- Chapter 2.3
- Linear first order differential equations

$$\frac{dy}{dx} + p(x)y = f(x)$$

(linear in y)

might be very non-linear functions of x

Simpler

$$y' + py = f$$

Simpler

$$\frac{dy}{dx} + p(x)y = 0$$

set $f=0$. Now separable...

$$\frac{dy}{dx} = -p(x)y$$

$$\frac{1}{y}\frac{dy}{dx} = -p(x)$$

integrate both sides $\int \cdots dx$

$$\int \frac{1}{y}\frac{dy}{\cancel{dx}}\,\cancel{dx} = -\int p(x)\,dx$$

$$\int \frac{1}{y}\,dy = -\int p(x)\,dx$$

$$\ln|y| = -\int p(x)\,dx + c_1$$

add in the const. of integration...

$$|y| = e^{-\int p(x)\,dx + c_1} = e^{c_1} e^{-\int p(x)\,dx}$$

$$y = \boxed{\pm e^{c_1}}\, e^{-\int p(x)\,dx} = C_2 e^{-\int p(x)\,dx}$$

this const has the $\pm$ in it.

Now use the solution to simpler eq. to simplify the original:

$$y' + py = f$$

set $c_2 = 1$ for convenience...

$$y = u(x) \cancel{c_2} e^{-\int p(x)dx}$$

plug it in

$$y = u e^{-\int p(x)dx}$$

$$y' = u' e^{-\int p(x)dx} + u \frac{d}{dx} e^{-\int p(x)dx}$$

I know from before

$$\frac{d}{dx} e^{-\int p(x)dx} + p(x) e^{-\int p(x)dx} = 0$$

check this, just to be sure...

$$y' = u' e^{-\int p(x)dx} - u\, p(x) e^{-\int p(x)dx}$$

$$y' + py = f$$

$$u' e^{-\int p(x)dx} - u\, p(x) e^{-\int p(x)dx} + p u e^{-\int p(x)dx} = f$$

Thus

$$u' e^{-\int p(x)dx} = f$$

$$u' = e^{\int p(x)dx} f(x)$$

$$u = \int \boxed{\left(e^{\int p(x)dx}\right) f(x)}\, dx$$

call this the integrating factor

$$\mu = e^{\int p(x)dx}$$

$$y = u e^{-\int p(x)dx}$$

Example

$$y' + 4y = x$$

$p(x) = 4$
$f(x) = x$

$$\mu = e^{\int 4dx} = e^{4x}$$

when computing integrating factor take const. of integration equal anything that's convenient.

Integrating factor trick...
multiply entire equation by μ.

$$\mu y' + 4\mu y = \mu x$$

$$\underbrace{e^{4x}y' + 4e^{4x}y} = \boxed{e^{4x}x}$$

Can write these as the derivative of a product

$$\boxed{\frac{d}{dx}\left(e^{4x}y\right) = 4e^{4x}y + e^{4x}y'}$$

$$\frac{d}{dx}\left(e^{4x}y\right) = e^{4x}x$$

$$\cancel{\int}\cancel{\frac{d}{dx}}\left(e^{4x}y\right)dx = \int e^{4x}x\,dx$$

Now solve for y

$$e^{4x}y = \underbrace{\int e^{4x}x\,dx}_{\text{by parts... or guess and check}} = \left(\frac{x}{4}-\frac{1}{16}\right)e^{4x} + C$$

Solution

$$\boxed{y = e^{-4x}\left[\left(\frac{x}{4}-\frac{1}{16}\right)e^{4x} + C\right]}$$

$$\int e^{4x}x\,dx = (Ax+B)e^{4x} = \left(\frac{x}{4}-\frac{1}{16}\right)e^{4x} + C$$

$$\frac{d}{dx}(Ax+B)e^{4x} = Ae^{4x} + (Ax+B)\cdot 4e^{4x}$$

$$= \left(4Ax + (A+4B)\right)e^{4x} \approx e^{4x}x$$

So $4A = 1$ and $A+4B=0$

$A = \frac{1}{4}$ $\quad\frac{1}{4}+4B=0\quad$ $B = -\frac{1}{16}$

Thus $\int e^{4x}x\,dx = \left(\frac{x}{4}-\frac{1}{16}\right)e^{4x} + C$

and so $y = e^{-4x}\left[\left(\frac{x}{4}-\frac{1}{16}\right)e^{4x}+c\right] = \boxed{\frac{x}{4}-\frac{1}{16}+ce^{-4x}}$

Example

15. $y\,dx - 4(x + y^6)\,dy = 0$

$4(x+y^6)\frac{dy}{dx} = y$ $\qquad$ $\frac{dy}{dx} = \frac{y}{4(x+y^6)}$

not linear in y

divide by dy

$$y\frac{dx}{dy} = 4(x+y^6)$$

$$\frac{dx}{dy} = \frac{4}{y}x + 4y^5$$

$$\frac{dx}{dy} - \frac{4}{y}x = 4y^5$$

but is linear in x