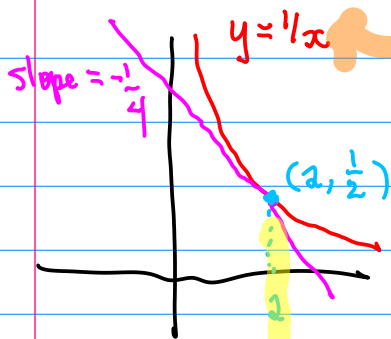


Another example

$$f(x) = \frac{1}{x}$$

$$x_0 = 2$$



what is the slope of that tangent line?

$$m = \text{slope of the tangent} = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$$

algebraic simplifications

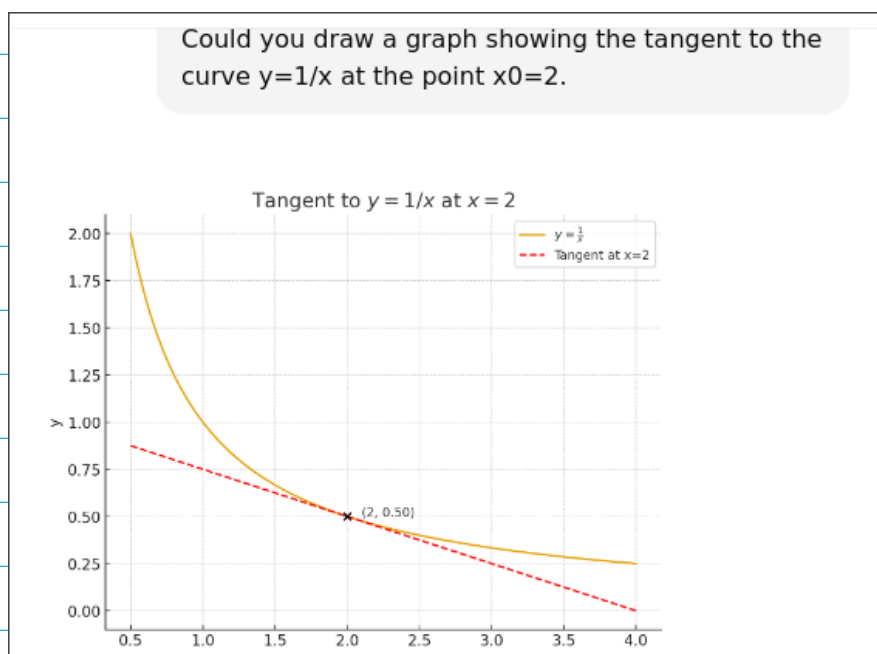
$$\frac{1}{2+h} - \frac{1}{2} = \frac{2 - (2+h)}{2(2+h)} = \frac{-h}{2(2+h)}$$

$$\frac{\frac{-h}{2(2+h)}}{h} = \frac{-1}{2(2+h)} = \frac{-1}{4+2h}$$

Therefore (since $4+2h \rightarrow 4$ as $h \rightarrow 0$ we have)

$$m = \text{slope of the tangent} = \lim_{h \rightarrow 0} \frac{-1}{4+2h} = -\frac{1}{4}$$

graph
that's
more
to
scale..



Here's the graph: the blue curve is $y = \frac{1}{x}$, and the dashed red line is the tangent at $x = 2$. The black dot marks the point $(2, \frac{1}{2})$.

We are talking about derivative, that is, the slope of tangent lines

$$\text{slope} = \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h}$$

difference quotient (slope of secant)

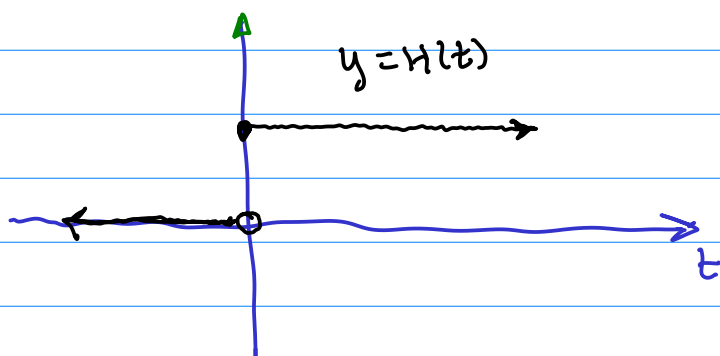
$$Q(h) = \frac{f(x_0+h) - f(x_0)}{h}$$

$$\text{slope} = \lim_{h \rightarrow 0} Q(h)$$

In general can use limits for anything that can be approximated by a function.

