If $f(x) = u(x)\,v(x)$

a product of two functions. Note that $h(x)$ is a confusing notation for a function since h is already used as the parameter in the secant line approximations...

difference between two products...

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\overbrace{u(x+h)\,v(x+h)} - \overbrace{u(x)\,v(x)}}{h}$$

Do some algebra to figure out the limit

intermediate point of comparison

$$\frac{u(x+h)\,v(x+h) - u(x)\,v(x)}{h} = \frac{u(x+h)\,v(x+h) \overbrace{- u(x+h)\,v(x) + u(x+h)\,v(x)} - u(x)\,v(x)}{h}$$

$$= \frac{u(x+h)\,v(x+h) - u(x+h)\,v(x)}{h} + \frac{u(x+h)\,v(x) - u(x)\,v(x)}{h}.$$

$$= u(x+h)\,\frac{v(x+h) - v(x)}{h} + \frac{u(x+h) - u(x)}{h}\,v(x)$$

$$f'(x) = \lim_{h \to 0} \left(u(x+h)\,\frac{v(x+h) - v(x)}{h} + \frac{u(x+h) - u(x)}{h}\,v(x) \right)$$

$$= \lim_{h \to 0} \underbrace{u(x+h)}_{u \text{ is continuous}} \; \underbrace{\lim_{h \to 0} \frac{v(x+h) - v(x)}{h}}_{\text{derivative of } v} + \underbrace{\lim_{h \to 0} \frac{u(x+h) - u(x)}{h}}_{\text{derivative of } u} \; \lim_{h \to 0} \underbrace{v(x)}_{\text{const with respect to } h.}$$

$$= u(x)\,v'(x) + u'(x)\,v(x).$$

Thus if $f(x) = u(x)\,v(x)$ then $f'(x) = u(x)\,v'(x) + u'(x)\,v(x)$.

$f'(x) = u'(x)\,v(x) + u(x)\,v'(x)$

Leibnitz notation $\frac{d}{dx}\left(u(x)v(x)\right) = u'(x)\,v(x) + u(x)\,v'(x)$

product rule

Chain rule: $f(x) = u(v(x))$

$$f'(x) = \lim_{h\to 0} \frac{u(v(x+h)) - u(v(x))}{h}$$

simplifying assumption

- $v(x+h) \neq v(x)$ when $h \neq 0$.
- Consequently the only time $v(x+h) = v(x)$ when h is close to 0 is when $h = 0$.

$$f'(x) = \lim_{h\to 0}\left(\frac{u(v(x+h)) - u(v(x))}{v(x+h) - v(x)} \cdot \frac{v(x+h) - v(x)}{h}\right)$$

$$= \lim_{h\to 0} \frac{u(v(x+h)) - u(v(x))}{v(x+h) - v(x)} \quad \underbrace{\lim_{h\to 0} \frac{v(x+h) - v(x)}{h}}_{v'(x)\text{ derivative of } v}$$

work on the first term separately...

$$v(x+h) = v(x) + k$$

$y = v(x)$ and $k = v(x+h) - v(x)$

$k \to 0$ is equivalent to $h \to 0$ by the assumption...

$$\lim_{h\to 0} \frac{u(v(x+h)) - u(v(x))}{v(x+h) - v(x)} = \underbrace{\lim_{k\to 0} \frac{u(y+k) - u(y)}{k}}_{\text{derivative of } u} = u'(y) = u'(v(x))$$

3.4 The Chain Rule

Chain rule...

If $f(x) = u(v(x))$ then $f'(x) = u'(v(x))\,v'(x)$

removing the simplifying assumption is possible and done at the end of the section at the end of §3.4

PROOF OF THE CHAIN RULE Suppose $u = g(x)$ is differentiable at a and $y = f(u)$ is differentiable at $b = g(a)$. If Δx is an increment in x and Δu and Δy are the corresponding increments in u and y, then we can use Equation 7 to write

8 $$\Delta u = g'(a)\,\Delta x + \varepsilon_1\,\Delta x = [g'(a) + \varepsilon_1]\,\Delta x$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$. Similarly

