means $x = \tan y$

$\frac{d \arctan x}{dx} = \frac{dy}{dx}$

implicit differentiation

$x = \tan y$ think of $y = y(x)$ differentiate both sides of equation

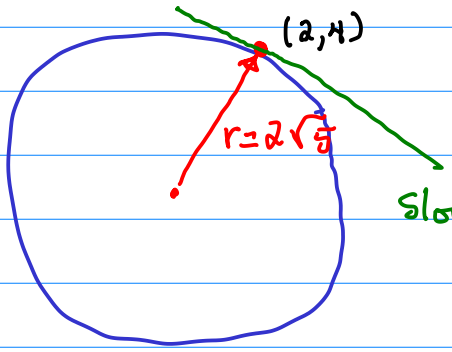
$\frac{dx}{dx} = \frac{d}{dx} \tan y$ chain rule

$1 = (\sec^2 y) \frac{dy}{dx}$ $\quad \sec^2 y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec^2 y}$ end with implicit differentiation

Consider any function $y = y(x)$ implicitly defined by an equation.

$x^2 + y^2 = r^2$ defines a function $y = y(x)$



$r = \sqrt{2^2 + 4^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

Slope of the line tangent to the circle at (2,4)

$x^2 + y^2 = r^2$

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(r^2)$ const radius

Power rule

$\frac{dx^2}{dx} = 2x$

$\frac{dx^2}{dx} + \frac{dy^2}{dx} = 0$

Chain rule

$\frac{dy^2}{dx} = \frac{d}{dx} u(v(x)) = u'(v(x)) v'(x) = 2 v(x) v'(x) = 2y \frac{dy}{dx}$

where $v(x) = y(x)$ $\quad v'(x) = y'(x) = \frac{dy}{dx}$

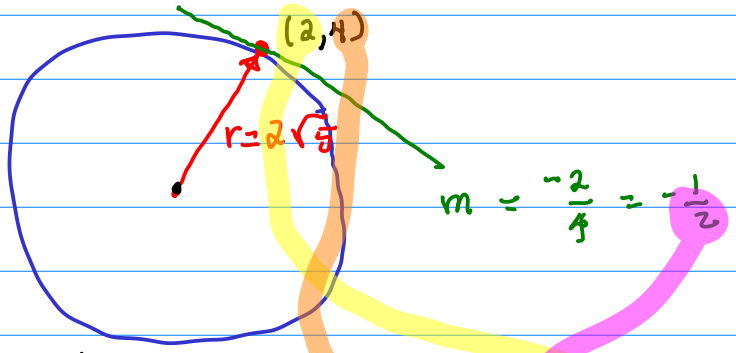
$u(x) = x^2$ $\quad u'(x) = 2x$

because of the way the chain rule works this is always a linear equation in dy/dx

$2x + 2y \frac{dy}{dx} = 0$ ← solve for $\frac{dy}{dx}$

$2y\frac{dy}{dx} = -2x$ → divide

$\frac{dy}{dx} = \frac{-x}{y}$.



Find equation for tangent line..

point-slope form

$y - y_0 = m(x - x_0)$

$y - 4 = -\frac{1}{2}(x - 2)$

25–32 Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

28. $x^2 + 2xy + 4y^2 = 12$, (2, 1) (ellipse)

supposed to be on the ellipse

I thought it would nice to have a graph of the ellipse...

$2^2 + 2\cdot2\cdot1 + 4\cdot1^2 = 4+4+4 = 12$

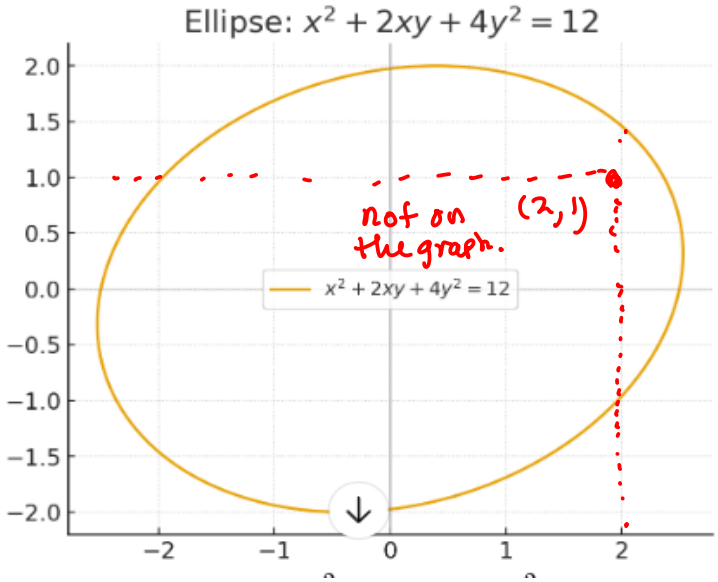
Can you complete the square to write the ellipse given by $x^2+2xy+4y^2=12$ into standard form?

Can you solve for X and Y in terms of the original variables?

