How about $f(x) = \dfrac{g(x)}{x}$ what is $f'(x)$?

note $x=0$ doesn't make sense in $f(x)$ so we don't try to find $f'(x)$ when $x=0$

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} = \lim_{h \to 0} \frac{\frac{g(x+h)}{x+h} - \frac{g(x)}{x}}{h}$$

limit definition of derivative

Algebra so we can see what happens when $h \to 0$.

common denominator | multiply out

$$\frac{\frac{g(x+h)}{x+h} - \frac{g(x)}{x}}{h} = \frac{x g(x+h) - (x+h) g(x)}{h(x+h)x}$$

$$= \frac{x g(x+h) - x g(x) - h g(x)}{h(x+h)x} = \frac{x(g(x+h) - g(x))}{h(x+h)x} - \frac{h g(x)}{h(x+h)x}$$

$$= \frac{1}{x+h} \frac{g(x+h) - g(x)}{h} - \frac{g(x)}{(x+h)x}$$

Therefore

$$f'(x) = \lim_{h \to 0} \left(\frac{1}{x+h} \frac{g(x+h) - g(x)}{h} - \frac{g(x)}{(x+h)x} \right)$$

limit laws

assumed $x \neq 0$

$$= \lim_{h \to 0} \frac{1}{x+h} \underbrace{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}_{\text{derivative of } g} - \lim_{h \to 0} \frac{g(x)}{(x+h)x}$$

$$= \frac{1}{x} \cdot g'(x) + \frac{g(x)}{x^2}$$

In summary...

$$\boxed{\text{If } f(x) = \frac{g(x)}{x} \text{ then } f'(x) = \frac{1}{x} g'(x) - \frac{1}{x^2} g(x)}$$

From before

$$\boxed{\text{If } f(x) = x g(x) \text{ then } f'(x) = x g'(x) + g(x)}$$

the explanation from before...

$$f'(x) = x \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} \boxed{g(x+h)}$$

assuming g is cont.

$$= x\, g'(x) + g(x)$$

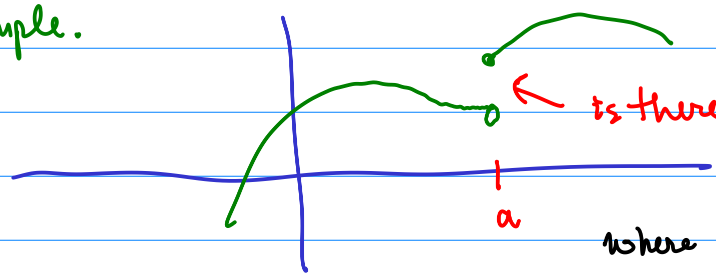
looks like an extra assumption, but it's not why?

It always the case that a differentiable function is continuous...

> **4 Theorem** If f is differentiable at a, then f is continuous at a.

Intuitive... if f is not continuous at a then

Example.



is there a tangent to the curve $y=f(x)$ at $x=a$..

where would I even draw that tangent:.

there is not a tangent at a.

> **4 Theorem** If f is differentiable at a, then f is continuous at a. ☑

Direct explanation using limits...

Need to show $\lim_{x\to a} f(x) = f(a)$.

Hypothesis $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists.

$$\lim_{x\to a} f(x) = \lim_{x\to a}\big(f(x)-f(a)+f(a)\big) = f(a) + \lim_{x\to a}\big(f(x)-f(a)\big)$$

$$= f(a) + \lim_{x\to a} \underbrace{\frac{f(x)-f(a)}{x-a}(x-a)}_{\text{because } x\neq a\ldots} = f(a) + \underbrace{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}_{\text{hypothesis}} \underbrace{\lim_{x\to a}(x-a)}_{\text{Cont. func.}}$$

$$= f(a) + f'(a)\underbrace{(a-a)}_{\text{zero}} = f(a)$$

In summary...

$$\boxed{\text{If } f(x)=\frac{g(x)}{x} \text{ then } f'(x)=\frac{1}{x}g'(x)-\frac{1}{x^2}g(x)}$$

From before

$$\boxed{\text{If } f(x)=xg(x) \quad \text{then} \quad f'(x)=xg'(x)+g(x)}$$

If $f(x)=\dfrac{1}{g(x)}$ How are the derivatives $f'(x)$ and $g'(x)$ related?

$$\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}} = \lim_{h\to 0}\frac{\frac{1}{g(x+h)}-\frac{1}{g(x)}}{h}$$

do algebra...

common denominator

$$\frac{\frac{1}{g(x+h)}-\frac{1}{g(x)}}{h} = \frac{g(x)-g(x+h)}{h\,g(x)\,g(x+h)} = \frac{-1}{g(x)g(x+h)}\cdot\frac{g(x+h)-g(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{-1}{g(x)g(x+h)}\cdot\frac{g(x+h)-g(x)}{h} = \frac{-1}{\lim_{h\to 0} g(x)g(x+h)}\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

need that $g(x)\neq 0$. that's okay because $f(x)=1/g(x)$ had to make to begin with

since $g'(x)$ is assumed to exist then we know g is continuous at x

$$=\frac{-1}{g(x)g(x)}\cdot g'(x) = \frac{-g'(x)}{(g(x))^2}$$

In summary

$$\boxed{\text{If } f(x)=\frac{1}{g(x)} \quad \text{then} \quad f'(x)=\frac{-g'(x)}{(g(x))^2}}$$

And also ... summary...

$$\boxed{\text{If } f(x) = \frac{g(x)}{x} \text{ then } f'(x) = \frac{1}{x} g'(x) - \frac{1}{x^2} g(x)}$$

From before

$$\boxed{\text{If } f(x) = x\,g(x) \qquad \text{then} \qquad f'(x) = x\,g'(x) + g(x)}$$

two functions multiplied together...

If $f(x) = u(x)\,v(x)$

a product of two functions. Note that $h(x)$ is a confusing notation for a function since h is already used as the parameter in the secant line approximations...

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\overbrace{u(x+h)v(x+h) - u(x)v(x)}^{\text{difference between two products...}}}{h}$$

need algebra

from lecture 9 ...

introduce an intermediate point of comparison...

$$|\underset{\text{engine}}{f(x)}\underset{\text{tires}}{g(x)} - h_1 h_2| = |f(x)g(x) - f(x)h_2 + f(x)h_2 - h_1 h_2|$$

$$\leq |f(x)g(x) - f(x)h_2| + |f(x)h_2 - h_1 h_2| = |f(x)|\,|g(x) - h_2| + |f(x) - h_1|\,|h_2|$$

$$< |f(x)|\,\varepsilon_2 + \varepsilon_1 |h_2| < (|h_1|+1)\varepsilon_2 + \varepsilon_1(|h_2|+1) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$$\frac{u(x+h)v(x+h) - u(x)v(x)}{h} = \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}$$

$$\frac{u(x+h)v(x+h) - u(x+h)v(x)}{h} + \frac{u(x+h)v(x) - u(x)v(x)}{h}.$$