Definition of continuity

For every $\varepsilon > 0$ there is a $\delta > 0$ such that $x \in D$ and $|x-c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.

Prop 4.3

Theorem: Composition of continuous functions is continuous. Namely: Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ such that $f(A) \subseteq B$. Suppose f is continuous at $c \in A$ and g is continuous at $f(c)$ then $g \circ f: A \to \mathbb{R}$ is continuous at c.

Proof: Let $\varepsilon > 0$

Since g is continuous at $f(c)$ then for $\varepsilon_2 = \boxed{\varepsilon} > 0$

(*) there exists $\delta_2 > 0$ such that $y \in B$ and $|y - f(c)| < \delta_2$ implies $|g(y) - g(f(c))| < \varepsilon_2$

Since f is continuous at c then for $\varepsilon_1 = \boxed{\delta_2} > 0$

(**) there exists $\delta_1 > 0$ such that $x \in A$ and $|x-c| < \delta_1$ implies $|f(x) - f(c)| < \varepsilon_1$

Choose $\delta = \delta_1$. Then $x \in A$ and $|x - c| < \delta$ implies by (**) that

$$|f(x) - f(c)| < \varepsilon_1 = \delta_2.$$

Since $f(x) \in f(A) \subseteq B$ and $|f(x) - f(c)| < \delta_2$ then by (*) holds

$$|g(f(x)) - g(f(c))| < \varepsilon_2 = \varepsilon.$$

Thus $g \circ f$ is continuous at c.

Lemma 4.1 Let $f: D \to \mathbb{R}$ be continuous at a point c in D. If $f(c) \neq 0$, then there exist an $\varepsilon > 0$ and a neighborhood U of c such that $|f(x)| > \varepsilon$ for all x in $U \cap D$.

Proof:

Since f is continuous at c then for $\varepsilon_1 = \boxed{|f(c)|/2} > 0$ there is $\delta_1 > 0$ such that $x \in D$ and $|x - c| < \delta_1$ implies $|f(x) - f(c)| < \varepsilon_1$.

Estimate (from below)

$$|f(x)| = |f(x)-f(c)+f(c)| \geq |f(c)| - |f(x)-f(c)|$$

or (from above)

$$|f(c)| \leq |f(c)-f(x)+f(x)| \leq |f(x)-f(c)| + |f(x)|$$

For contradiction suppose

not then there exist an $\varepsilon > 0$ and a neighborhood U of c such that $|f(x)| > \varepsilon$ for all x in $U \cap D$.

Then for all $\varepsilon_2 > 0$ and all neighborhoods U of c

then $|f(x)| \leq \varepsilon_2$ for some $x \in U \cap D$.

all I need is a contradiction of some sort...

Choose $\varepsilon_2 = |f(c)|/2$. Then

let U be the δ_1 neighborhood of c given by $(c-\delta_1, c+\delta_1)$.

Then for some $x \in U \cap D$, $|f(x)| \leq \varepsilon_2$ and

$|x-c| < \delta_1$ implies $|f(x)-f(c)| < \varepsilon_1$. Therefore.

$$|f(c)| \leq |f(c)-f(x)+f(x)| \leq |f(x)-f(c)| + |f(x)|$$

$$< \varepsilon_1 + \varepsilon_2 = \frac{|f(c)|}{2} + \frac{|f(c)|}{2} = |f(c)|$$

want this to be a contradiction,
so how do I choose ε_1 and ε_2.

which is a contradiction. Therefore

then there exist an $\varepsilon > 0$ and a neighborhood U of c such that $|f(x)| > \varepsilon$ for all x in $U \cap D$.

Lemma 4.1 Let $f : D \to \mathbb{R}$ be continuous at a point c in D. If $\boxed{f(c) \neq 0,}$ then there exist an $\varepsilon > 0$ and a neighborhood U of c such that $|f(x)| > \varepsilon$ for all x in $U \cap D$.

Proposition 4.2 Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be functions. Let c be an element of D and assume that f and g are continuous at c. Then the functions $f + g$, $f - g$, $f \cdot g$, and kf (for any k in $\mathbb{R}$) are continuous at c. Also, if $\boxed{g(c) \neq 0,}$ then f/g is continuous at c.

- Note the lemma is used in the proof that $\frac{f}{g}$ is continuous so the denominator is never 0.

Read proof of Prop 4.2 at home.. Proofs are similar to the proofs of the limit laws...

Theorem 4.1 Let $f : D \to \mathbb{R}$ and let c be in D. Then f is continuous at c if and only if whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in D that converges to c, then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$.

Proposition 4.4 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ with $f(x) = 0$ for all x in $\mathbb{Q}$. Then $f(x) = 0$ for all x in $\mathbb{R}$.

Proposition 4.5 Let $f : D \to \mathbb{R}$ with c in D. If f is continuous at c, then there is a neighborhood U of c such that f is bounded on $U \cap D$.

Lemma 4.2 Let $f : D \to \mathbb{R}$ with c in D. Make the following suppositions:

1. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Q} \cap D$ converging to c, then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$;

and

2. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $(\mathbb{R} \setminus \mathbb{Q}) \cap D$ converging to c, then $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(c)$.

Then f is continuous at c.

after the spring break ...