Limit Laws Suppose that c is a constant and the limits

$$① \lim_{x \to a} f(x) = L_1 \quad \text{and} \quad ② \lim_{x \to a} g(x) = L_2$$

exist. Then

1. $③ \lim_{x \to a} [f(x) + g(x)] = \overbrace{\lim_{x \to a} f(x)}^{L_1} + \overbrace{\lim_{x \to a} g(x)}^{L_2} = L_1 + L_2$

Let $\varepsilon > 0$ be arbitrary.

arbitrary so I can choose it

Since $\lim_{x \to a} f(x) = L_1$, then for $\varepsilon_1 = \underline{\varepsilon/2}$

There exists $\delta_1 > 0$ such that

$0 < |x-a| < \delta_1$ implies $|f(x) - L_1| < \varepsilon_1$

Since $\lim_{x \to a} g(x) = L_2$, then for $\varepsilon_2 = \underline{\varepsilon/2}$

There exists $\delta_2 > 0$ such that

$0 < |x-a| < \delta_2$ implies $|g(x) - L_2| < \varepsilon_2$

Choose $\delta = \min(\delta_1, \delta_2)$

Then $0 < |x-a| < \delta$ implies $|f(x) - L_1| < \varepsilon_1$ and $|g(x) - L_2| < \varepsilon_2$

Therefore

triangle inequality

$$|f(x)+g(x)-(L_1+L_2)| \leq |f(x)-L_1| + |g(x)-L_2| < \varepsilon_1 + \varepsilon_2 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Also in the book §2.4.

PROOF OF THE SUM LAW Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) + g(x) - (L + M)| < \varepsilon$$

Using the Triangle Inequality we can write

Limit Laws Suppose that c is a constant and the limits

$$① \lim_{x\to a} f(x) = L_1 \quad \text{and} \quad ② \lim_{x\to a} g(x) = L_2$$

Then $\boxed{\lim_{x\to a} f(x)g(x) = L_1 L_2}$.

Let $\varepsilon > 0$ be arbitrary.

arbitrary so I can choose it

Since $\lim_{x\to a} f(x) = L_1$, then for $\varepsilon_1 = \min\left(1, \dfrac{\varepsilon}{2(|L_2|+1)}\right)$

There exists $\delta_1 > 0$ such that

$0 < |x-a| < \delta_1$ implies $|f(x) - L_1| < \varepsilon_1$

Since $\lim_{x\to a} g(x) = L_2$, then for $\varepsilon_2 = \dfrac{\varepsilon}{2(|L_1|+1)}$

There exists $\delta_2 > 0$ such that

$0 < |x-a| < \delta_2$ implies $|g(x) - L_2| < \varepsilon_2$

Choose $\delta = \min(\delta_1, \delta_2)$. Then $0 < |x-a| < \delta$ implies

$|f(x) - L_1| < \varepsilon_1$ so $-\varepsilon_1 < f(x) - L_1 < \varepsilon_1$

$-|L_1| - \varepsilon_1 < L_1 - \varepsilon_1 < f(x) < L_1 + \varepsilon_1 < |L_1| + \varepsilon_1$

Thus $|f(x)| < |L_1| + \varepsilon_1 < |L_1| + 1$

introduce an intermediate point of comparison...

$$|\underbrace{f(x)}_{\text{engine}}\underbrace{g(x)}_{\text{tires}} - L_1 L_2| = |f(x)g(x) - f(x)L_2 + f(x)L_2 - L_1 L_2|$$

$$\le |f(x)g(x) - f(x)L_2| + |f(x)L_2 - L_1 L_2| = |f(x)||g(x) - L_2| + |f(x) - L_1||L_2|$$

$$< |f(x)|\varepsilon_2 + \varepsilon_1 |L_2| < (|L_1|+1)\varepsilon_2 + \varepsilon_1(|L_2|+1) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Continuity:

2.5 Continuity

If f is continuous at $x=a$, then $\lim_{x\to a} f(x) = f(a)$

definition of continuity.

The follow are continuous

means for every $\varepsilon>0$ there is a $\delta>0$ such that

if $\boxed{0<}|x-a|<\delta$ then $|f(x)-f(a)|<\varepsilon$.

don't need this condition

already know where the bullseye is...

If $x=a$ then $f(x)=f(a)$ and $|f(x)-f(a)|<\varepsilon$.

Simplified definition of continuity:

f is continuous at a

means for every $\varepsilon>0$ there is a $\delta>0$ such that

if $|x-a|<\delta$ then $|f(x)-f(a)|<\varepsilon$.

made hypothesis weaker means continuity is a stronger notion than existance of a limit.

from lecture 5

One more property of continuous functions:

If ① $\lim_{x\to b} g(x) = a$ and f is continuous at a,

Then ② $\lim_{x\to b} f(g(x)) = f(a)$ ~~$= f(\lim_{x\to b} g(x))$~~

If $\lim_{x\to b} g(x) = a$ and f is continuous at a

then $\lim_{x\to b} f(g(x)) = f(a)$.

Let $\varepsilon > 0$ be arbitrary. (The ε for hitting $f(a)$ target)

Since f is continuous at a then for $\varepsilon_2 =$ ________

there is $\delta_2 > 0$ such that $|y-a| < \delta_2$ implies $|f(y) - f(a)| < \varepsilon_2$

Since $\lim_{x\to b} g(x) = a$ then for $\varepsilon_1 =$ ________

there is $\delta_1 > 0$ such that $0 < |x-b| < \delta_1$ implies $|g(x) - b| < \varepsilon_1$

Finish this next time...