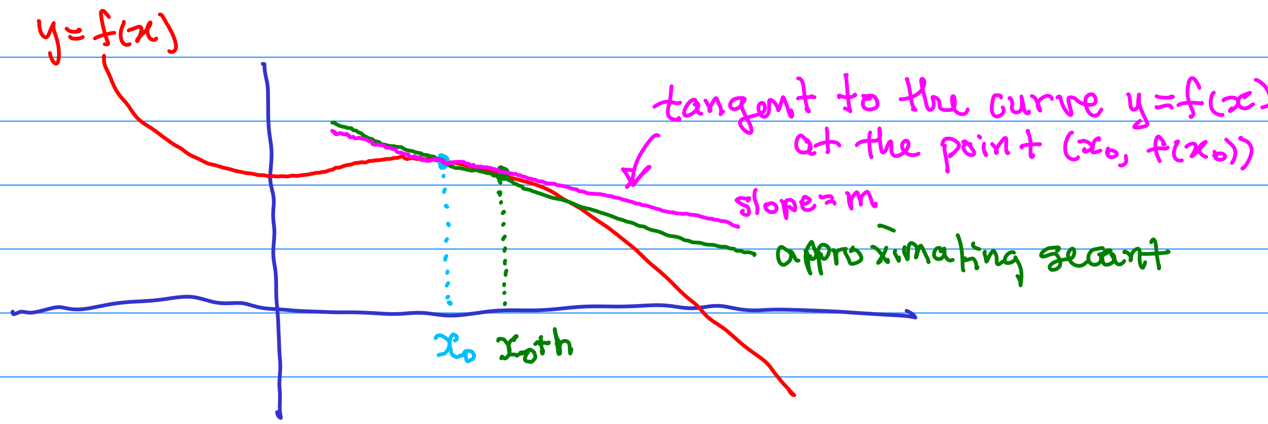
Need control over the quality of the approximations.

Example Tangent lines are defined as the limit of the approximating secant lines.



Note h is the control over the quality of the approx.
As h gets closer to zero the approxating secant line gets closer to the tangent.

So to find the tangent we need the point $(x_0, f(x_0))$ and the slope m of the tangent



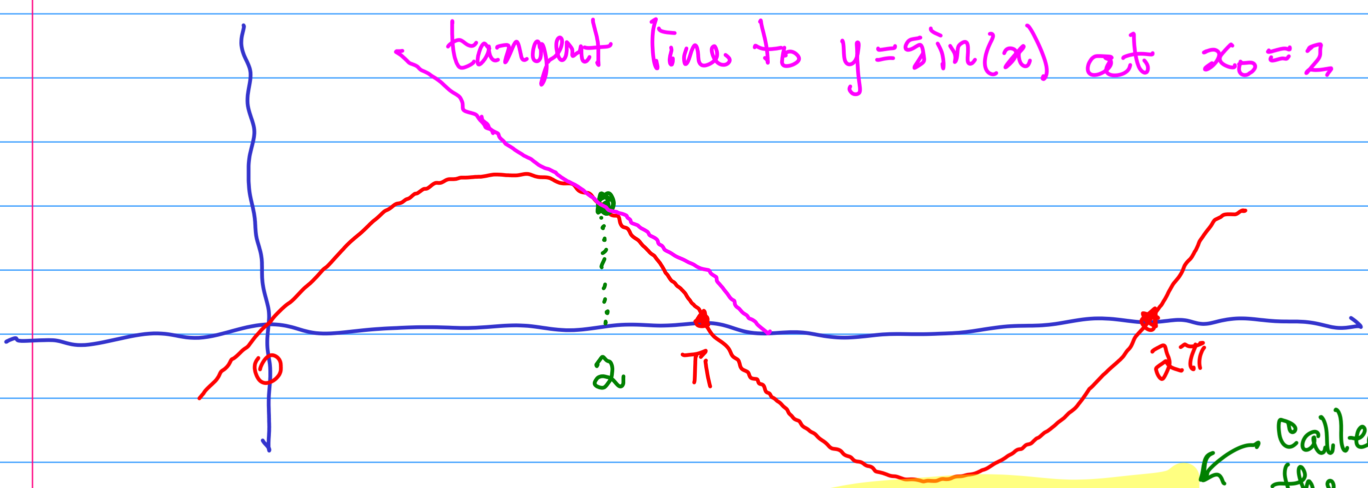
The slopes of the approximating secant lines approximate the slope m of the tangent.

Note the slope of the tangent is called the derivative.

$$\text{slope of the tangent} = m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{(\cancel{x_0}+h) - \cancel{x_0}}$$

h is the parameter controlling the quality of approximation. In this case as h gets close to 0 the approximation gets better.

Example from last time



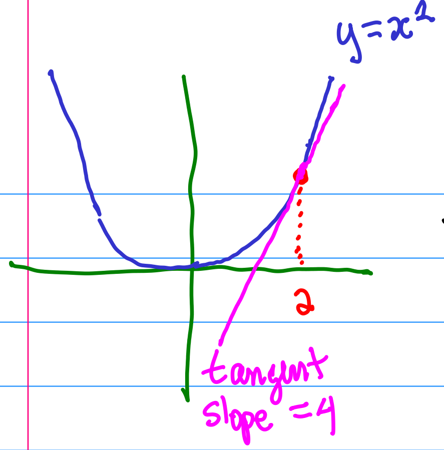
$$\text{slope of tangent} = m = \lim_{h \to 0} \boxed{\frac{f(x_0+h) - f(x_0)}{h}}$$

called the difference quotient

$$= \lim_{h \to 0} \frac{\sin(2+h) - \sin(2)}{h} \approx -0.416\ldots \approx \cos(2)$$

use trig. to figure this out later.

Instead of trigonometry an example that only involves algebra...



$f(x)=x^2$

line tangent to $y=x^2$ at $x_0=2$

in terms of derivatives

$\underbrace{\text{slope of the line tangent}}_{\text{called the derivative of } y=x^2 \text{ at } x_0=2}$ to $y=x^2$ at $x_0=2$

$$\text{slope of tangent} = m = \lim_{h\to 0} \underbrace{\frac{f(x_0+h)-f(x_0)}{h}}$$

↑ limit of slopes of approx. secant lines

$$= \lim_{h\to 0} \frac{(x_0+h)^2 - x_0^2}{h} = \lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h}$$

↑ difference quotient

Could use numerics to see what happens when h gets close to 0. Or use algebra to simplify things so we can see what happens when h gets close to 0.

Simpify the difference quotient

$$\frac{(2+h)^2 - 2^2}{h} = \frac{\cancel{2^2} + 4h + h^2 - \cancel{2^2}}{h} = \frac{4h+h^2}{h} = \frac{(4+h)\cancel{h}}{\cancel{h}} = 4+h$$

As $h \to 0$ then $4+h \to 4$

$$\text{slope of tangent} = \lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h\to 0} 4+h = 4$$

Next time examples for

$$f(x) = \frac{1}{x}, \quad f(x) = \sqrt{x}, \quad f(x) = \frac{1}{x^2}$$