Monotone subsequence theorem:

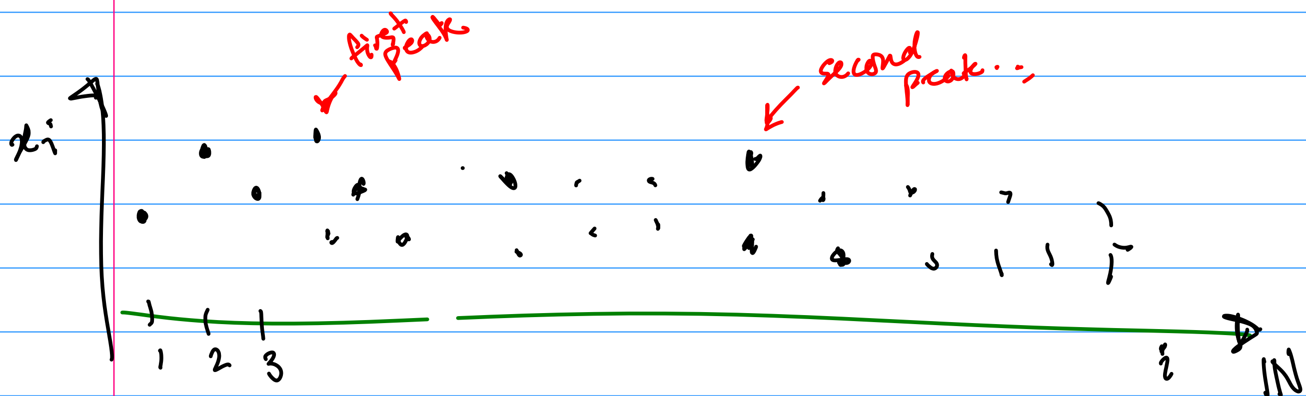
Theorem 3.9 (Monotone Subsequence Theorem) Every sequence in $\mathbb{R}$ has a monotone subsequence.

Proof Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. For the purpose of this proof, we call the mth term x_m a *peak* if $x_m \geq x_n$ for all $n \geq m$. That is, x_m is a peak if x_m is never exceeded by any term that follows it.

finish next time...

definition of a peak only for this proof...

Consider a sequence (x_n) $n\in\mathbb{N}$.



One of Two things could happen

① There are an infinite # of peaks

② There are an finite # of peaks

Case: The number of peaks is infinite. Make a sequence of peaks as follows...

Let m_1 be the smallest positive integer such that x_{m_1} is a peak.

Let m_2 be the smallest positive integer such that $m_2 > m_1$ and x_{m_2} is a peak.

⋮

Let m_{n+1} be the smallest positive integer such that $m_{n+1} > m_n$ and $x_{m_{n+1}}$ is a peak

Since by assumption there are an infinite number of peaks, then this construction yield a sequence of peaks x_{m_n} for $n \in \mathbb{N}$.

Since x_{m_1} is a peak and $m_2 > m_1$ then $x_{m_1} \geq x_{m_2}$

Since x_{m_2} is a peak and $m_3 > m_2$ then $x_{m_2} \geq x_{m_3}$

In summary $(x_{m_k})_{k \in \mathbb{N}}$ is a monotone decreasing subsequence of $(x_n)_{n \in \mathbb{N}}$.

Case there are a finite numer of peaks.. Then the construction of creating a sequence of peaks terminates and $x_{m_1} \geq x_{m_2} \geq x_{m_3} \geq \cdots \geq x_{m_r}$. Alternatively there are no peaks at all

$$n_1 = \begin{cases} 1 & \text{if no peaks at all} \\ m_r + 1 & \text{otherwise.} \end{cases}$$

Since x_{n_1} is not a peak there is $n_2 > n_1$ such that $x_{n_2} > x_{n_1}$

Since x_{n_2} is not a peak there is $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$

$\vdots$

Since x_{n_k} is not a peak there is $n_{k+1} > n_k$ such that $x_{n_{k+1}} > x_{n_k}$

and since the sequence $(x_n)_{n \in \mathbb{N}}$ has infinitely many terms the above yield a subsequence such that

$$x_{n_1} < x_{n_2} < x_{n_3} < \cdots < x_{n_k} < x_{n_{k+1}} < \cdots$$

That is, a strictly monotone increasing subsequence

3.5 Bolzano Weierstrass Theorem

page 55

Theorem 3.10 (Bolzano-Weierstrass Theorem for sequences) A bounded sequence in $\mathbb{R}$ has a convergent subsequence (that is, a subsequence that converges to a real number).

Proof Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence.
By the Monotone Subsequence Theorem there is a monotone subsequence $(x_{n_k})_{k\in\mathbb{N}}$.
Since $(x_n)_{n\in\mathbb{N}}$ is bounded then so is $(x_{n_k})_{k\in\mathbb{N}}$.

By the Monotone Convergence Theorem

recall

Monotone Convergence Theorem: A bounded monotone sequence is convergent.

then $(x_{n_k})_{k\in\mathbb{N}}$ is convergent.

Note there is another proof of this theorem in the book Proof 2 on page 55 that generalized to sequences of vectors... of course vectors are not until 311 but anyway please read over the weekend...

Bolzano–Weierstrass theorem for sets

Theorem 3.11 (Bolzano-Weierstrass Theorem for sets) Every bounded infinite subset of $\mathbb{R}$ has an accumulation point in $\mathbb{R}$.

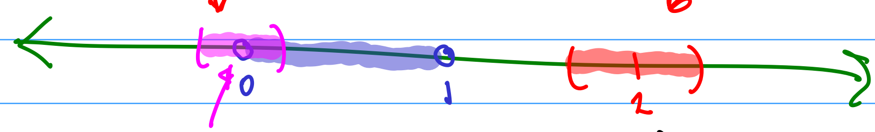
What's an accumulation point?

Let $A \subseteq \mathbb{R}$. Then $\boxed{x \in \mathbb{R}}$ is an accumulation point of A is every neighborhood of x contains at least one point in A other than x.

Example: $A = (0,1)$ then set of all accumulation points $[0,1]$.

any neighborhood

neighborhood around 2.



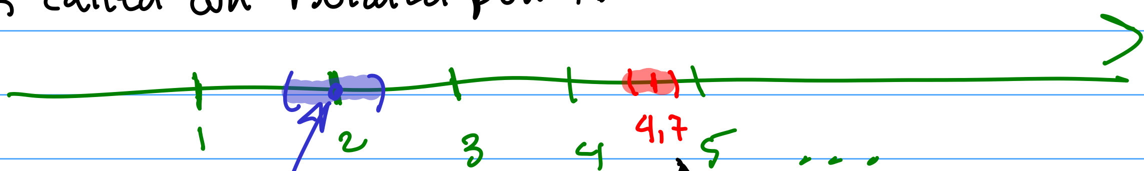
Is 0 an accumulation point

So 2 is not an accumulation point of A

any neighborhood of 0 contains points other than 0 in A.

Example: $A = \mathbb{N}$ has no accumulation points

Def. If $x \in A$ and x is not an accumulation point it is called an isolated point.



Is 2 an accumulation point?

Not an accumulation point because nothing other than 2 is in A.

not an accumulation point of A because there is a neighborhood that doesn't have any points of A in it