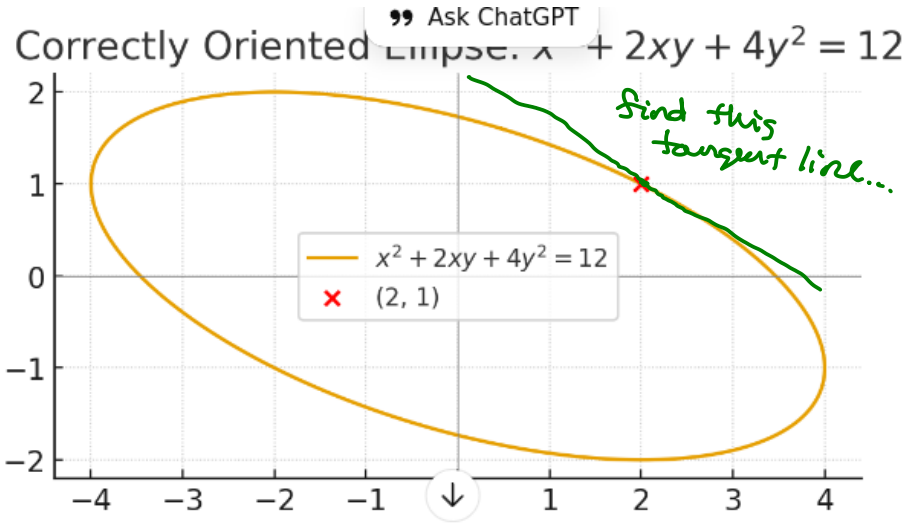
Could you please draw a graph of the ellipse?

Is the point (2,1) on the ellipse?

It does not look like (2,1) is on the ellipse in the graph. Could you check whether the graph is correct.



Please replot it correctly.



Actually wrote a program to draw the graph algorithmically...

```
import numpy as np
import matplotlib.pyplot as plt

# Given coefficients
A, B, C = 1, 2, 4
rhs = 12

# Correct rotation angle
theta = 0.5 * np.arctan2(B, A - C)

# Compute sin and cos
cos_t, sin_t = np.cos(theta), np.sin(theta)

# Compute rotated coefficients
A prime = A*cos t**2 + B*cos t*sin t + C*sin t**2
```

$$\frac{d\,2xy}{dx} = \frac{d}{dx} f(x)g(x)$$

$f(x) = 2x$ $\quad g(x) = y(x)$

$f'(x) = 2$ $\quad g'(x) = dy/dx$

$$= f'(x)g(x) + f(x)g'(x)$$

$$= 2\,y(x) + 2x\,dy/dx$$

Implicit differentiation

$$x^2 + 2xy + 4y^2 = 12,$$

$$\frac{d}{dx}\left(x^2 + 2xy + 4y^2\right) = \frac{d}{dx}\,12,$$

prod rule

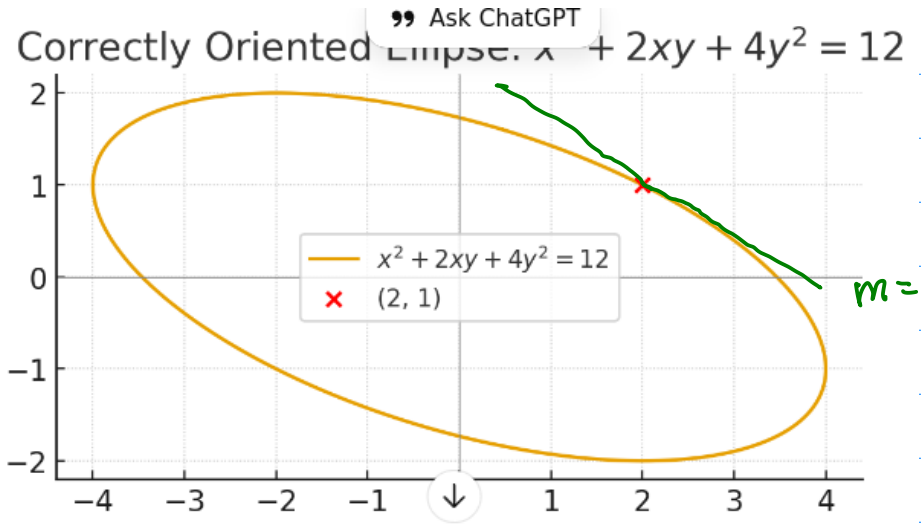
$$2x + 2y + 2x\frac{dy}{dx} + 4\cdot 2y\frac{dy}{dx} = 0$$

linear equation in dy/dx

Solve for dy/dx

$$\frac{dy}{dx} = -\frac{2x + 2y}{2x + 8y}$$

$$\left.\frac{dy}{dx}\right|_{(2,1)} = -\frac{2\cdot 2 + 2\cdot 1}{2\cdot 2 + 8\cdot 1} = -\frac{6}{12} = -\frac{1}{2}$$



$m = -\frac{1}{2}$

Tangent line: $y - 1 = -\frac{1}{2}(x - 2)$

Related Rates Story Problems using the chain rule.

derivative with respect to time.

EXAMPLE 1 Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm³/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

$\frac{dV}{dt}$ dV/dt

$\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$

$r = \frac{50 \text{ cm}}{2} = 25 \text{ cm}$

dr/dt

presumably memorized formula

$$V = \frac{4}{3}\pi r^3$$

$V = V(t)$ $r = r(t)$ both V and r are viewed as functions of time.

$$\frac{d}{dt}V = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt}$$

plug in the data from the story

$$100 = 4\pi(25)^2 \frac{dr}{dt}$$

Now solve for dr/dt.

$$\frac{dr}{dt} = \frac{100}{4\pi(25)^2} = \frac{1}{\pi}\frac{10^2}{2^2 25^2} = \frac{1}{\pi}\left(\frac{10}{2 \cdot 25}\right)^2 = \frac{1}{25\pi} \text{ cm/s}$$

5.5