$$M\,dx + N\,dy = 0$$

Solve for $f(x,y)$ such that $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$

Then $f(x,y) = C$ is a family of solutions...

Example

$$2xy\,dx + (x^2-1)\,dy = 0$$

$M = 2xy$ $\qquad\qquad$ $N = x^2 - 1$

Check that it's possible to find f. How:

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2-1) = 2x$$

since these agree we say the equation is exact.

Since exact we can find the f.

$\frac{\partial f}{\partial x} = M = 2xy$ $\qquad\qquad$ $f(x,y) = \int 2xy\,dx = x^2y + C$

holding y constant...

since y is constant then C could be a function of y

constant

$$f(x,y) = x^2y + g(y)$$

$$\frac{\partial f}{\partial y} = N = \cancel{x^2} - 1 = \frac{\partial}{\partial y}\left(x^2y + g(y)\right) = \cancel{x^2} + g'(y)$$

so $g'(y) = -1$ $\qquad\qquad$ $g(y) = -y + C_1$

$f(x,y) = x^2y - y + C_1$ take $C_1 = 0$ for convenience

$f(x,y) = x^2y - y$

different c

General solutions are $f(x,y) = c$

or $\boxed{x^2y - y = c}$

General solution is always $f(x,y) = 0$, since this is what we started with...

Example:

$$\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}$$

put into the form $M\,dx + N\,dy = 0$

$$y(1-x^2)\,dy = (xy^2 - \cos x \sin x)\,dx$$

$$\underbrace{(xy^2 - \cos x \sin x)}_{M}dx \underbrace{- y(1-x^2)}_{N}dy = 0$$

$M = xy^2 - \cos x \sin x$ $\quad\quad N = y(x^2-1)$.

Is finding f possible?, Is the eqn. exact?

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(xy^2 - \cos x \sin x) = 2xy$$

YES! So the eqn. is exact.

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\, y(x^2-1) = 2xy$$

partial derivative hold x constant here, so x is constant and can appear here

$$\frac{\partial f}{\partial y} = N = y(x^2-1). \qquad f = \int y(x^2-1)\,dy = \boxed{\tfrac{1}{2}y^2(x^2-1) + h(x)}$$

$$\frac{\partial f}{\partial x} = M = xy^2 - \cos x \sin x = \frac{\partial}{\partial x}\left(\frac{1}{2}y^2(x^2-1) + h(x)\right)$$

Therefore $xy^2 - \cos x \sin x = \frac{1}{2}y^2(2x-0) + h'(x)$

$$h'(x) = -\cos x \sin x$$

$$h(x) = -\int \cos x \sin x \, dx = -\int u\,du = -\frac{1}{2}u^2$$

$u = \sin x$ $du = \cos x \, dx$

$$= -\frac{1}{2}\sin^2 x$$

Solution: $f(x,y) = C$ where $f(x,y) = \frac{1}{2}y^2(x^2-1) - \frac{1}{2}\sin^2 x$

$$\boxed{\frac{1}{2}y^2(x^2-1) - \frac{1}{2}\sin^2 x = C}$$

Integrating factors to make equations exact..

$M\,dx + N\,dy = 0$ but $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Idea multiply by $\mu = \mu(x,y)$ both sides.

$$\boxed{\mu M}\,dx + \boxed{\mu N}\,dy = 0$$

Solve for μ so that this is exact.

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$ ← solve this...

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\mu_x N - \mu_y M = \mu(M_y - N_x)$$

← solve for μ, but this is a partial differential equation.

Simplify by considering $\mu = \mu(x)$

$$\mu_x N - \cancel{\mu_y M} = \mu (M_y - N_x)$$

$$\mu_x = \boxed{\frac{(M_y - N_x)}{N}} \mu .$$

should be eq of x

shouldn't have any y's.

OR// Simplify by considering $\mu = \mu(y)$

$$\cancel{\mu_x N} - \mu_y M = \mu (M_y - N_x)$$

$$\mu_y = \boxed{\frac{N_x - M_y}{M}} \mu$$

should by eq of y

no x's here