If $\lim_{x\to b} g(x) = a$ and f is continuous at a

then $\lim_{x\to b} f(g(x)) = f(a)$.

Let $\varepsilon > 0$ be arbitrary. (The ε for hitting $f(a)$ target)

Since f is continuous at a then for $\varepsilon_2 = \underline{\ \varepsilon\ }$

there is $\delta_2 > 0$ such that $|y-a| < \delta_2$ implies $|f(y)-f(a)| < \varepsilon_2$

Since $\lim_{x\to b} g(x) = a$ then for $\varepsilon_1 = \underline{\ \delta_2\ }$

there is $\delta_1 > 0$ such that $0 < |x-b| < \delta_1$ implies $|g(x)-a| < \varepsilon_1$

Choose $\delta = \delta_1$. Then $0 < |x-b| < \delta$ implies $|g(x)-a| < \varepsilon_1 = \delta_2$

Let $y = g(x)$. Then $|y-a| < \delta_2$ implies $|f(y)-f(a)| < \varepsilon_2 = \varepsilon$.

Since $0 < |x-b| < \delta$ implies $|y-a| < \delta_2$ implies $|f(y)-f(a)| < \varepsilon$,

therefore $|f(g(x))-f(a)| = |f(y)-f(a)| < \varepsilon$.

8 Theorem If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

finished.

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Why?

take $b = g(a)$ in Theorem 8.

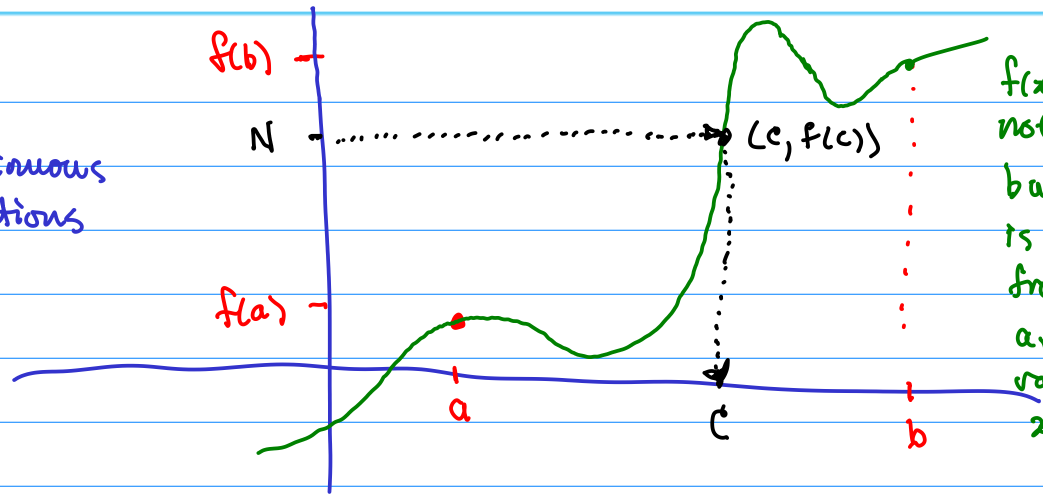
$\lim_{x\to a} f(g(x)) = f(b). = f(g(a))$

Intermediate value theorem

function is defined on entire interval [a,b]

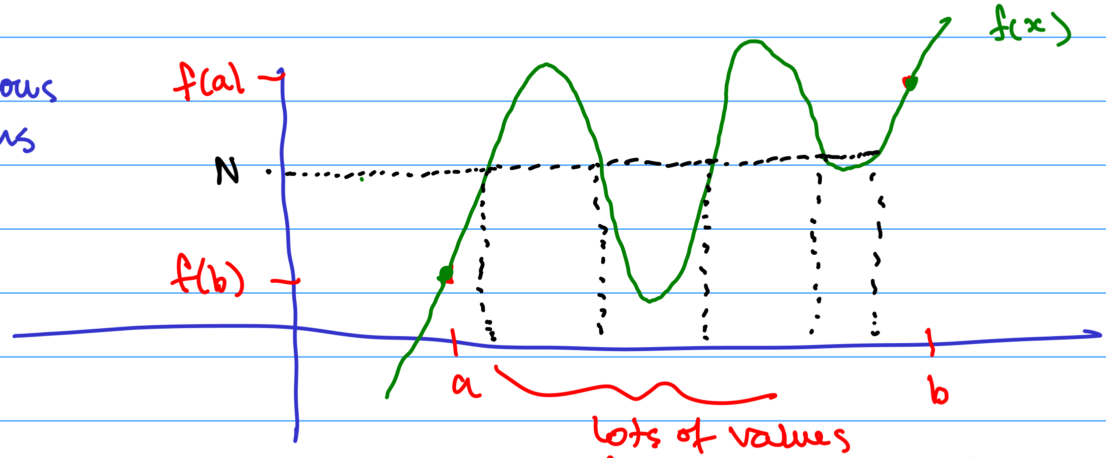
> **10 The Intermediate Value Theorem** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Continuous functions



