2 **Precise Definition of a Limit** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

3 **Definition of Left-Hand Limit**

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \qquad a - \delta < x < a \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

4 **Definition of Right-Hand Limit**

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \qquad a < x < a + \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

6 **Precise Definition of an Infinite Limit** Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad f(x) > M$$

a way to approximate ∞

no matter how big... it's bigger than that

M

f(x)

7 Definition Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that for every negative number N there is a positive number δ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad f(x) < N$$

no matter how negative... it's smaller than that

N

f(x)

7 Precise Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

8 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

(limit at $-\infty$)

$$\lim_{x \to -\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x < N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

9 Definition of an Infinite Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

infinity means successively approximating by bigger and bigger...

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \quad \text{then} \quad f(x) > M$$

→ use one intuitive fact $\lim_{x\to\infty} \frac{1}{x} = 0$

§2.6

15. $\lim_{x\to\infty} \frac{3x-2}{2x+1}$

use algebra then apply limit laws...

$$\frac{3x-2}{2x-1}\cdot\frac{1/x}{1/x} = \frac{3-\frac{2}{x}}{2-\frac{1}{x}} = \frac{3-2\cdot\boxed{\frac{1}{x}}}{2-\boxed{\frac{1}{x}}}$$

limit laws...

15. $\lim_{x\to\infty} \frac{3x-2}{2x+1} = \lim_{x\to\infty} \frac{3-2\cdot\frac{1}{x}}{2-\frac{1}{x}} = \frac{\lim_{x\to\infty}\left(3-2\cdot\frac{1}{x}\right)}{\lim_{x\to\infty}\left(2-\frac{1}{x}\right)}$

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty} 2\cdot\frac{1}{x}}{\lim_{x\to\infty} 2 - \lim_{x\to\infty}\frac{1}{x}} = \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty} 2 \lim_{x\to\infty}\cdot\frac{1}{x}}{\lim_{x\to\infty} 2 - \lim_{x\to\infty}\frac{1}{x}}$$

$$= \frac{3-2\cdot 0}{2-0} = \frac{3}{2}$$

Short cut related to orders of infinity

$\alpha > 1/2 \quad \beta > \alpha \quad b > 2$

$$\log x \ll \sqrt{x} \ll \underbrace{x^{\alpha} \ll x^{\beta}}_{\text{need this right now}} \ll 2^{x} \ll b^{x} \ll x^{x}$$

ranking of how fast these function tend to ∞ as $x\to\infty$.

15. $\lim_{x\to\infty} \frac{3x-2}{2x+1} =$

← pick function that goes fastest to ∞ in numerator and denominator

$$\lim_{x\to\infty}\frac{3x-2}{2x+1} = \lim_{x\to\infty}\frac{3\cancel{x}}{2\cancel{x}} = \frac{3}{2}$$

22. $\displaystyle\lim_{x\to\infty}\frac{x^2}{\sqrt{x^4+1}}$ ← square root here...

algebra... mult to prevent the numerator and denomitor both going to ∞.

$$\frac{x^2}{\sqrt{x^4+1}}\cdot\frac{1/x^2}{1/x^2} = \frac{1}{\sqrt{1+1/x^4}}$$

Now

Limit law

$$\lim_{x\to\infty}\frac{x^2}{\sqrt{x^4+1}} = \lim_{x\to\infty}\frac{1}{\sqrt{1+1/x^4}} = \frac{1}{\lim_{x\to\infty}\sqrt{1+1/x^4}}$$

Since $\sqrt{\cdot}$ is continuous

$$\lim_{x\to\infty}\sqrt{1+1/x^4} = \sqrt{\lim_{x\to\infty}(1+1/x^4)}$$

also

$$\lim_{x\to\infty}(1+1/x^4) = 1+\lim_{x\to\infty}\frac{1}{x^4} = 1+\lim_{x\to\infty}\left(\frac{1}{x}\right)^4$$

Since x^4 is continuous...

$$\lim_{x\to\infty}\left(\frac{1}{x}\right)^4 = \left(\lim_{x\to\infty}\frac{1}{x}\right)^4 = 0^4 = 0$$

22. $\lim_{x \to \infty} \dfrac{x^2}{\sqrt{x^4+1}} = \dfrac{1}{\lim_{x\to\infty} \sqrt{1 + 1/x^4}} = \dfrac{1}{\sqrt{1+0}} = 1.$

27. $\lim_{x \to \infty} \left(\sqrt{9x^2+x} - 3x\right)$ (do at home)

Similar

$$\lim_{x\to\infty} \left(\sqrt{x^2+3x} - x\right)$$

can't apply the limit law $\lim(f(x)-g(x)) = \lim f(x) - \lim g(x)$

algebra and then use limit laws...

make a difference of square...

$$\left(\sqrt{x^2+3x} - x\right) \cdot \frac{\sqrt{x^2+3x}+x}{\sqrt{x^2+3x}+x} = \frac{x^2+3x-x^2}{\sqrt{x^2+3x}+x}$$

$$= \frac{3x}{\sqrt{x^2+3x}+x} \cdot \frac{1/x}{1/x} = \frac{3}{\sqrt{1+3/x}+1}$$

(big jump)

$$\lim_{x\to\infty} \left(\sqrt{x^2+3x} - x\right) = \lim_{x\to\infty} \frac{3}{\sqrt{1+3/x}+1} = \frac{3}{2}$$

(could put additional discussion of limit laws here...)

With better idea of limits... revisit finding slopes of tangent line...

Sections 2.7, 2.8 look like lectures 2, 3, 4

Next time!