Definition. $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is a Cauchy sequence if for every $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that $m,n\geq n_0$ implies $|x_n-x_m|<\varepsilon$.

Proposition 3.5 A convergent sequence is Cauchy.

Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence. Claim that it's Cauchy.

Since $(x_n)_{n\in\mathbb{N}}$ is convergent there is a limit $x\in\mathbb{R}$ such that $\lim_{n\to\infty} x_n = x$.

Let $\varepsilon>0$ be arbitrary

Set $\varepsilon_1=\varepsilon/2 > 0$ there is n_0 such that

$|x-x_n|<\varepsilon_1$ for every $n\geq n_0$

Let $n,m\geq n_0$. Then estimate

$$|x_n-x_m| \leq |x_n - x + x - x_m|$$

introduce the limit value

$$\leq |x_n-x| + |x_m-x| < \varepsilon_1+\varepsilon_1 = 2\varepsilon_1 = 2\frac{\varepsilon}{2} = \varepsilon.$$

Shorter form of the proof:

Let $\varepsilon > 0$ be arbitrary

There is n_0 such that

$|x - x_n| < \varepsilon/2$ for every $n \geq n_0$

Let $n, m \geq n_0$. Then estimate

condition for a sequence to be Cauchy...

$$|x_n - x_m| \leq |x_n - x + x - x_m|$$

introduce the limit value

$$\leq |x_n - x| + |x_m - x| < \varepsilon/2 + \varepsilon/2 = 2\frac{\varepsilon}{2} = \varepsilon.$$

Lemma 3.3 A Cauchy sequence is bounded.

Proof: Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence.

Let $\varepsilon = 1$. Then there is n_0 such that

$|x_m - x_n| < 1$ for all $m, n \geq n_0$

Estimate. For $m, n \geq n_0$ we have

$$(*) \quad |x_n| = |x_n - x_m + x_m| \leq |x_n - x_m| + |x_m| < \underbrace{1 + |x_m|}$$

bound that doesn't depend on n.

Let $m \geq n_0$ be fixed. For example take $m = n_0$.

Define $B = \max\{|x_1|, |x_2|, \ldots, |x_{n_0-1}|, 1+|x_{n_0}|\}$

Claim that $|x_n| \leq B$ for all $n \in \mathbb{N}$.

Case $n \geq n_0$ then $|x_n| \leq 1+|x_{n_0}| \leq B$ by $(*)$

Case $n < n_0$ then $|x_n| \leq B$ since the $|x_n|$ appears in the max defining B.

Theorem 3.12 A sequence in $\mathbb{R}$ is Cauchy if and only if the sequence converges.

"$\Leftarrow$" Was already done as prop 3.5.

"$\Rightarrow$" Suppose $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Claim that the sequence converges.

By lemma 3.3 $(x_n)_{n\in\mathbb{N}}$ is bounded

By the Bolzano-Weierstrass theorem there is a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} x_{n_k} = x \quad \text{for some } x \in \mathbb{R}.$$

Claim $\lim_{n\to\infty} x_n = x$.

Let $\varepsilon > 0$, we need to find $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $|x_n - x| < \varepsilon$. But how?

Choose $\varepsilon_1 = \boxed{\varepsilon/2}$. Then since $(x_n)_{n\in\mathbb{N}}$ is Cauchy there is n_1 such that

$$|x_n - x_m| < \varepsilon_1 \quad \text{for all } m, n \geq n_1.$$

Choose $\varepsilon_2 = \boxed{\varepsilon/2}$. Then since $\lim_{k\to\infty} x_{n_k} = x$

there is n_2 such that

$$|x_{n_k} - x| < \varepsilon_2 \quad \text{for all} \quad k \geq n_2$$

Let $n_0 = \max(n_1, n_2)$, Estimate. Suppose $n \geq n_0$

Choose $k \geq n_0$ then $k \geq n_2$ and $n_k \geq k \geq n_1$

property of subsequences

$$|x_n - x| = |x_n - x_{n_k} + x_{n_k} - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x|$$

$$< \varepsilon_1 + \varepsilon_2 = \varepsilon$$