31. **(a)** Verify that $y = -1/(x + c)$ is a one-parameter family of solutions of the differential equation $y' = y^2$.

(b) Since $f(x, y) = y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere, the region R in Theorem 1.2.1 can be taken to be the entire xy-plane. Find a solution from the family in part (a) that satisfies $y(0) = 1$. Then find a solution from the family in part (a) that satisfies $y(0) = -1$. Determine the largest interval I of definition for the solution of each initial-value problem.

(c) Determine the largest interval I of definition for the solution of the first-order initial-value problem $y' = y^2$, $y(0) = 0$. [*Hint*: The solution is not a member of the family of solutions in part (a).]

To verify that $y = -\frac{1}{x+c}$ is a solution to the differential equation $y' = y^2$, evaluate both the left-hand side (y') and the right-hand side (y^2) of the differential equation using the given expression for y.

Step 1: Compute the derivative y' (Left-Hand Side)

Rewrite y using negative exponents:

$$y = -(x + c)^{-1}$$

Differentiate with respect to x using the power rule and chain rule:

$\int$

$$y' = \frac{d}{dx}\left[-(x+c)^{-1}\right] = -(-1)(x+c)^{-2} \cdot \frac{d}{dx}(x+c)$$

Since $\frac{d}{dx}(x+c) = 1$:

$$y' = \frac{1}{(x+c)^2}$$

Step 2: Compute y^2 (Right-Hand Side)

Square the expression for y:

$$y^2 = \left(-\frac{1}{x+c}\right)^2 = \frac{(-1)^2}{(x+c)^2} = \frac{1}{(x+c)^2}$$

Step 3: Compare both sides

Since both sides yield the same expression:

$$y' = \frac{1}{(x+c)^2} = y^2$$

The left-hand side equals the right-hand side, verifying that $y = -\frac{1}{x+c}$ is a one-parameter family of solutions to $y' = y^2$.

(b) Since $f(x, y) = y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere, the region R in Theorem 1.2.1 can be taken to be the entire xy-plane. Find a solution from the family in part (a) that satisfies $y(0) = 1$. Then find a solution from the family in part (a) that satisfies $y(0) = -1$. Determine the largest interval I of definition for the solution of each initial-value problem.

$y(x) = -\frac{1}{x+c}$

$y(0) = -1 = -\frac{1}{0+c} = -\frac{1}{c}$

so $c = 1$

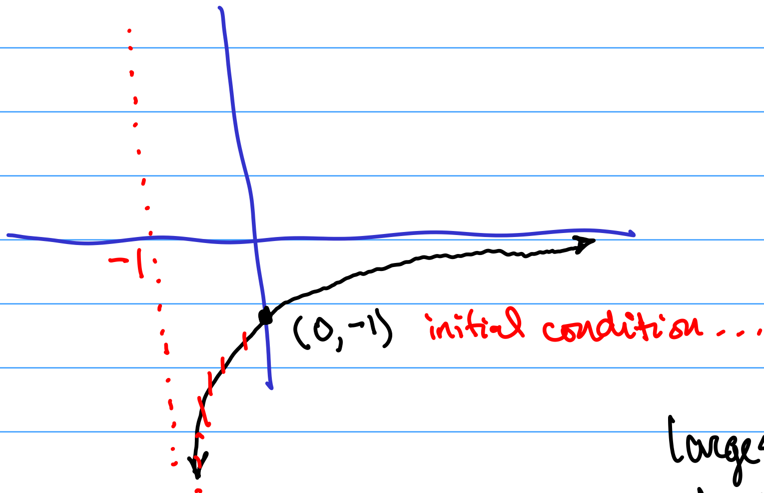
unique solution is $y(x) = -\frac{1}{x+1}$

also ... $\boxed{y(0) = 1} = -\frac{1}{0+c} = -\frac{1}{c}$

so $c = -1$

unique solution is $\boxed{y(x) = -\frac{1}{x-1}}$

Try $y(x) = -\frac{1}{x+1}$ first



Largest interval of existence of the solution is $(-1, \infty)$.

Similar for the other solution: Largest interval of existence of the solution is $(-\infty, 1)$.

(c) Determine the largest interval I of definition for the solution of the first-order initial-value problem $y' = y^2$, $y(0) = 0$. [*Hint*: The solution is not a member of the family of solutions in part (a).]

Start at $(0,0)$ so $y'(0) = 0$ so $y(0) = 0$ so it's a horizontal line and $y = 0$ everywhere.

plug $y=0$ into the ODE

$\frac{dy}{dx} = \frac{d0}{dx} = 0$ $\qquad y^2 = 0^2 = 0$

solution

The solution is $y=0$ and it exists on $(-\infty, \infty)$.

In fluid flow the convective term uses the velocity field to transport the velocity and this is a square term. A Grey institute prize problem worth \$1,000,000 asks do the solutions of the 3D fluid flow equations exist for all time and are they unique? The same difficulty is this square term.

32. **(a)** Show that a solution from the family in part (a) of Problem 31 that satisfies $y' = y^2$, $y(1) = 1$, is $y = 1/(2 - x)$.

(b) Then show that a solution from the family in part (a) of Problem 31 that satisfies $y' = y^2$, $y(3) = -1$, is $y = 1/(2 - x)$.

(c) Are the solutions in parts (a) and (b) the same?

$y(x) = \dfrac{-1}{x+c}$ $\qquad y(1) = 1 = \dfrac{-1}{1+c}$ $\qquad 1+c = -1$

$c = -2$

Therefore the unique solution is

$$y(x) = \frac{-1}{x-2} = \boxed{\frac{1}{2-x}}$$

$y(3) = -1 = \dfrac{-1}{3+c}$ $\qquad -3-c = -1$

$c = 2$

Therefore the unique solution is

$$y(x) = \frac{-1}{x-2} = \frac{1}{2-x}$$

(c) Are the solutions in parts (a) and (b) the same?

What about interval of existence.

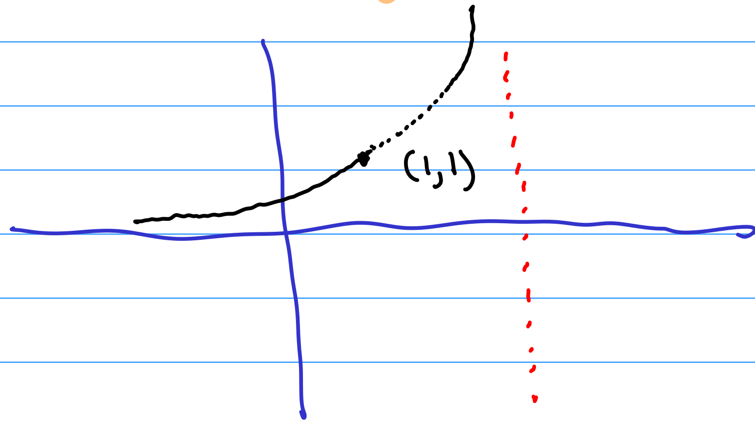
First solution

$y(1)=1=\frac{-1}{1+c}$ $\qquad 1+c=-1$

$\qquad c=-2$

Therefore the unique solution is

$y(x)=\frac{-1}{x-2}=\boxed{\frac{1}{2-x}}$ on the interval $(-\infty,2)$



The other solution

$y(3)=-1=\frac{-1}{3+c}$ $\qquad -3-c=-1$

$\qquad c=2$

Unique solution:

$y(x)=\frac{-1}{x-2}=\boxed{\frac{1}{2-x}}$ on the interval $(2,\infty)$

$f(x,y) = 0.2xy$ $\qquad y(2) = 2$

Unique solution in green