Try the function $H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$

- Interpretation of this function is a switch that turn on at time $t=0$.



at the end of
Chapter 2.2
for next time...



Teach me calculus please.



I'd love to help you get started with calculus! To begin, could you tell me if you've studied algebra, geometry, or pre-calculus before? Do you have a particular calculus topic you want to start with, such as limits, derivatives, or integrals?

If you're not sure, we can start with the basics: understanding what calculus is and the two main ideas—differentiation and integration. Would you like to do that?



Generate Examples

Let's start with differentiation. What is that?

Great choice—differentiation is the first big idea in Calculus !!

Think of it as a very precise way to talk about how fast something is changing at an exact moment.

1. Linking to what you already know

- In algebra you found the slope of a straight line with $m = (y_2 - y_1)/(x_2 - x_1)$.
- That slope tells you the constant rate of change of the line.

2. What if the graph isn't a straight line?

- Curves don't have one single slope; the steepness keeps changing.
- Intuitively, we still want "the slope right there."

3. The key Calculus idea

- Take two points on the graph, find the slope of the secant line, then slide the points closer until they merge.
- The limiting value of those slopes—if it exists—is called the derivative at that point.
- In symbols, for a function $f(x)$ at $x = a$,
$$f'(a) = \lim_{h \rightarrow 0} [f(a + h) - f(a)] / h.$$

(We read $f'(a)$ as "f prime of a.")

4. Why it matters

- Physics: velocity is the derivative of position.
- Economics: marginal cost is the derivative of cost.
- Everyday life: the steeper parts of a hill on a bike ride correspond to larger derivatives.

Let's make this interactive:

- Can you picture why shrinking the distance h between the two points should capture an "instantaneous" slope?
- Pick a simple curve you know—say $y = x^2$. What do you guess the slope is at $x = 2$ just by looking at its shape?
- How would you describe, in your own words, why a limit is necessary in the definition above?

Share your thoughts on one (or all) of those questions, and we'll build from there!

We did that last lecture. The slope of the tangent line to the curve $y=x^2$ is 4 at $x_0=2$. Could you explain that to me again because I didn't understand on Wednesday what exactly happened?

Absolutely—let's slow the process down and unpack each little move that turns the “average” slope into that single-point (tangent) slope.

1. Start with the basic picture

- Fix the point P on the curve $y = x^2$ at $x_0 = 2$.
- Pick a nearby point Q whose x-coordinate is $2 + h$. (So h is just a tiny horizontal step.)

2. Write the “average” slope for the secant PQ

- Can you fill in what y-values belong to P and Q?
 - For P: $y = (2)^2 = 4$.
 - For Q: $y = (2 + h)^2 = \dots$ (try expanding this yourself).
- The slope of PQ is
$$m_{\text{secant}} = [y_Q - y_P] / [(2 + h) - 2] = [\dots] / h.$$

Question: after you expand $(2 + h)^2$, what does the numerator simplify to? (Hint: a lot cancels.)

3. Simplify before you “take the limit”

- Once you cancel common factors, you should get an expression that still depends on h .
- What is that simplified expression? Think about whether h is still in the denominator after canceling.

4. Let h shrink to 0

- The tangent slope m_{tan} is the limit of m_{secant} as $h \rightarrow 0$.
- When you plug $h = 0$ into your simplified expression, what number do you get?

Reflect: why is it safe to plug in $h = 0$ only *after* we've canceled the h in the denominator?

5. General insight

- Notice that this particular computation is just a special case of a pattern for $y = x^2$.
- If you repeated the algebra but started at a generic x_0 instead of 2, what function of x_0 do you think you'd get for the derivative?

Give steps 2–4 a try on paper (or in your head if you're comfortable) and tell me:

- a) What simplified expression do you get just before taking the limit?
- b) What number pops out when $h \rightarrow 0$?

Once you post those, we can verify them together and then talk about why that same algebraic trick works for lots of other functions.

Okay. That's fine. Now I want to find the slope of the tangent to $y=1/x$ at $x_0=2$. Could you do that for me?

Here's a road map you can follow without anyone giving away the ending.

1. Underlying concepts and principles • Tangent-line slope at a point is the limit of all nearby “average” slopes (secant slopes).

- For a curve $y = f(x)$ and a fixed x_0 , the average slope between $P(x_0, f(x_0))$ and $Q(x_0 + h, f(x_0 + h))$ is

$$m_{\text{secant}} = [f(x_0 + h) - f(x_0)] / h.$$

- The tangent slope is what m_{secant} approaches as $h \rightarrow 0$:

$$f'(x_0) = \lim_{h \rightarrow 0} m_{\text{secant}}.$$

- The algebra goal is always: (i) substitute your specific f , (ii) simplify the fraction so the pesky h in the denominator cancels, (iii) take the limit.

