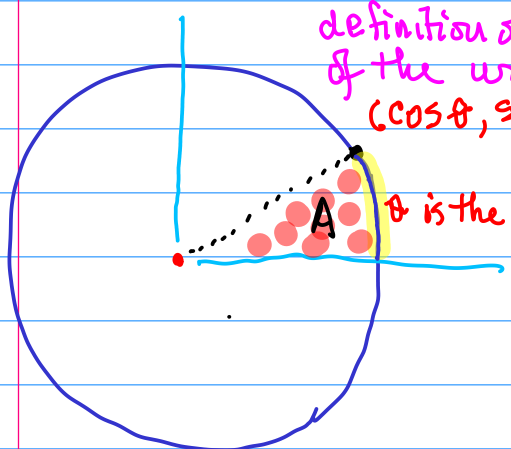
Derivatives of sine and cosine

remember trigonometry: Angle addition formula.

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

definition of cosine and sine ... the parameterization of the unit circle in terms of arc length

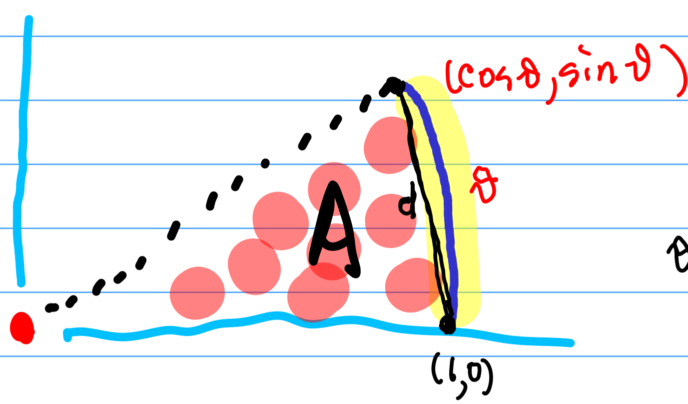


Area $A = \frac{1}{2}\theta$

since this is unit circle the distance from any point on the circle to the center is 1. Thus

$$\sqrt{(\cos\theta)^2 + (\sin\theta)^2} = 1$$

$$\cos^2\theta + \sin^2\theta = 1^2 = 1$$



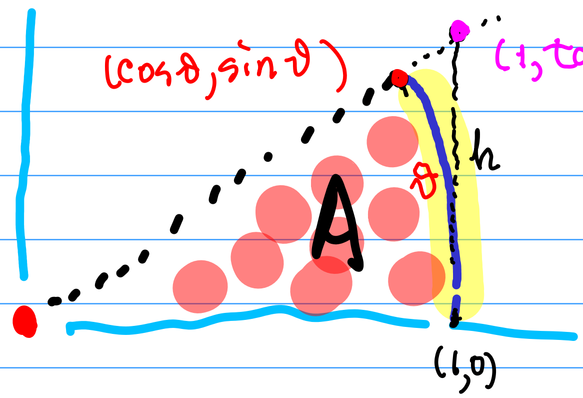
the arclength is longer than the straight line

$$\theta \geq d = \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}$$

$$\theta^2 \geq \underbrace{(1-\cos\theta)^2}_{\text{positive}} + \underbrace{(\sin\theta)^2}_{\text{positive}} \geq (\sin\theta)^2$$

Therefore

$\sin\theta \leq \theta$ or $\frac{\sin\theta}{\theta} \leq 1$ for $\theta > 0$ and θ not too big.



$$\frac{(\cos\theta, \sin\theta)}{\cos\theta} = \left(1, \frac{\sin\theta}{\cos\theta}\right)$$

$$= (1, \tan\theta)$$

length of the curve is less than the length h of the other line.

$$\theta \leq h = \sqrt{(1-1)^2 + (\tan\theta - 0)^2} = \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Therefore

$$\cos\theta \leq \frac{\sin\theta}{\theta} \qquad \text{for } \theta > 0 \text{ and not too big.}$$

From before

$$\frac{\sin\theta}{\theta} \leq 1 \qquad \text{for } \theta > 0 \text{ and not too big.}$$

