Theorem: Let $(x_n)_{n\in\mathbb{N}}$ and $x \in \mathbb{R}$

then $\lim\limits_{n\to\infty} x_n \neq x$ if and only if

Definition A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is bounded if there exists $B>0$ such that $|x_n| \leq B$ for all $n \in \mathbb{N}$.

Definition 3.8 A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is *monotone increasing* (respectively, *strictly increasing*) if $x_n \leq x_{n+1}$ (respectively, $x_n < x_{n+1}$) for all n in $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is *monotone decreasing* (respectively, *strictly decreasing*) if $x_n \geq x_{n+1}$ (respectively, $x_n > x_{n+1}$) for all n in $\mathbb{N}$. A sequence is *monotone* (or *monotonic*) if it is either monotone increasing or monotone decreasing.

Monotone Convergence Theorem: A bounded monotone sequence is convergent.

Proof: Let $(x_n)_{n\in\mathbb{N}}$ be a bounded monotone sequence.

Since it's bounded,

there is a $B>0$ such that $|x_n| \leq B$ for all $n \in \mathbb{N}$.

Since it monotone it's either monotone increasing or monotone decreasing.

Case x_n is monotone increasing.

Then $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$,

Let $A = \{x_k : k \in \mathbb{N}\}$. Then A is a bounded set of real numbers

By the completeness axiom $\sup A \in \mathbb{R}$ and by the corollary to that axiom $\inf A \in \mathbb{R}$,

Let $\alpha = \sup A$.

Claim $\lim_{n\to\infty} x_n = \alpha$. Need to show for every $\varepsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $|x_n - \alpha| < \varepsilon$.

Since α is the least upper bound of A then $\alpha - \varepsilon$ is not an upper bound of $A = \{x_k : k \in \mathbb{N}\}$.

Thus there is an x_k such that $\alpha - \varepsilon < x_k$.

Let $n_0 = k$. Then for $n \geq n_0$ then $x_n \geq x_{n_0}$

monotone increasing

so $-x_n \leq -x_{n_0}$

$$|x_n - \alpha| = \alpha - x_n \leq \alpha - x_{n_0}$$

Since α is an upper bound

Since $\alpha - \varepsilon \leq x_{n_0}$ then $-x_{n_0} \leq \varepsilon - \alpha$

Therefore

$$|x_n - \alpha| < \alpha - x_{n_0} \leq \cancel{\alpha} + \varepsilon - \cancel{\alpha} = \varepsilon. \qquad \square$$

Monotone subsequence theorem:

Theorem 3.9 (Monotone Subsequence Theorem) Every sequence in $\mathbb{R}$ has a monotone subsequence.

Proof Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. For the purpose of this proof, we call the mth term x_m a *peak* if $x_m \geq x_n$ for all $n \geq m$. That is, x_m is a peak if x_m is never exceeded by any term that follows it.

finish next time...