9 $$\Delta y = f'(b)\,\Delta u + \varepsilon_2\,\Delta u = [f'(b) + \varepsilon_2]\,\Delta u$$

Leibnitz notation

$$\boxed{\frac{d}{dx}\left(u(v(x))\right) = u'(v(x))\,v'(x)}$$

What is the chain in the chain rule?

$$\frac{d}{dx}\left(u(v(x))\right) = u'(v(x))\,v'(x)$$

$$\frac{d}{dx}\left(u(v(w(x)))\right) = u'(v(w(x)))\,v'(w(x))\,w'(x)$$

$$\frac{d}{dx}\left(u(v(w(p(x))))\right) = u'(v(w(p(x))))\,v'(w(p(x)))\,w'(p(x))\,p'(x)$$

Exponential function (3.1) Chapter 1 first

Recall e was defined in term of the continuous limit of the compound interest formula in a different class...

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

how many times per year you compound the interest...

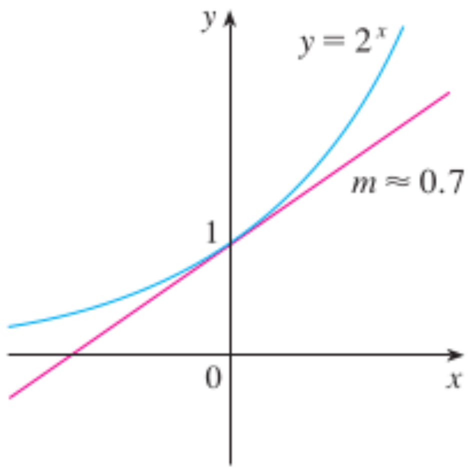

FIGURE 13

tangent to exponential function that have a known base.

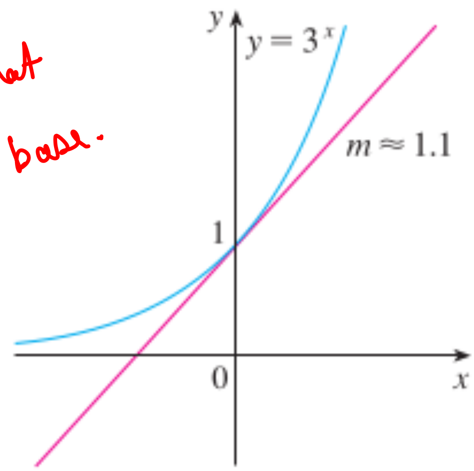

FIGURE 14

Since the slope of the tangent to a^x gets steeper when a gets larger, then there is some number between 2 and 3 where the tangent at $x=0$ has slope equal 1.

by definition that number is called e.

$$\boxed{f(x) = e^x \quad \text{then} \quad f'(0) = 1}$$

defines what e is...

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^h - e^0}{h} = \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

we can use this to compute $f'(x)$ for any x...

properties of exponents

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x$$

Therefore $\boxed{\text{if } f(x) = e^x \text{ then } f'(x) = e^x}$ ☑

Find the derivatives of other exponential functions

$y=2^x$ $\quad$ $y=3^x$ $\quad$ using rules of calculus...

(chain rule)

$$y=2^x$$
$$\ln y = \ln 2^x = x\ln 2$$
$$y = e^{\ln y} = e^{x\ln 2}$$

view as a composition of two functions.

$$f(x)=2^x = \boxed{e^{x\ln 2}}$$
$$= u(x\ln 2) = u(v(x))$$

$u(x)=e^x$ $\qquad$ $u'(x)=e^x$

$v(x)=x\underbrace{\ln 2}_{\text{const...}}$ $\qquad$ $v'(x)=\ln 2$

$$f'(x) = u'(v(x))\, v'(x) = u'(x\ln 2)\ln 2 = e^{x\ln 2}\ln 2 = 2^x \ln 2$$

Leibnitz

$$\boxed{\frac{d}{dx}2^x = 2^x \ln 2}$$

$$\frac{d}{dx}3^x = 3^x \ln 3$$

in general $\quad \dfrac{d}{dx}a^x = a^x \ln a$.