Combining these gives

$$\underbrace{\cos\theta}_{f} \leq \underbrace{\frac{\sin\theta}{\theta}}_{g} \leq \underbrace{1}_{h}$$

from lecture 5

hypothesis of squeeze theorem

If $f(x) \leqslant g(x) \leqslant h(x)$ whe

$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

$\lim_{x\to a} g(x) = L$

$\lim_{x\to 0} h(x) = \lim_{x\to 0} |x|$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} -|x$

Since cosine is continuous

$$\lim_{\theta\to 0^+} \cos\theta = \cos 0 = 1$$

$$\lim_{\theta\to 0^+} 1 = 1$$

So $L = 1$ and it follows from the squeeze theorem that

$$\boxed{\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1}$$

What about when $\theta < 0$? What happens.

$t = -\theta$ then $t \to 0^+$ as $\theta \to 0^-$

$\theta = -t$

Then
$$\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = \lim_{t \to 0^+} \frac{\sin(-t)}{-t} = \lim_{t \to 0^+} \frac{-\sin t}{-t} = \lim_{t \to 0^+} \frac{\sin t}{t} = 1$$

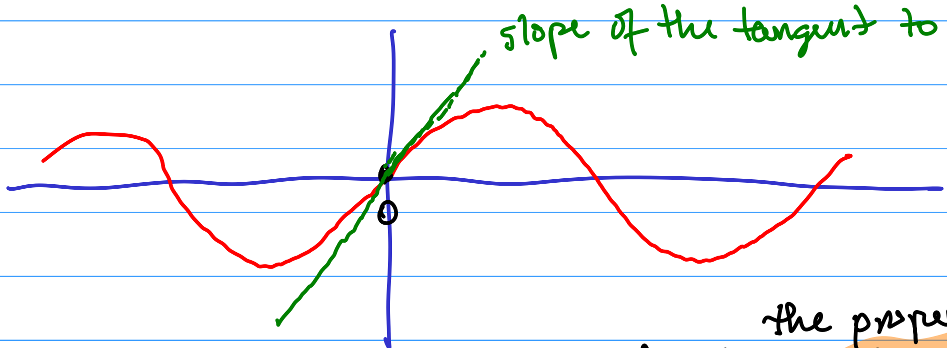
We know sine function has odd symmetry so $\sin(-t) = -\sin(t)$

It follows that
$$\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$$

Let $f(x) = \sin x$ and find the derivative at $x = 0$.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\sin h - \sin 0}{h} = \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

slope of the tangent to $\sin(x)$ at $(0,0)$ is 1



the properties of exponential functions allowed determining the derivative of e^x at points other that $x=0$

properties of exponents

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$
$$= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x$$

angle addition formula...

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

recall

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$f'(x) = \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \boxed{\sin x} \frac{\cos h - 1}{h} + \lim_{h \to 0} \frac{\sin h}{h} \boxed{\cos x}$$

$$= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \boxed{\lim_{h \to 0} \frac{\sin h}{h}}$$

this is the limit we were working on...

Do some algebra to find the other limit

mult. to make a difference of squares...

$$\frac{\cos h - 1}{h} = \frac{(\cos h - 1)}{h} \cdot \frac{(\cos(h) + 1)}{\cos(h) + 1} = \frac{\cos^2 h - 1}{h(\cos(h) + 1)}$$

by the distance formula

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta - 1 = -\sin^2\theta$$

$$\frac{\cos h - 1}{h} = \frac{-\sin^2 h}{h(\cos h + 1)} = -\frac{\sin h}{h} \cdot \frac{\sin h}{\cos h + 1}$$

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = -\lim_{h \to 0} \left(\frac{\sin h}{h}\right) \lim_{h \to 0} \frac{\sin h}{\cos h + 1} = -1 \cdot \frac{\sin 0}{\cos 0 + 1} = -1 \cdot \frac{0}{2} = 0$$

cont at $h = 0$

$$f'(x) = \sin x \boxed{\lim_{h \to 0} \frac{\cos h - 1}{h}} + \cos x \boxed{\lim_{h \to 0} \frac{\sin h}{h}} = \cos x$$

1

Leibnitz notation $\frac{d \sin x}{dx} = \cos x$