1.2 Initial Value Problems

Example:

$$\underbrace{y' + 5y = 7}_{\text{first order linear ODE.}} \qquad \underbrace{y(2) = 0}_{\text{initial value..}}$$

Solving an equation with one derivative give one constant of integration. The initial value can be used to solve for that constant.

Abstract:

$$y' = f(x,y) \qquad y(x_0) = y_0$$

In the above case $f(x,y) = 7 - \underbrace{5y}$, $x_0 = 2$, $y_0 = 0$

linear in this case but doesn't need to be...

Another example:

$$\underbrace{y'' + 5(y')^2 - 4y = e^x}_{\text{2nd order nonlinear ODE}} \qquad \underbrace{y(3) = 1, \quad y'(3) = 4}_{\text{initial value}}$$

Solving an equation with two derivatives give two constants of integration. The initial value can be used to solve for those constants.

Abstract

$$y'' = f(x, y, y') \qquad y(x_0) = y_0, \quad y'(x_0) = y_1$$

Here $f(x,y,y') = e^x - 5(y')^2 + 4y$, $x_0 = 3$, $y_0 = 1$, $y_1 = 4$.

Curious fact: all differential equation of any order can be written as a system of first order differential equation:

$$y'' = f(x,y,y') \qquad y(x_0) = y_0, \quad y'(x_0) = y_1$$

Let $v = y'$ then $v' = y''$

We get the system

$$\begin{cases} y' = v \\ v' = f(x,y,v) \end{cases}$$

Idea of vector algebra is to turn a system of equations into one vector equation...

$$Y = \begin{bmatrix} y \\ v \end{bmatrix}, \quad Y' = \begin{bmatrix} y' \\ v' \end{bmatrix} \qquad Y(x_0) = \begin{bmatrix} y(x_0) \\ v(x_0) \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

$$F(x,Y) = \begin{bmatrix} v \\ f(x,y,v) \end{bmatrix}$$

Vector-valued first order ordinary differential equation

$$\underbrace{Y' = F(x,Y)}_{\text{ODE}} \qquad \underbrace{Y(x_0) = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}}_{\text{initial value..}} = Y_0$$

THEOREM 1.2.1 Existence of a Unique Solution

Let R be a rectangular region in the xy-plane defined by $a \le x \le b$, $c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f/\partial y$ are continuous on R, then there exists some interval I_0: $(x_0 - h, x_0 + h)$, $h > 0$, contained in $[a, b]$, and a unique function $y(x)$, defined on I_0, that is a solution of the initial-value problem (2).

This existence and uniqueness. theorem applies to the initial value problem

$y' = f(x,y)$ such that $y(x_0) = y_0$.

as well as to the vector valued first order ODE above.

We can prove the above theorem using calculus and it takes about a day. ($\frac{1}{2}$ a day for each part) existence & uniqueness

In Problems 1 and 2, $y = 1/(1 + c_1 e^{-x})$ is a one-parameter family of solutions of the first-order DE $y' = y - y^2$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition.

1. $y(0) = -\frac{1}{3}$ **2.** $y(-1) = 2$

$\underbrace{y' = y - y^2}_{\text{ODE}}$ $\underbrace{y(0) = -\frac{1}{3}}_{\text{initial value}}$

I know because the book told me that

$y = \frac{1}{1+c_1 e^{-x}}$ is a general solution to the ODE.

Goal is to find a unique solution so $y(0) = -\frac{1}{3}$.

$$y(0) = \frac{1}{1+c_1 e^{-0}} = \frac{1}{1+c_1} = -\frac{1}{3}$$

Thus $1 + c_1 = -3$ so $c_1 = -4$.

Let's verify that $y = \frac{1}{1-4e^{-x}}$ is a solution to the ODE $y' = \underbrace{y - y^2}$.

Plug it in and check

left side

$$y' = \frac{d}{dx}\frac{1}{1-4e^{-x}} = \frac{d}{dx}(1-4e^{-x})^{-1}$$

$$= -1\,(1-4e^{-x})^{-2}\frac{d}{dx}(1-4e^{-x})$$

$$= -1\,(1-4e^{-x})^{-2}\,(0+4e^{-x}) = \frac{-4e^{-x}}{(1-4e^{-x})^2}$$

right side

$$y - y^2 = \frac{1}{1-4e^{-x}} - \frac{1}{(1-4e^{-x})^2}$$

$$= \frac{1-4e^{-x}}{1-4e^{-x}}\cdot\frac{1}{1-4e^{-x}} - \frac{1}{(1-4e^{-x})^2}$$

$$= \frac{\cancel{1}-4e^{-x}\;\cancel{-1}}{(1-4e^{-x})^2} = \frac{-4e^{-x}}{(1-4e^{-x})^2}$$