Theorem: The countable union of countable sets is countable. If I is countable and for every $\alpha \in I$ the set A_α is also countable, then

$$\bigcup_{\alpha \in I} A_\alpha \quad \text{is countable.}$$

Applications

$\mathbb{Z}$ is countable.

$$\mathbb{Z} = \underbrace{\underset{\text{finite}}{\{0\}} \cup \underset{\substack{\mathbb{N} \\ \text{countable}}}{\{1,2,3,\dots\}} \cup \underset{\substack{-\mathbb{N} \\ \text{also countable}}}{\{-1,-2,-3,\dots\}}}_{\text{finite union of countable sets}}$$

$$\mathbb{Q} = \bigcup_{n \in \mathbb{N}} \underbrace{\left\{ \frac{m}{n} : m \in \mathbb{Z} \right\}}_{\text{countable sets}}$$

countable union

$\left\{ \frac{m}{n} : m \in \mathbb{Z} \right\} \sim \mathbb{Z}$

bijection $f(m) = \frac{m}{n}$.

Thus $\mathbb{Q}$ is countable...

$$\mathbb{Q} \times \mathbb{Q} = \mathbb{Q}^2 = \bigcup_{q \in \mathbb{Q}} \{ (p,q) : p \in \mathbb{Q} \}$$

countable set

(p,q)

countable union

$\{ (p,q) : p \in \mathbb{Q} \} \sim \mathbb{Q}$

bijection $f((p,q)) = p$

Thus $\mathbb{Q} \times \mathbb{Q}$ is countable...

$\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q} = \mathbb{Q}\times(\mathbb{Q}\times\mathbb{Q})$

(p,q,r) $\quad (p,(q,r))$

a little different but never mind

$$= \mathbb{Q}^3 = \bigcup_{r\in\mathbb{Q}} \underbrace{\mathbb{Q}^2\times\{r\}}$$

countable since $\mathbb{Q}$ is countable.

Countable union.

$$\mathbb{Q}^{n+1} = \bigcup_{r\in\mathbb{Q}} \mathbb{Q}^n\times\{r\}$$

by induction $\mathbb{Q}^n$ is countable for all $n\in\mathbb{N}$

Theorem 2.17

The closed interval $[0,1]$ is uncountable...

Proof: For contradiction, suppose $[0,1]$ were countable..

Then there would be a bijection

$$f: \mathbb{N}\to[0,1]$$

Countable means can write them as a sequence.

$f(1) = y_1 = 0.a_{11}a_{12}a_{13}\dots = 0.246\dots$

$f(2) = y_2 = 0.a_{21}a_{22}a_{23}\dots = 0.597\dots$

$f(3) = y_3 = 0.a_{31}a_{32}a_{33}\dots = 0.693\dots$

where $a_{ij}\in\{0,1,\dots 9\}$ are digits in the expansion.

Idea find a number in $[0,1]$ thats not in the list..

Let $b_n = \begin{cases} 3 & \text{if } a_{nn}\neq 3 \\ 4 & \text{if } a_{nn}=3 \end{cases}$

For example on the diagonal shown

$$b_1 = 3, \quad b_2 = 3, \quad b_3 = 4, \quad \ldots$$

Let $b = 0.b_1 b_2 b_3 \ldots$

Then $b \neq y_n$ for any n because it's decimal expansion differs in the n^{th} place.

Since $b \in [0,1]$ and $b \notin f(\mathbb{N})$ this is a contradiction.

Thus $[0,1]$ is uncountable..

Note we used 3's and 4's to avoid the non-uniqueness of decimal representation with tails of 9's or 0's.

