Claim 2. $\lim_{n\to\infty} x_n y_n = xy = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n$;

Let $\varepsilon > 0$

Suppose $x_n \to x$ and $y_n \to y$.

Let $\varepsilon_1 = \boxed{\dfrac{\varepsilon}{3|y|+1}} > 0$. Then by definition of limit there is $n_1 \in \mathbb{N}$ such that $n \geq n_1$ implies $|x - x_n| < \varepsilon_1$

Let $\varepsilon_2 = \frac{2\varepsilon}{3} \Big/ \boxed{\left(\frac{\varepsilon}{3|y|+1} + |x|\right)} > 0$. Then by definition of limit there is $n_2 \in \mathbb{N}$ such that $n \geq n_2$ implies $|y - y_n| < \varepsilon_2$.

Need to show there is n_0 such that $|x_n y_n - xy| < \varepsilon$ for all $n \geq n_0$

Let $n_0 = \max(n_1, n_2)$. Then for $n \geq n_0$

Estimate

$$|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy|$$

$$\leq |x_n y_n - x_n y| + |x_n y - xy|$$

$$= |x_n|\,|y_n - y| + |x_n - x|\,|y|$$

$$< |x_n|\,\varepsilon_2 + \varepsilon_1 |y|$$

⇖ since trying to obtain a bound for all $n \geq n_0$ then can't depend on n here

$$\leq |x_n - x + x|\,\varepsilon_2 + \varepsilon_1 |y|$$

$$\leq |x_n - x|\,\varepsilon_2 + |x|\,\varepsilon_2 + \varepsilon_1 |y|$$

Thus,

$$|x_n y_n - xy| < \underbrace{\varepsilon_1 \varepsilon_2}_{\varepsilon/3} + \underbrace{|x|\,\varepsilon_2}_{\varepsilon/3} + \underbrace{\varepsilon_1 |y|}_{\varepsilon/3} \quad \text{for } n \geq n_0$$

Need $\varepsilon_1 \varepsilon_2 \leq \varepsilon/3$, $|x|\varepsilon_2 \leq \varepsilon/3$, $|y|\varepsilon_1 \leq \varepsilon/3$

Case $|y| > 0$ then $\varepsilon_1 = \dfrac{\varepsilon}{3|y|}$

if $|y| = 0$ then $\varepsilon_1 = 1$

Simpler $\boxed{\varepsilon_1 = \dfrac{\varepsilon}{3|y|+1}}$

Thus

$$|x_n y_n - xy| \leq \frac{\varepsilon}{3|y|+1}\varepsilon_2 + |x|\varepsilon_2 + \frac{\varepsilon}{3}$$

$$< \underbrace{\varepsilon_2\left(\frac{\varepsilon}{3|y|+1} + |x|\right)}_{2\varepsilon/3} + \frac{\varepsilon}{3}$$

$$\varepsilon_2 = \frac{2\varepsilon}{3} \Big/ \left(\frac{\varepsilon}{3|y|+1} + |x|\right)$$

It follows that

$$|x_n y_n - xy| < \varepsilon \quad \text{for } n \geq n_0$$

Therefore $\lim_{n\to\infty} x_n y_n = xy$

Please look at the proof in the book to see how it is different...

Also read how they prove

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y} \quad \text{for } y \neq 0.$$

Subsequences.

Def: The function $h: \mathbb{N} \to \mathbb{N}$ is strictly increasing if for $n, m \in \mathbb{N}$ with $n < m$ then $h(n) < h(m)$.

(*) Claim $h(n) \geq n$ for all $n \in \mathbb{N}$. Why?

$n_k \geq k$ by induction

Base case: $n=1$ then $h(n) \in \mathbb{N}$ implies $h(n) \geq 1 = 1$.

Induction step. Suppose $h(n) \geq n$.

Need to show $h(n+1) \geq n+1$.

since h is strictly increasing then

$$h(n+1) > h(n)$$

since they are integers this means

$$h(n+1) \geq h(n)+1 \geq n+1$$

by induction hypothesis

Def: $(y_n)_{n\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ if there exists a strictly increasing $h: \mathbb{N} \to \mathbb{N}$ such that $y_n = x_{h(n)}$ for all $n \in \mathbb{N}$.

Draw a picture in the book....

Equivalently, $y = x \circ h$, or the diagram

$$\begin{tikzcd} \mathbb{N} \arrow[r, "x"] & \mathbb{R} \\ \mathbb{N} \arrow[u, "h"] \arrow[ur, "y"'] \arrow[ur, bend left=80, "x \circ h"] \end{tikzcd}$$

$x_{h(n)}$ end

y_n

start

commutes.

If you follow the arrows you get the same result no matter which path you take

Notation: $y_n = x_{h(n)}$

Write $n_k = h(k)$ then $y_k = x_{n_k}$ ← with this notation the subsequence look like a stacking of subscripts...

Theorem If $x_n \to x$ then every subsequence of $(x_n)_{n\in\mathbb{N}}$ also converges to x.

Let $(y_n)_{n\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$. Then there is a strictly monotone $h : \mathbb{N} \to \mathbb{N}$ such that $y_n = x_{h(n)}$.

Notation: Let $(x_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$

Let $\varepsilon > 0$.

Since $x_n \to x$ then by definition there is $n_0 \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for $n \geq n_0$.

Suppose $k \geq n_0$ Estimate. $n_k \geq k \geq n_0$ by Claim (*)

Thus $|x_{n_k} - x| < \varepsilon$ for all $k \geq n_0$

So $x_{n_k} \to x$ as $k \to \infty$.

means not $\left(\lim_{n\to\infty} x_n = x\right)$

Theorem: Let $(x_n)_{n\in\mathbb{N}}$ and $x \in \mathbb{R}$

then $\boxed{\lim_{n\to\infty} x_n \neq x}$ if and only if

$\exists\, \varepsilon > 0$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$

such that $|x_{n_k} - x| \geq \varepsilon$ for all $k \in \mathbb{N}$.

Please read the proof of this in the book...

Monotone sequences next time...