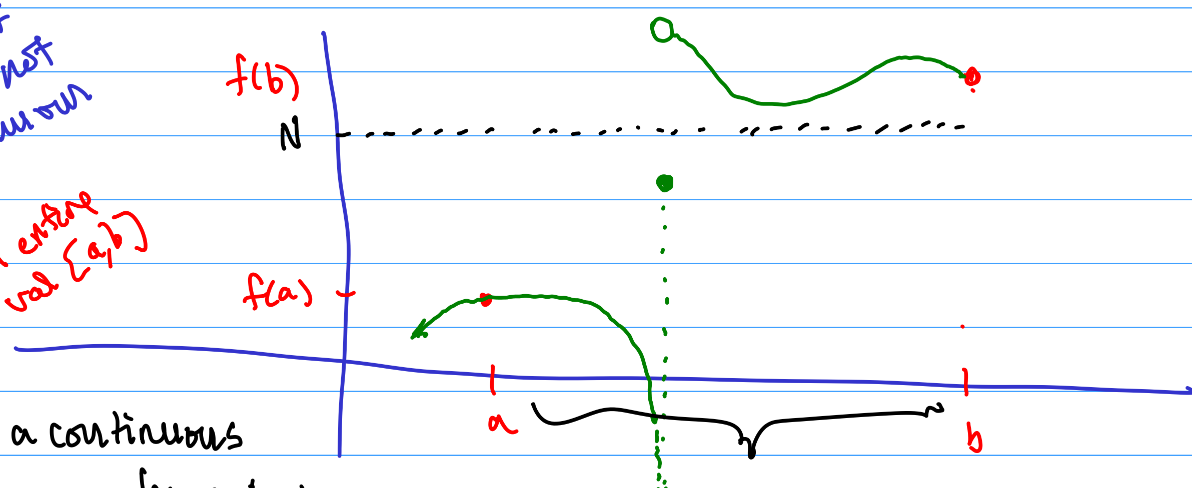
f(x) is not invertible..., but the argument is like starting from the y-axis and finding the value c on the x-axis.

Continuous functions



lots of values for c that give f(c) = N.

What if f is not continuous

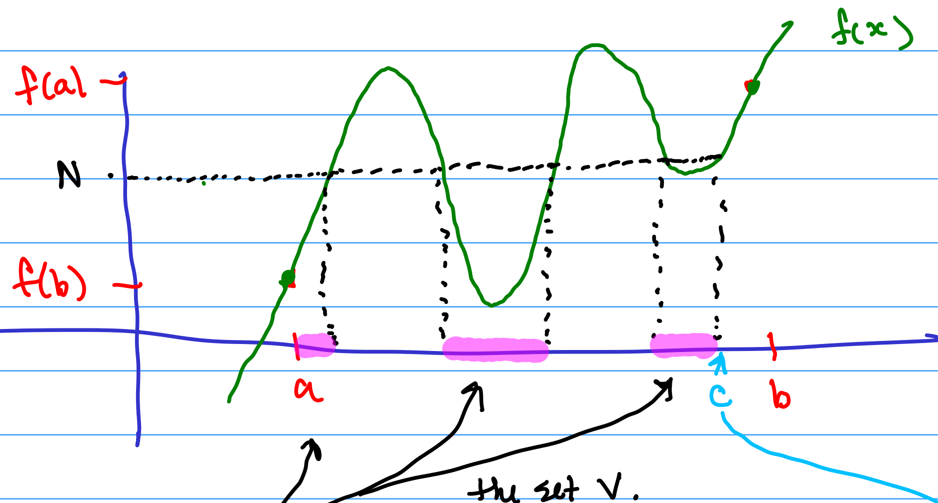


function is defined on entire interval [a,b]

Intuitively a continuous function is one whose graph can be drawn without lifting my pencil.

no value of c such that f(c) = x.

Continuous functions



Idea define a set

$$V = \{ x \in (a,b) : f(x) < N \}$$

let c be the least upper bound of V

Is it really true that $f(c) = N$?

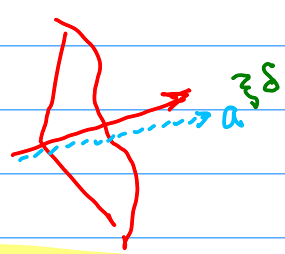
use limits for this

Think of approximations for $x < c$

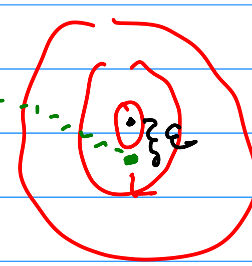
and approximations for $x > c$.

Limit in terms of inequalities involving ε and δ

$$\lim_{x \to a} f(x) = L$$



aim with precision δ
to hit the bullseye
within a tolerance of ε



For every $\varepsilon > 0$ there is $\delta > 0$ such that

If $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

approximating parameter

approximation.

3 Definition of Left-Hand Limit

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a \quad \text{then} \quad |f(x) - L| < \varepsilon$$

$0 < |x-a| < \delta$

$x < a$

4 Definition of Right-Hand Limit

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a < x < a + \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

$0 < |x-a| < \delta$

$x > a$