Limit Laws Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$

4. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

If f is continuous at $x=a$, then $\lim_{x\to a} f(x) = f(a)$

definition of continuity.

The follow are continuous

x^n is continuous everywhere (here $n=1,2,\ldots$)

☑ polynomial is continuous everywhere

$\left.\begin{array}{l}\sin(x)\\ \cos(x)\end{array}\right\}$ are continuous everywhere

$\ln x$ continuous for $x>0$.

rational functions continuous except where $q(x)=0$

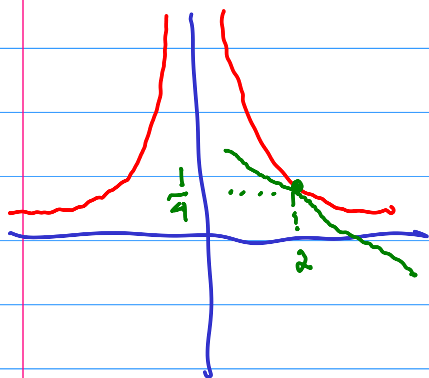
quotient of polynomials $\dfrac{p(x)}{q(x)}$ both p and q are polynomials

$\sqrt[n]{x}$ continuous for $x>0$ when n is even

continuous everywhere when n is odd

$|x|$ is continuous everywhere

Derivative of $f(x) = \frac{1}{x^2}$ at $x_0 = 2$.



slope m of the tangent $= -\frac{1}{4}$

$$m = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{\frac{1}{(2+h)^2} - \frac{1}{2^2}}{h}$$

Simplify

$$\frac{\frac{1}{(2+h)^2} - \frac{1}{2^2}}{h} = \frac{1}{h}\left(\frac{1}{(2+h)^2}\frac{2^2}{2^2} - \frac{1}{2^2}\frac{(2+h)^2}{(2+h)^2}\right)$$

$$= \frac{2^2 - (2+h)^2}{h(2+h)^2 2^2} = \frac{\cancel{4} - (\cancel{4} + 4h + h^2)}{h(2+h)^2 2^2} = \frac{-\cancel{h}(4+h)}{\cancel{h}(2+h)^2 2^2} = \frac{-(4+h)}{4(2+h)^2}$$

Thus

$$m = \lim_{h\to 0} \frac{-(4+h)}{4(2+h)^2} \underset{\text{by 5}}{=} \frac{\lim_{h\to 0}(-1)(4+h)}{\lim_{h\to 0} 4(2+h)^2} = \frac{\overset{\text{by 3}}{(-1)}\lim_{h\to 0}(4+h)}{\underset{\text{by 3}}{4}\lim_{h\to 0}(2+h)^2}$$

Since polynomials are continuous

$$= \frac{(-1)}{4}\frac{(4+0)}{(2+0)^2} = \frac{-1}{2^2} = \frac{-1}{4}$$

One more property of continuous functions:

$$\boxed{\begin{array}{l}\text{If } \lim_{x\to b} g(x) = a \text{ and } f \text{ is continuous at } a, \\ \text{Then } \lim_{x\to b} f(g(x)) = f(a) = f\left(\lim_{x\to b} g(x)\right)\end{array}}$$

2 Theorem If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

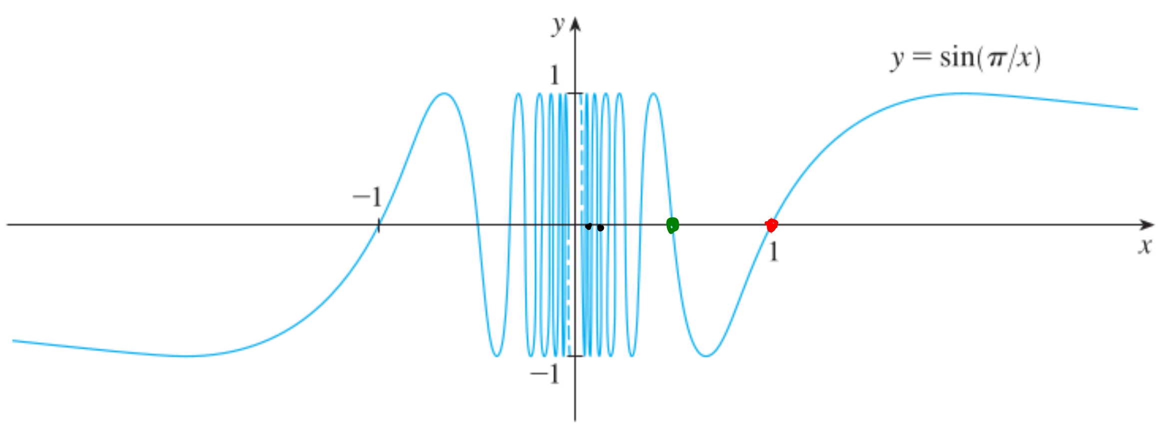
3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

Note that if the limits for f and h exist and are equal then it's guaranteed that the limit of g exists and the same.



x	$\sin\frac{\pi}{x}$
1	$\sin \pi = 0$
$\frac{1}{2}$	$\sin 2\pi = 0$
.1	$\sin 10\pi = 0$
.01	$\sin 100\pi = 0$
.001	$\sin 1000\pi = 0$

but it's clear from the graph that $\sin(\pi/x)$ oscillates between −1 and 1 as $x \to 0$ and so

$\lim_{x \to 0} \sin\frac{\pi}{x}$ = Does not exist

If $f(x) \leqslant g(x) \leqslant h(x)$ whe ← squeeze theorem

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

$$\lim_{x\to a} g(x) = L$$

$$\lim_{x\to 0} x \sin\frac{\pi}{x}$$

$$g(x) = x \sin\frac{\pi}{x}$$

Case $x>0$ then $g(x) = \underset{\text{positive}}{x} \sin\frac{\pi}{x} \leq x$

$$g(x) = x\sin\frac{\pi}{x} \geq -x$$

Thus

$$-x \leq g(x) \leq x \quad \text{for} \quad x>0$$

Case $x<0$ then $g(x) = \underset{\text{negative}}{x} \sin\frac{\pi}{x} \leq -x$

$$g(x) = \underset{\text{negative}}{x} \sin\frac{\pi}{x} \geq x$$

Thus

$$x \leq g(x) \leq -x \quad \text{for} \quad x<0$$

Therefore

$$-|x| = \begin{cases} -x & \text{for } x>0 \\ x & \text{for } x<0 \end{cases} \leq g(x) \leq \begin{cases} x & \text{for } x>0 \\ -x & \text{for } x<0 \end{cases} = |x|$$

If

$$f(x) = -|x|, \quad g(x) = x\sin\frac{\pi}{x}, \quad h(x) = |x|$$

Then $\underbrace{f(x) \leq g(x) \leq h(x)}_{\text{hypothesis of squeeze theorem}}$ for all $x \neq 0$.

If $f(x) \leqslant g(x) \leqslant h(x)$ whe

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

$$\lim_{x \to a} g(x) = L$$

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} |x| = |0| = 0$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} -|x| = 0$$

Therefore $\lim_{x \to 0} g(x) = \lim_{x \to 0} x \sin \frac{\pi}{x} = 0$.