2. Guiding Questions a) What are the exact y -coordinates of your two points when

$$f(x) = 1/x, \quad x_0 = 2, \quad \text{and} \quad Q \text{ is } (2 + h, f(2 + h))?$$

(Hint: write each y as one simple fraction.)

b) Using those, set up the secant slope formula. Do you see a common denominator that will let you combine the two fractions in the numerator?

c) After you combine them, you should find a single fraction whose top has a factor of h . Can you factor that h out so it cancels with the h in the denominator?

d) Once the h 's cancel, what expression is left to evaluate as h approaches 0?
(Notice you can safely set $h = 0$ now because there's no longer a division by zero.)

e) Before you actually compute that limit, pause: what do you expect the answer to look like—positive or negative? Why does that make sense from the shape of $y = 1/x$?

3. A couple of extra tips (optional) • If combining the two fractions feels messy, cross-multiply to get a common denominator quickly.

- Keep track of signs: $1/(2 + h) - 1/2$ will turn into something like $(2 - (2 + h)) / [2(2 + h)]$. Notice what cancels there!

Give those steps a try on paper and see what number pops out when $h \rightarrow 0$. Let me know what you get (or where it gets sticky) and we'll untangle it together!

I want to find the slope of the tangent to $y=1/x$ at $x_0=2$. Could you do that for me?

Slope of the tangent to $y = \frac{1}{x}$ at $x_0 = 2$:

We can differentiate $y = x^{-1}$:

$$\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}.$$

Evaluate at $x = 2$:

$$m = -\frac{1}{2^2} = -\frac{1}{4}.$$

(If you want the tangent line too: it passes through $(2, \frac{1}{2})$ with slope $-\frac{1}{4}$, so

$$y - \frac{1}{2} = -\frac{1}{4}(x-2) \quad \rightarrow \quad y = \frac{1}{2} - \frac{1}{4}(x-2).$$



Could you work the same problem using limits and an approximation with the slopes of the secant lines?

Thought for 6s

Nice — let's do it with limits and with secant-line approximations.

1) Limit (difference quotient)

Use the definition of derivative at $x_0 = 2$:

$$f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}.$$

Compute the fraction algebraically:

$$\frac{1}{2+h} - \frac{1}{2} = \frac{2 - (2+h)}{2(2+h)} = \frac{-h}{2(2+h)}.$$

Divide by h :

$$\frac{\frac{-h}{2(2+h)}}{h} = \frac{-1}{2(2+h)} = \frac{-1}{4+2h}.$$

Now take the limit as $h \rightarrow 0$:

$$f'(2) = \lim_{h \rightarrow 0} \frac{-1}{4+2h} = -\frac{1}{4}.$$

2) Symmetric difference quotient (optional check)

Using the symmetric secant slope

$$\frac{f(2+h) - f(2-h)}{2h} = \frac{\frac{1}{2+h} - \frac{1}{2-h}}{2h} = \frac{-1}{4-h^2} \xrightarrow{h \rightarrow 0} -\frac{1}{4},$$

same result.


3) Numerical secant approximations

Using the formula $\frac{f(2+h) - f(2)}{h} = -\frac{1}{4+2h}$, plug in small h :

- $h = 0.1 : -\frac{1}{4.2} \approx -0.2380952381$
- $h = 0.01 : -\frac{1}{4.02} \approx -0.2487562189$
- $h = 0.001 : -\frac{1}{4.002} \approx -0.2498750625$

You can see the secant slopes approach -0.25 .

Answer

The slope of the tangent at $x_0 = 2$ is .

(And the tangent line is $y - \frac{1}{2} = -\frac{1}{4}(x - 2)$.)

I wish you hadn't solved the entire problem. Could you ask me a similar problem to find the derivative using limits and the approximation with the slopes of the secant lines?

Got it 👍 I'll hold back on the full solution this time.

Here's a similar problem for you to try:

Problem:

Use the definition of the derivative (limits of secant slopes) to find the slope of the tangent line to

$$y = \frac{1}{x+1}$$

at $x_0 = 3$.

👉 Try setting it up first with the difference quotient, and then compute a few secant slopes for small values of h (like $h = 0.1, 0.01, 0.001$) to approximate the slope numerically.

Would you like me to check your work step by step after you try it?

