Called this part 2 of the Fundamental Theorem of Calculus.

Net Change Theorem The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Change of variables for integrals (Chain rule in reverse)

4 The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

chain...

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

20. $\displaystyle\int \frac{z^2}{z^3+1}\,dz$

Find an antiderivative.

derivative

$$\frac{z^2}{z^3+1} = f(g(z))g'(z)$$

For working the integral puzzles...

- Idea look for a function $g(z)$ such that $g'(z)$ is also in the expression

- Idea look for a composition of functions.

Try substituting $u = z^3+1$ or $g(z) = z^3+1$

$\frac{du}{dz} = 3z^2$ $\qquad g'(z) = 3z^2$

$$\ldots = \frac{1}{3}\int \frac{3z^2}{z^3+1}\,dz = \frac{1}{3}\int \frac{g'(z)}{g(z)}\,dz = \frac{1}{3}\int f(g(z))\,g'(t)\,dz$$

$f(u) = \frac{1}{u}$ $\qquad f(g(z)) = \frac{1}{g(z)}$

change of variables

$$= \frac{1}{3}\int f(u)\,du = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln u + C = \frac{1}{3}\ln(z^3+1) + C$$

answer.

improve this answer

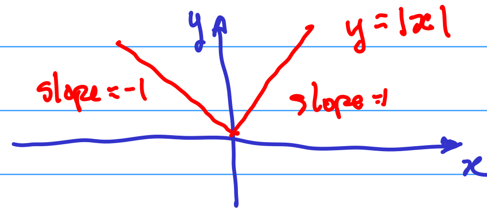
$$\int \frac{z^2}{z^3+1}\,dz = \frac{1}{3}\ln(z^3+1) + C$$

Where did the absolute values come from?

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

This says $\frac{d}{dx}(\ln|x| + C) = \frac{1}{x}$ for x in the domain that is $x \neq 0$.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



$$\frac{d|x|}{dx} = \begin{cases} \frac{dx}{dx} = 1 & \text{if } x > 0 \\ -\frac{dx}{dx} = -1 & \text{if } x < 0 \end{cases}$$

note function is not differentiable at $x = 0$

Chain rule..

$$\frac{d\ln|x|}{dx} = \frac{1}{|x|}\frac{d|x|}{dx} = \frac{1}{|x|}\begin{Bmatrix} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 1 \end{Bmatrix} = \begin{cases} \frac{1}{|x|} & \text{if } x > 0 \\ \frac{-1}{|x|} & \text{if } x < 0 \end{cases}$$

since $x > 0$ implies $|x| = x$

& if $x < 0$ then $|x| = -x$.

$$= \begin{Bmatrix} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{x} & \text{if } x < 0 \end{Bmatrix} = \frac{1}{x}$$

$$\frac{d\ln|x|}{dx} = \frac{1}{x} \quad \text{for any } x \neq 0.$$

Improved antiderivative that works as long as $z^3+1 \neq 0$.

$$\int \frac{z^2}{z^3+1}\,dz = \frac{1}{3}\ln|z^3+1| + C$$

Another example:

- Idea look for a function $g(z)$ such that $g'(z)$ is also in the expression
- Idea look for a composition of functions.

60. $\int_0^1 xe^{-x^2}\,dx$

see a composition e^u where $u = -x^2$

match

(or $f(u) = e^u$ and $g(x) = -x^2$)

$g'(x) = -2x$

$u = -x^2$ so $\frac{du}{dx} = -2x$

what to do with this...

Change of variables...

recall

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

The notation work out so you can substitute for dx using the $\frac{du}{dx} = \dots$ form.

$a=0$, $b=1$

$g(a) = -0^2 = 0$

$g(b) = -(1^2) = -1$

$\frac{du}{dx} = -2x$ so $du = -2x\,dx$

match

60. $\frac{1}{-2}\int_0^1 -2xe^{-x^2}\,dx = \frac{1}{-2}\int_0^{-1} e^u\,du = \frac{1}{-2}e^u\Big|_0^{-1} = \frac{1}{-2}\left(e^{-1} - e^0\right)$

$= \frac{-1}{2}\left(\frac{1}{e} - 1\right) = \frac{1}{2}\left(1 - \frac{1}{e}\right)$

what about this (extra

↓ not here...

$f(g(x))g'(x)$

68. $\displaystyle\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

$u = 1+2x$

solve here for x

$u-1 = 2x$

$\frac{u-1}{2} = x$

$\frac{du}{dx} = 2$ so $du = 2dx$

$dx = \frac{1}{2}du$

$u(0) = 1 \qquad u(4) = 1+2\cdot4 = 9$

$$= \int_1^9 \frac{\left(\frac{u-1}{2}\right)}{\sqrt{u}}\,\frac{1}{2}du$$

$$= \frac{1}{4}\int_1^9 \frac{u-1}{\sqrt{u}}\,du = \frac{1}{4}\int_1^9 \left(u^{1/2} - u^{-1/2}\right)du$$

$$= \frac{1}{4}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right)\Big|_1^9 = \frac{1}{4}\left(\left(\frac{2}{3}3^3 - 2\cdot3\right) - \left(\frac{2}{3} - 2\right)\right)$$

$$= \frac{1}{4}\left(18 - 6 - \frac{2}{3} + 2\right) = \frac{1}{4}\left(14 - \frac{2}{3}\right) = \frac{1}{4}\,\frac{40}{3} = \frac{10}{3}$$

$\begin{array}{r} {}^{1}14 \\ \times\; 3 \\ \hline 42 \end{array}$