Recall

$$0.3\bar{9} = \frac{3}{10} + \frac{9}{100}\underbrace{\left(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots\right)}_{\frac{10}{9}}$$

sum the geometric series..

$$0.3\bar{9} = \frac{3}{10} + \frac{\cancel{9}}{\cancel{100}}\frac{\cancel{10}}{\cancel{9}} = \frac{3}{10} + \frac{1}{10} = .4$$

$$S = 1 + \cancel{\frac{1}{10}} + \cancel{\frac{1}{10^2}} + \ldots$$

$$\frac{1}{10}S = \cancel{\frac{1}{10}} + \cancel{\frac{1}{10^2}} + \cancel{\frac{1}{10^3}} + \ldots$$

$$\left(1 - \frac{1}{10}\right)S = 1$$

$$S = \frac{1}{1 - \frac{1}{10}} = \frac{10}{9}$$

So that why we avoid 9's and 0's when constructing $b = 0.b_1 b_2 b_3 \ldots$ using the diagonal...

Chapter 3

Limits, sequences ...

<u>Definition</u>

Definition 3.1 A *sequence of real numbers* (or a *sequence in* $\mathbb{R}$) is a function whose domain is $\mathbb{N}$ and whose range is a subset of $\mathbb{R}$.

$f : \mathbb{N} \to \mathbb{R}$

the sequence f,

Notation write.

$f(1) = x_1$

$f(2) = x_2$

$f(3) = x_3$

$\vdots$

$f(n) = x_n$

Also talk about the sequence $(x_n)_{n \in \mathbb{N}}$

Definition 3.2 A sequence $(x_n)_{n \in \mathbb{N}}$ *eventually* has a certain property if there exists an n_0 in $\mathbb{N}$ such that

$$(x_n)_{n \geq n_0} = (x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots)$$

has this property.

tail of a sequence

$g : \mathbb{N} \to \mathbb{R}$ and $g(1) = x_{n_0} = f(n_0)$

$g(2) = x_{n_0+1} = f(n_0+1)$

$g(n) = x_{n_0+n-1} = f(n_0+n-1)$.

Thus g is the tail of f starting at n_0.

Definition 3.3 For x in $\mathbb{R}$ and $\varepsilon > 0$, the open interval

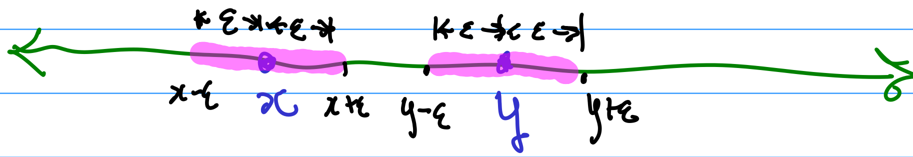
$$(x-\varepsilon, x+\varepsilon) = \{y \in \mathbb{R} : |y-x| < \varepsilon\}$$

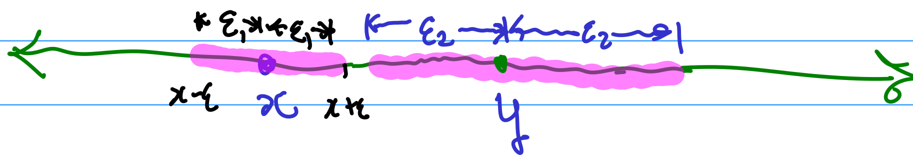
centered at x of radius ε is a *neighborhood of* x.

Sometimes called an ε-neighborhood of x

Separation property of the real numbers:

If $x \neq y$ then there is a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$.





Note that $\varepsilon_1 + \varepsilon_2 < |x-y|$ for the neighborhoods to be disjoint.

Proof Let x and y be in $\mathbb{R}$ with $x \neq y$. Let $\varepsilon = \frac{1}{2}|x-y|$. Let $U = (x-\varepsilon, x+\varepsilon)$ and $V = (y-\varepsilon, y+\varepsilon)$. We claim that $U \cap V = \emptyset$. Suppose that z is in $U \cap V$. Then, by the triangle inequality,

$$|x-y| \leq |x-z| + |z-y| < \varepsilon + \varepsilon = 2\varepsilon = |x-y|.$$

This is a contradiction, because a real number cannot be less than itself. Therefore, U and V are disjoint neighborhoods of x and y. (The reader should compare this with Exercise 11 in Section 2.1.) ■