Let $(x_n)_{\in\mathbb{N}}$ be a sequence of real numbers.

Define Convergence.

The sequence $(x_n)_{n\in\mathbb{R}}$ converges to $x$ if for every neighborhood of $x$ the sequence is eventually in that neighborhood. ?

Define Eventually

A sequence $(x_n)_{n\in\mathbb{N}}$ eventually has some property if some tail of that sequence has the property.

rewrite in mathematical shorthand...

recall

**Definition 3.2** A sequence $(x_n)_{n\in\mathbb{N}}$ *eventually* has a certain property if there exists an $n_0$ in $\mathbb{N}$ such that

$$(x_n)_{n\geq n_0} = (x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots)$$

has this property.

tail of a sequence

$$\lim_{n\to\infty} x_n = x$$

means

$$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \text{ s.t. } |x_n - x| < \varepsilon \text{ for all } n \geq n_0$$

equivalently

$$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \text{ s.t. } \forall n \geq n_0 \text{ holds } |x_n - x| < \varepsilon$$

Question: What does $\lim_{n\to\infty} x_n \neq x$ mean?

$$\text{not}\left(\lim_{n\to\infty} x_n = x\right)$$

$$\text{not}\left(\forall \varepsilon>0\ \exists n_0\in\mathbb{N} \text{ s.t. } \forall n\geq n_0 \text{ holds } |x_n-x|<\varepsilon\right)$$

$$\boxed{\exists \varepsilon>0 \text{ s.t. } \forall n_0\in\mathbb{N}\ \exists n\geq n_0 \text{ s.t. } |x_n-x|\geq\varepsilon.}$$

means $(x_n)_{n\in\mathbb{N}}$ does not have limit $x$

<u>Example:</u> $\lim_{n\to\infty}\frac{1}{n}=0$

since $\varepsilon>0$

Let $\varepsilon>0$. We want an $n_0$. By the Archimedian principle there is $n_0\in\mathbb{N}$ such that $\frac{1}{n_0}<\varepsilon$.

Thus if $n\geq n_0$ then $\frac{1}{n}\leq\frac{1}{n_0}$ and

$$\left|\frac{1}{n}-0\right| = \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon.$$

If $0<r<1$ then $\lim_{n\to\infty} r^n = 0$

Let $x$ be such that $r = \frac{1}{1+x}$.

solving $1+x = \frac{1}{r}$, $x = \frac{1}{r} - 1$

since $r<1$ then $x>0$.

Now, with this choice of $x$ we have

$$r^n = \frac{1}{(1+x)^n}$$

By Bernoulli's inequality ...

10. Prove Bernoulli's inequality: If $x > -1$, then $(1+x)^n \geq 1 + nx$ for each $n$ in $\mathbb{N}$.

We have that $(1+x)^n \geq 1+nx$ or $\frac{1}{(1+x)^n} \leq \frac{1}{1+nx}$

Therefore

$$r^n \leq \frac{1}{1+nx}.$$

Claim $\lim_{n\to\infty} r^n = 0$.

Proof: Let $\varepsilon>0$ and choose $n_0 > \frac{1}{x\varepsilon}$ since $\mathbb{N}$ is unbounded.

Then $n \geq n_0$ implies

$$|r^n - 0| = r^n = \frac{1}{(1+x)^n}$$ where $x = \frac{1}{r} - 1 > 0$

By Bernoulli's inequality $(1+x)^n \leq 1+xn$. Therefore

$$|r^n - 0| \leq \frac{1}{1+xn} \leq \frac{1}{xn} \leq \frac{1}{xn_0} < \frac{1}{x\left(\frac{1}{x\varepsilon}\right)} = \varepsilon$$

$$\frac{1}{xn_0} < \varepsilon$$

$$n_0 > \frac{1}{x\varepsilon}$$

or

$$\frac{1}{n_0} < x\varepsilon$$

by Archimedian principle

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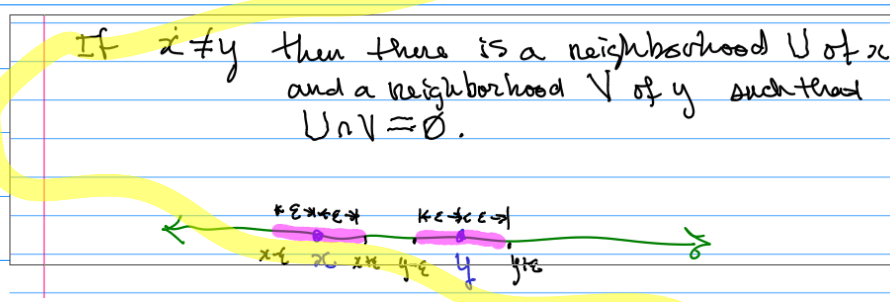
Proposition: Limits are unique..

If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$ then $x = y$

Proof: By Contradiction. Suppose $x \neq y$. Then by the separation property of real numbers there is a neighborhood $U$ of $x$ and a neighborhood $V$ of $y$ such that $U \cap V = \emptyset$,




- By definition $\lim_{n\to\infty} x_n = x$ means $(x_n)_{n\in\mathbb{N}}$ is eventually in $U$, $\exists\, n_0 \in \mathbb{N}$ such $x_n \in U$ for all $n \geq n_0$

• By definition $\lim_{n\to\infty} x_n = y$ means $(x_n)_{n\in\mathbb{N}}$ is eventually in $V$,

$\exists\, n_1 \in \mathbb{N}$ such $x_n \in V$ for all $n \geq n_1$

Let $n_2 = \max(n_0, n_1)$. Then

$n \geq n_2$ implies $x_n \in U$ and $x_n \in V$.

Thus $x_n \in U \cap V$. But since $U \cap V = \emptyset$ this is a contradiction.

## Section 3.2 Limit Laws:

Define. A sequence $(x_n)_{\in\mathbb{N}}$ is said to be bounded if $\exists\, B > 0$ s.t. $|x_n| \leq B$ for all $n \in \mathbb{N}$.

Proposition: If $\lim_{n\to\infty} x_n = x$ then $(x_n)_{n\in\mathbb{N}}$ is bounded.

Lemma: if $\lim_{n\to\infty} x_n = x$ and $x \neq 0$ then there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $|x_n| \geq \varepsilon$ for all $n \geq n_0$